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Solutions of Coupled KdV-Type Equations

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We consider a family of coupled (Ito type) KdV equations in 1+1 dimensions and use Hlavatý's technique to obtain a class of explicit wave-type solutions.

In a paper by Hlavatý (1985), a systematic approach for finding a class of wave-type solutions of the KdV-type equations has been reported. The objective of the present note is to extend this procedure to coupled systems to derive a similar set of solutions. As is well known, coupled nonlinear equations in which a KdV structure is embedded occur naturally in shallow water wave problems (Whitham, 1974). Further, such equations also possess infinitely many symmetries and conservation laws (Hirota and Satsuma, 1981; Ito, 1982; Antonowicz and Fordy, 1987).

Let us consider the family of coupled KdV equations

$$f_t + \alpha g g_x + \beta f f_x + \gamma f_{xxx} = 0 \tag{1a}$$

$$g_t + \delta (fg)_x = 0 \tag{1b}$$

where α, β, γ, and δ are arbitrary parameters and subscripts denote partial derivatives. In the following we shall show that the solutions of (1a), (1b) can be dealt with exactly by adopting the technique of Hlavatý (1985). It is worth remarking that for $\alpha = -2$, $\beta = -6$, $\gamma = -1$, and $\delta = -2$, the set (1) represents Ito's equation, which has infinitely many conserved quantities (Ito, 1982). It may be noticed that this choice of parameters is not unique (see, e.g., Antonowicz and Fordy, 1987).

We have shown elsewhere (Guha-Roy *et al.*, 1986) that if one of the solutions of (1a), (1b) is of the traveling wave type, then the other must

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also exhibit the same form. Suppose that the variables $f(x, t)$ and $g(x, t)$ depend on the same traveling wave variable s in the following manner:

$$s = x - ct, \qquad f(x, t) = f(s), \qquad g(x, t) = g(s) \tag{2}$$

where c (>0) is the wave velocity.

Introducing (2) into (1a), (1b) and then integrating once, we have

$$-cf + \tfrac{1}{2}\alpha g^2 + \tfrac{1}{2}\beta f^2 + \gamma f_{ss} = k_1 \tag{3a}$$

$$-cg + \delta fg = k_2 \tag{3b}$$

where k_1 and k_2 are integration constants.

Inserting (3b) into (3a), we obtain

$$-cf + \frac{1}{2}\frac{\alpha k_2^2}{(\delta f - c)^2} + \frac{1}{2}\beta f^2 + \gamma f_{ss} = k_1 \tag{4}$$

Integrating (4) yields

$$-\frac{1}{2}cf^2 - \frac{1}{2}\frac{\alpha k_2^2}{\delta(\delta f - c)} + \frac{1}{6}\beta f^3 + \frac{1}{2}\gamma f_s^2 = k_1 f + k_3 \tag{5}$$

where k_3 is another integration constant.

For convenience, we set $\psi = (\delta f - c)$ to rewrite (5) as

$$\psi\psi_s^2 = \zeta_4\psi^4 + \zeta_3\psi^3 + \zeta_2\psi^2 + \zeta_1\psi + \zeta_0 \tag{6}$$

where the parameters ζ are given by

$$\zeta_4 = -\frac{\beta}{3\gamma\delta}$$

$$\zeta_3 = \frac{1}{\gamma}\left(1 - \frac{\beta}{\delta}\right)c$$

$$\zeta_2 = \frac{2}{\gamma}\left(c^2 + k_1\delta - \frac{\beta c^2}{2\delta}\right)$$

$$\zeta_1 = \frac{1}{\gamma}\left(c^3 + 2k_1\delta c + 2k_3\delta^2 - \frac{\beta c^3}{3\delta}\right)$$

$$\zeta_0 = \frac{\alpha k_2^2\delta}{\gamma}$$

The form of equation (6) suggests that if we define (Takahashi and Satsuma, 1988; Guha-Roy, 1989) an independent variable τ such that

$$\tau = \int^s \frac{1}{\psi^{1/2}}\, ds \tag{7}$$

then a potential function $\varphi(\psi)$ may be introduced as

$$\psi_\tau^2 = \sum_{r=0}^{4} \zeta_r \psi^\gamma = -\varphi(\psi) \tag{8}$$

It may checked that the solutions of equation (8) are expressible in terms of the elliptic functions (see, for instance, Wadati, 1975). However, more interesting results of (8) can be obtained if one follows the procedure described by Hlavatý (1985).

To this end, let us consider a class of potentials of the following type:

$$-\varphi(\psi) \equiv \zeta_4(\psi+\lambda_1)^2(\psi^2+\lambda_2\psi+\lambda_3) \tag{9}$$

where λ_1, λ_2, and λ_3 are constants dependent on the parameters ζ_r and $\zeta_4 \neq 0$. This means that by introducing

$$\psi = \frac{1}{\theta} - \lambda_1 \tag{10}$$

we can recast equation (8) as

$$\theta_\tau = (m\theta^2 + l\theta + k)^{1/2} \tag{11}$$

in which

$$m = \zeta_4(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_3)$$

$$l = \zeta_4(\lambda_2 - 2\lambda_1)$$

$$k = \zeta_4$$

It is obvious that the solutions of (11) are crucially dependent upon the parameters m, l, and k. Let us choose (Hlavatý, 1985)

$$\theta(\tau;\, k, l, m \neq 0) = \frac{1}{2m}\{\sigma(\tau;\, m, \Delta) - l\} \tag{12}$$

such that

$$\sigma_\tau^2 = m(\sigma^2 - \Delta), \qquad \Delta = (l^2 - 4\,km) \tag{13}$$

We are now in a position to consider the following possibilities for the solutions of (11), depending on the signs of m and Δ:

For $m > 0$ and $\Delta > 0$,

$$\sigma(\tau;\, m, \Delta) = \mu\sqrt{\Delta}\cosh(\sqrt{m}\,\tau) \tag{14a}$$

For $m > 0$ and $\Delta < 0$,

$$\sigma(\tau;\, m, \Delta) = \mu(-\Delta)^{1/2}\sinh(\sqrt{m}\,\tau) \tag{14b}$$

For $m > 0$ and $\Delta = 0$,

$$\sigma(\tau; m, \Delta) = \mu \exp(\pm\sqrt{m}\,\tau) \tag{14c}$$

And for $m < 0$ and $\Delta > 0$,

$$\sigma(\tau; m, \Delta) = \mu\sqrt{\Delta}\sin[(-m)^{1/2}\tau] \tag{14d}$$

Here $\mu = \pm 1$.

Substitutions of (14a)-(14d) into (12) then give different forms of θ. It may be noted that one can also obtain other suitable forms of θ by writing

$$\theta(\tau; k, l \neq 0, m = 0) = \frac{1}{l}\left(\tfrac{1}{4}l\tau^2 - k\right) \tag{15}$$

$$\theta(\tau; k \neq 0, l, m = 0) = \sqrt{k}\,\tau \tag{16}$$

Finally, the corresponding solutions for ψ may be determined by plugging the above expressions for θ into (10).

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