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Proof Theory for Minimal Quantum Logic: A Remark

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It is remarked that the inference rule $(' \to ')$ is superfluous for the sequential system GMQL introduced by H. Nishimura for the minimal quantum logic.

1. INTRODUCTION

The purpose of this note is to show that the inference rule

$$(' \to '):\quad \frac{\Gamma \to \Delta}{\Delta' \to \Gamma'}$$

is superfluous for the sequential system GMQL introduced by Nishimura (1994) for the minimal quantum logic. Namely, our goal is to prove the following theorem.

Theorem. If a sequent is provable in GMQL, then it is provable in GMQL without $(' \to ')$.

Remember that the antecedent Γ and the succedent Δ of the sequent $\Gamma \to \Delta$ are finite sets of formulas.

2. AUXILIARY SYSTEM AND LEMMAS

For the proof of the above theorem, we introduce the auxiliary system GMQL$^{\#}$, which is obtained from GMQL by deleting the inference rule $(' \to ')$ as well as

$$(' \to):\quad \frac{\Gamma \to \Delta}{\Delta', \Gamma \to} \qquad \text{and} \qquad (\to '):\quad \frac{\Gamma \to \Delta}{\to \Delta, \Gamma'}$$

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while by supplying any sequent of the form $\alpha', \alpha \to$ or $\to \alpha, \alpha'$ as an additional axiom sequent, and the rules $(\wedge' \to)^{\#}$ and $(\to \vee')^{\#}$ which are described below as the additional ones. Namely, the axiom sequents of GMQL# are those having the form: $\alpha \to \alpha$; $\alpha', \alpha \to$; or $\to \alpha, \alpha'$; while the inference rules of GMQL# are as follows:

$$\text{(extension):}\quad \frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Sigma}$$

$$(\wedge \to):\quad \frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}, \quad \frac{\beta, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}$$

$$(\to \wedge):\quad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}; \qquad ('\to \wedge):\quad \frac{\alpha' \to \Delta \quad \beta' \to \Delta}{\to \Delta, \alpha \wedge \beta}$$

$$(\vee \to):\quad \frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta}; \qquad (\vee \to '):\quad \frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\alpha \vee \beta, \Gamma \to}$$

$$(\to \vee):\quad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta}, \quad \frac{\Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha \vee \beta}$$

$$('' \to):\quad \frac{\alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta}; \qquad (\to ''):\quad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha''}$$

$$(\wedge' \to):\quad \frac{\alpha' \to \Delta \quad \beta' \to \Delta}{(\alpha \wedge \beta)' \to \Delta}; \qquad (\wedge' \to)^{\#}:\quad \frac{\Gamma \to \alpha \quad \Gamma \to \beta}{(\alpha \wedge \beta)', \Gamma \to}$$

$$(\to \wedge'):\quad \frac{\Gamma \to \Delta, \alpha'}{\Gamma \to \Delta, (\alpha \wedge \beta)'}, \quad \frac{\Gamma \to \Delta, \beta'}{\Gamma \to \Delta, (\alpha \wedge \beta)'}$$

$$(\vee' \to):\quad \frac{\alpha', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta}, \quad \frac{\beta', \Gamma \to \Delta}{(\alpha \vee \beta)', \Gamma \to \Delta}$$

$$(\to \vee'):\quad \frac{\Gamma \to \alpha' \quad \Gamma \to \beta'}{\Gamma \to (\alpha \vee \beta)'}; \qquad (\to \vee')^{\#}:\quad \frac{\alpha \to \Delta \quad \beta \to \Delta}{\to \Delta, (\alpha \vee \beta)'}$$

We will prove the following lemmas in the next section.

Lemma 1. (1) If the sequent $\alpha'', \Gamma \to \Delta$ is provable in GMQL#, then so is $\alpha, \Gamma \to \Delta$.

(2) If the sequent $\Gamma \to \Delta, \alpha''$ is provable in GMQL#, then so is $\Gamma \to \Delta, \alpha$.

Lemma 2. If the sequent $\Gamma \to \Delta$ is provable in GMQL#, then so is $\Delta' \to \Gamma'$.

Lemma 3. (1) If the sequent $\Gamma \to \Delta$ is provable in GMQL#, then so is $\Delta', \Gamma \to$.

(2) If the sequent $\Gamma \to \Delta$ is provable in GMQL$^{\#}$, then so is $\to \Delta, \Gamma'$.

Proof of Theorem. Suppose that the sequent S is provable in GMQL. By Lemmas 2 and 3, S is provable in GMQL$^{\#}$. Since additional axiom sequents of GMQL$^{\#}$ are obtainable from axiom sequents of GMQL by $(' \to)$ or $(\to ')$, and since the additional inference rule $(\wedge' \to)^{\#}$ $[(\to \vee')^{\#}]$ is justified by $(\to \wedge)$ and $(' \to)$ $[(\vee \to)$ and $(\to ')]$, S is provable in GMQL without $(' \to ')$. ∎

3. PROOF OF LEMMAS

Proof of Lemma 1. We will prove this by induction on the length of the given proof. We will mention only (1), and denote by S the sequent $\alpha'', \Gamma \to \Delta$.

Case 1. The case where S is an axiom sequent: We divide this case into three subcases according to the form of S.

Subcase 1.1. The subcase where S is $\alpha'' \to \alpha''$: The sequent $\alpha \to \alpha''$ is obtainable from the axiom sequent $\alpha \to \alpha$ by $(\to '')$, and so is provable.

Subcase 1.2. The subcase where S is $\alpha'', \alpha' \to$: The sequent $\alpha, \alpha' \to$ is an axiom, and so is provable.

Subcase 1.3. The subcase where S is $\alpha'', \alpha''' \to$: The sequent $\alpha, \alpha''' \to$ is obtainable from the axiom sequent $\alpha, \alpha' \to$ by $('' \to)$, and so is provable.

In the rest of this proof, we let I be the last inference of the given proof of S.

Case 2. The case where I is (extension): The inference I has one of the following two forms:

$$\frac{\Gamma_1 \to \Delta_1}{\alpha'', \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2}; \qquad \frac{\alpha'', \Gamma_1 \to \Delta_1}{\alpha'', \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2}$$

In the former case, by applying (extension) to $\Gamma_1 \to \Delta_1$, the sequent $\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2$ is provable; while in the latter case, by the induction hypothesis, $\alpha, \Gamma_1 \to \Delta_1$ is provable, and so is $\alpha, \Gamma_2, \Gamma_1 \to \Delta_1, \Delta_2$ by (extension).

Case 3. The case where I is $('' \to)$: We divide this case into two subcases according as the principal formula of I is α'' or not.

Subcase 3.1. The subcase where the principal formula of I is α'': The inference I has one of the following two forms:

$$\frac{\alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta}; \qquad \frac{\alpha'', \alpha, \Gamma \to \Delta}{\alpha'', \Gamma \to \Delta}$$

In the former case, $\alpha, \Gamma \to \Delta$ is provable clearly; while in the latter case, it is provable, too, by the induction hypothesis.

Subcase 3.2. The subcase where the principal formula of I is not α'': The inference I has the form

$$\frac{\alpha'', \beta, \Gamma_1 \to \Delta}{\alpha'', \beta'', \Gamma_1 \to \Delta}$$

The sequent $\alpha, \beta, \Gamma_1 \to \Delta$ is provable by the induction hypothesis, and so is $\alpha, \beta'', \Gamma_1 \to \Delta$ by $('' \to)$.

Case 4. The case where I is not (extension) nor $('' \to)$: Similar to Subcase 3.2. ■

Proof of Lemma 2. The proof is by induction of the length of the given proof. We will denote by S the sequent $\Gamma \to \Delta$.

Case 1. The case where S is an axiom sequent: The sequent S has one of the following three forms: $\alpha \to \alpha$; $\alpha', \alpha \to$; and $\to \alpha, \alpha'$. The sequents $\alpha' \to \alpha'$; $\to \alpha', \alpha''$; and $\alpha'', \alpha' \to$ are axioms, and so are provable.

In the rest of this proof, we let I be the last inference of the given proof of S.

Case 2. The case where I is either (extension), $(\wedge \to)$, $(\to \wedge)$, $(\vee \to)$, $(\to \vee)$, $('' \to)$, or $(\to '')$: All the cases can be dealt with similarly, so we deal only with the case where I is $(\to \wedge)$. The inference I has the form

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}$$

By the induction hypothesis, $\alpha' \to \Gamma'$ and $\beta' \to \Gamma'$ are provable, and so is $(\alpha \wedge \beta)' \to \Gamma'$ by $(\wedge' \to)$.

Case 3. The case where I is either $(\wedge' \to)^{\#}$ or $(\to \vee')^{\#}$: Similar to Case 2, by applying $(\to '')$ or $('' \to)$ in addition.

Case 4. The case where I is either $(' \to \wedge)$ or $(\vee \to ')$: Suppose that I is $(' \to \wedge)$ and has the form

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{\to \Delta_1, \alpha \wedge \beta}$$

By the induction hypothesis and Lemma 1, $\Delta_1' \to \alpha$ and $\Delta_1' \to \beta$ are provable, and so is $(\alpha \wedge \beta)', \Delta_1' \to$ by $(\wedge' \to)^{\#}$.

Case 5. The case where I is either $(\wedge' \to)$, $(\to \wedge')$, $(\vee' \to)$, or $(\to \vee')$: Similar to Case 4, by applying $(\to '')$ or $('' \to)$ in addition. ■

Proof of Lemma 3. The proof is by induction on the length of the given proof, too. We will mention only (1), and denote by S the sequent $\Gamma \to \Delta$.

Case 1. The case where S is an axiom sequent: The sequent S has one of the following three forms: $\alpha \to \alpha$; $\alpha', \alpha \to$; and $\to \alpha, \alpha'$. The sequents $\alpha', \alpha \to$ and $\alpha'', \alpha' \to$ are axioms, and so are provable.

In the rest of this proof, we let I be the last inference of the given proof of S.

Case 2. The case where I is either (extension), ($\wedge \to$), ($\to \vee$), ($'' \to$), ($\to ''$), or ($\vee' \to$): All the cases can be dealt with similarly, so we suppose that I is ($\to \vee$) and has the form

$$\frac{\Gamma \to \Delta_1, \alpha}{\Gamma \to \Delta_1, \alpha \vee \beta}$$

By the induction hypothesis, $\alpha', \Delta_1', \Gamma \to$ is provable, and so is $(\alpha \vee \beta)', \Delta_1', \Gamma \to$ by ($\vee' \to$).

Case 3. The case where I is ($\to \wedge'$): Suppose that I has the form

$$\frac{\Gamma \to \Delta_1, \alpha'}{\Gamma \to \Delta_1, (\alpha \wedge \beta)'}$$

By the induction hypothesis and Lemma 1, $\alpha, \Delta_1', \Gamma \to$ is provable, and so is $(\alpha \wedge \beta)'', \Delta_1', \Gamma \to$ by ($\wedge \to$) and ($'' \to$).

Case 4. The case where I is ($\to \wedge$): Suppose that I has the form

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}$$

By applying $(\wedge' \to)^{\#}$ to $\Gamma \to \alpha$ and $\Gamma \to \beta$, the sequent $(\alpha \wedge \beta)', \Gamma \to$ is provable.

Case 5. The case where I is ($\to \vee'$): Similar to Case 4, by applying ($'' \to$) in addition.

Case 6. The case where I is either ($' \to \wedge$) or ($\wedge' \to$): The inference I has one of the following two forms:

$$\frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{\to \Delta_1, \alpha \wedge \beta}; \qquad \frac{\alpha' \to \Delta_1 \quad \beta' \to \Delta_1}{(\alpha \wedge \beta)' \to \Delta_1}$$

In either case, by applying Lemmas 2 and 1 to $\alpha' \to \Delta_1$ and $\beta' \to \Delta_1$, the sequents $\Delta_1' \to \alpha$ and $\Delta_1' \to \beta$ are provable, and so is $(\alpha \wedge \beta)', \Delta_1' \to$ by $(\wedge' \to)^{\#}$.

Case 7. The case where I is ($\vee \to$): Suppose that I has the form

$$\frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta}$$

By applying Lemma 2 to $\alpha \to \Delta$ and $\beta \to \Delta$, the sequents $\Delta' \to \alpha'$ and $\Delta' \to \beta'$ are provable, and so is $\Delta', \alpha \vee \beta \to$ by $(\vee \to ')$.

Case 8. The case where I is $(\to \vee')^{\#}$: Similar to Case 7, by applying $('' \to)$ in addition.

Case 9. The case where I is either $(\vee \to ')$ or $(\wedge' \to)^{\#}$: Clear, since the succedent of S is empty. ■

REFERENCE

Nishimura, H. (1994). *International Journal of Theoretical Physics*, **33**, 103–113, 1443–1459.