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q-Permutations and q-Combinations

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In this paper we create q-combinatorics and investigate the meaning of q-permutations and q-combinations. Here we modify the task of selecting r objects among n objects into the task of q-selecting, which will be fully discussed later. Moreover, in defining the q-combination, we introduce the idea of q-selecting in a q-order, which is also explained later. The addition principle and multiplication principle are preserved and are widely used in developing our new combinatorics.

First we start with the following question: *In how many ways can we q-select one element from a set of n different elements denoted by* $1, 2, 3, \ldots, n$?

To answer the above question, we consider the following situation. Arrange n different elements in a line as follows:

$$1, 2, 3, \ldots, n$$

Here we impose the following two conditions on the rule of q-selecting one element from the above arrangement.

(I) We can only select an element through traveling from the left.

(II) When we jump one element in a line, we pay with a factor q in the number of ways.

When we select an element in a line according to the above two conditions, we will say that we select q-select an object in a line.

For example, when we q-select 1 in a line from the left, we need not pay any more because we do not jump. However, when we q-select 2 in a

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line from the left, the number of ways of q-selecting 2 is not the same as that of q-selecting 1, because we jump once for the purpose of approaching 2. Instead, the number of ways of q-selecting 2 is q times the number of ways of q-selecting 1. Similarly, the number of ways of q-selecting r in a line from the left is q^{r-1} because we jump $(r-1)$ times.

Therefore we say that the number of ways of q-selecting one object from a set of n different objects is

$$
\begin{aligned}
{}_nP_1^q &= 1 + q + q^2 + \cdots + q^{n-1} \\
&= \frac{1-q^n}{1-q} = [n]
\end{aligned}
$$

where $[n]$ is called a q-number.

Now we can easily show that the number of q-permutations of a set of n different objects, taken r at a time, without repetition, is

$$ {}_nP_r^q = \frac{[n]!}{[n-r]!} $$

where

$$ [n]! = [n][n-1][n-2] \cdots [2][1] $$

and

$$ [0]! = 1 $$

The proof is simple by virtue of application of the multiplication principle. Suppose that we have r spaces to fill and n objects from which to q-select. The first space can be filled with any one of the n objects, and so in $[n]$ ways. After the first space has been filled with any one to be q-selected from n objects, there remain $n-1$ objects, any one of which can be q-selected and put in the second space. Thus the second space can be filled in $[n-1]$ ways. Similarly, the third space can be filled in $[n-2]$ ways, the fourth space in $[n-3]$ ways, and so on. The pattern shows that the tenth space can be filled in $[n-9]$ ways, and in general, the rth space in $[n-(r-1)]$ ways. From the multiplication principle, the r spaces can be filled in

$$ {}_nP_r^q = [n][n-1] \cdots [n-r+1] $$

ways. This takes another convenient form if we multiply by $[n-r]!/[n-r]!$, since then we can write

$$ {}_nP_r^q = \frac{[n]!}{[n-r]!} $$

Now we discuss the meaning of q-combination. Let us define ${}_nC_r^q$ as a q-combination of n objects in a certain order taken r at a time. To begin with, we present a simple example: let us consider the case that we q-select one object from four objects in a line

$$1, 2, 3, 4$$

(I) The case that 1 is q-selected: We do not jump, so we can q-select 1 in one way.

(II) The case that 2 is q-selected: We jump once, so we can q-select 2 in q ways.

(III) The case that 3 is q-selected: We jump twice, so we can q-select 3 in q^2 ways.

(IV) The case that 4 is q-selected: We jump three times, so we can q-select 4 in q^3 ways.

From the addition principle, the number of q-combinations of four objects in a certain order taken one at a time is

$${}_4C_1^q = 1 + q + q^2 + q^3 = [4]$$

Now consider the case that two objects are q-selected in a certain order called a q-order and that the object at the left is first q-selected.

(I) The case that 1 and 2 are q-selected: First we should q-select 1 and then we should q-select 2 because 1 lies at the left in relation to 2. When we q-select 1 in a line, we do not jump and can q-select 1 in one way. After 1 is q-selected, we have

$$2, 3, 4$$

When we then q-select 2 in the line where 1 is deleted, we also do not jump and can q-select 2 in one way. Therefore the number of ways of q-selecting 1 and 2 in a q-order is $1 \times 1 = 1$.

(II) The case that 1 and 3 are q-selected: We can q-select 1 in a line without jumping, and so in one way. After 1 is q-selected, we have

$$2, 3, 4$$

When we then q-select 3 in the line where 1 is deleted, we jump once and can q-select in q ways. Therefore the number of ways of q-selecting 1 and 3 in a q-order is $1 \times q = q$.

(III) The case that 1 and 4 are q-selected: We can q-select 1 in a line without jumping, and so in one way. After 1 is q-selected, we have

$$2, 3, 4$$

When we then q-select 4 in the line where 1 is deleted, we jump twice and can q-select 4 in q^2 ways. Therefore the number of ways of q-selecting 1 and 4 in a q-order is $1 \times q^2 = q^2$.

(IV) The case that 2 and 3 are q-selected: When we q-select 2 in a line, we jump once and can q-select 2 in q ways. After 2 is q-selected, we have

$$1, 3, 4$$

When we then q-select 3 in the line where 2 is deleted, we jump once and can q-select 3 in q ways. Therefore the number of ways of q-selecting 2 and 3 in a q-order is $q \times q = q^2$.

(V) The case that 2 and 4 are q-selected: First we q-select 2 in q ways and then we have

$$1, 3, 4$$

Then, in order to q-select 4, we should jump twice. Therefore the number of ways of q-selecting 2 and 4 in a q-order is $q \times q^2 = q^3$.

(VI) The case that 3 and 4 are q-selected: First we q-select 3 in q^2 ways and then we have

$$1, 2, 4$$

Then, in order to q-select 4, we jump twice. Therefore the number of ways of q-selecting 3 and 4 in a q-order is $q^2 \times q^2 = q^4$.

Therefore the q-combination of four objects, taking two in a q-order, is

$$\begin{aligned} {}_4C_2^q &= 1 + q + 2q^2 + q^3 + q^4 \\ &= \frac{[4]!}{[2]!\,[2]!} \end{aligned}$$

This can be generalized to the more general case. Generally, the q-combination of n objects, taking r in a q-order, is given by

$${}_nC_r^q = \frac{[n]!}{[r]!\,[n-r]!}$$

Now we will prove the above statement by means of mathematical induction. Let us consider the number of ways of q-selecting n objects, taking r in a q-order. Then r objects can be written in the sequence

$$\{i_1, i_2, \ldots, i_r\}$$

where we arrange the r objects according to the following rule:

$$i_1 < i_2 < \cdots < i_r$$

First, in order to q-select i_1, we jump $i_1 - 1$ times. Therefore we q-select i_1

in $q^{i_1 - 1}$ ways and then we have

$$\{\cdots, \hat{i}_1, \cdots\}$$

where the caret means that the element is deleted in the sequence $\{1, 2, \ldots, n\}$.

Second, in order to q-select i_2 in the sequence where i_1 is deleted, we jump $i_2 - 2$ times because i_1 was to the left in relation to i_2. Therefore we can q-select i_2 in $q^{i_2 - 2}$ ways. Similarly we can q-select i_k $(k = 1, 2, \ldots, r)$ in $q^{i_k - k}$ ways.

The multiplication principle indicates that we can q-select $\{i_1, i_2, \ldots, i_r\}$ in

$$q^{i_1 - 1} q^{i_2 - 2} \cdots q^{i_r - r}$$

ways. Since we can choose the r elements $\{i_1, i_2, \ldots, i_r\}$ in ${}_nC_r$ ways, we can therefore q-select r objects among n objects in

$$\sum_{i_1 < i_2 < \cdots < i_r} q^{i_1 - 1} q^{i_2 - 2} \cdots q^{i_r - r}$$

ways. Our next task is to prove that

$$\begin{aligned} {}_nC_r^q &= \sum_{i_1 < i_2 < \cdots < i_r} q^{i_1 - 1} q^{i_2 - 2} \cdots q^{i_r - r} \\ &= \sum_{i_1 < i_2 < \cdots < i_r} q^{\Sigma_{k=1}^r i_k - r(r+1)/2} \end{aligned}$$

In order to use mathematical induction, we assume that the above relation holds for n objects. Let the ways of q-selecting r objects among $n + 1$ objects be K. Then K is written as

$$K = \sum_{I_1 < I_2 < \cdots < I_r} q^{\Sigma_{j=1}^r I_j - r(r+1)/2}$$

where

$$\{I_1, I_2, \ldots, I_r\} \subset \{1, 2, \ldots, n, n+1\}$$

Then we have

$$\begin{aligned} K = &\sum_{i_1 < i_2 < \cdots < i_r} q^{\Sigma_{j=1}^r i_j - r(r+1)/2} \\ &+ \sum_{i_1 < i_2 < \cdots < i_{r-1} < i_r = n+1} q^{\Sigma_{j=1}^{r-1} i_j + n + 1 - r(r+1)/2} \end{aligned}$$

where

$$\{i_1, i_2, \ldots, i_{r-1}, i_r\} \subset \{1, 2, \ldots, n\}$$

Hence we have

$$K = {}_nC_r^q + q^{n+1-r}\,{}_nC_{r-1}^q$$
$$= {}_{n+1}C_r^q$$

which completes the proof by induction.

On the other hand, we have the simple relation of q-combination

$${}_nC_r^q = {}_nC_{n-r}^q$$

The relation between q-permutation and q-combination is as follows:

$${}_nC_r^q = \frac{{}_nP_r^q}{[r]!}$$

In this paper we have discussed the meaning of q-permutations and q-combination by introducing the ideas of q-selecting and q-order. In particular, we have been concerned with q-distributions such as the q-binomial distribution, the q-normal distribution, and so forth. We think that these and related topics will become clear in the near future. We hope that this type of new combinatorics will shed light on various areas of q-physics and q-mathematics.

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