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Kantowski–Sachs Cosmological Model and an Inflationary Era

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We study an empty Kantowski–Sachs universe model with cosmological constant Λ. The characteristic feature of an inflationary era is found. This universe model emerges from a pointlike, stringlike, membranelike singularity and develops toward an isotropic de Sitter universe.

1. INTRODUCTION

A qualitative study of Kantowski–Sachs cosmological models, (Kantowski, 1966, 1969; Kantowski and Sachs, 1966) has recently been performed by Weber (1984, 1985). Grøn (1986) found an exact solution of the Einstein vacuum field equation with a cosmological constant Λ. These models are spatially homogeneous, have shear, and have no rotation.

Such cosmological models are of particular interest due to the possible existence of a GUT phase transition producing a vacuum-dominated inflationary era in the very early history of the universe.

In this paper we find some additional exact solutions of KS cosmological models. These are used to investigate if such a universe model will show a transition into an inflationary era at the GUT time.

2. FIELD EQUATIONS AND SOLUTIONS

The Kantowski–Sachs metric takes the form (McCallum, 1917; McCallum *et al.*, 1970)

$$ds^2 = dt^2 - A^2\, dr^2 - B^2[d\theta^2 + f^2(\theta)\, d^2\phi\,] \tag{2.1}$$

where $f(e) = \sin\theta(K>0)$, $\theta(K=0)$, or $\sinh\theta(K<0)$ with K the curvature.

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A and B are functions of the cosmic time t. The field equations are readily calculated and are given by (McCallum, 1917; McCallum *et al.*, 1970)

$$2\frac{\dot{A}\dot{B}}{AB}+\frac{k+\dot{B}^2}{B^2}=\Lambda \tag{2.2}$$

$$2\frac{\ddot{B}}{B}+\frac{k+\dot{B}^2}{B^2}=\Lambda \tag{2.3}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}=\Lambda \tag{2.4}$$

where $k=+1, 0, -1$ if $K>0$, $K=0$, $K<0$ $(K=k/B^2)$.

Integration of Equation (2.3) gives

$$\dot{B}^2=H^2B^2-k+c_1/B, \qquad H^2=\Lambda/3, \qquad c_1=\text{const} \tag{2.5}$$

Putting the integration constant c_1 equal to zero and integrating once more leads to

$$B=B_0\cosh Ht, \qquad B_0=H^{-1}, \qquad k=1 \tag{2.6}$$

$$B=B_0\sinh Ht, \qquad B_0=H^{-1}, \qquad k=-1 \tag{2.6$'$}$$

Equations (2.2) and (2.4) give

$$A=c_2\dot{B} \tag{2.7}$$

Inserting the solutions (2.6), (2.6′), and (2.7) into the field equations, one finds that they are satisfied for $B_0=c_2=H^{-1}$ after a constant adjustment of the r coordinate.

The solutions may now be written

$$ds^2=dt^2-H^{-2}\sinh^2 Ht\,dr^2-H^{-2}\cosh^2 Ht\,(d^2\theta+\sin^2\theta\,d^2\phi), \qquad k=1 \tag{2.8}$$

$$ds^2=dt^2-H^{-2}\cosh^2 Ht\,dr^2-H^{-2}\sinh^2 Ht\,(d^2\theta+\sinh^2\theta\,d^2\phi) \qquad k=-1 \tag{2.8$'$}$$

Setting $k=0$, equation (2.5) changes to

$$B\dot{B}^2=H^2B^3+c_1 \tag{2.9}$$

Integration of equations (2.9) gives

$$B=B_0\sinh^{2/3}(\tfrac{3}{2}Ht), \qquad c_1>0 \tag{2.10}$$

$$B=B_0\cosh^{2/3}(\tfrac{3}{2}Ht), \qquad c_1<0 \tag{2.10$'$}$$

$$B=Be^{Ht}, \qquad c_1=0 \tag{2.10$''$}$$

$$A=c_2\dot{B}, \qquad c_1=0 \tag{2.10$'''$}$$

After a constant adjustment of the r, one has

$$B_0 = c_2 = H^{-1} \tag{2.11}$$

The solutions may now be written

$$ds^2 = dt^2 - H^{-2}\sinh^{-2/3}(\tfrac{3}{2}Ht)\cosh^2(\tfrac{3}{2}Ht)\,dr^2 - H^{-2}\sinh^{4/3}(\tfrac{3}{2}Ht)\,d\Omega^2, \qquad c_1 > 0 \tag{2.12}$$

$$ds^2 = dt^2 - H^{-2}\cosh^{-2/3}(\tfrac{3}{2}Ht)\sinh^2(\tfrac{3}{2}Ht)\,dr^2 - H^2\cosh^{4/3}(\tfrac{3}{2}Ht)\,d\Omega^2, \qquad c_1 < 0 \tag{2.12'}$$

$$ds = dt^2 - H^{-2}e^{2Ht}\,dr^2 - H^{-2}e^{2Ht}\,d\Omega^2, \qquad c_1 = 0 \tag{2.12''}$$

where $d\Omega^2 = d\theta^2 + f^2(\theta)\,d\phi^2$.

For arbitrary c_1 the solution of equation (2.5) is expressed in terms of elliptic functions. Applying a semiquantitative method, algebraic solutions can be obtained. According to the discriminant Δ of a cubic function,

$$f(B) = B^3 - kH^{-2}B + c_1H^{-2}$$

Equation (2.5) can be changed to

$$\frac{B^{1/2}\,dB}{(H^2B^3 - kB + c_1)^{1/2}} = dt \tag{2.13}$$

and

$$-\Delta = -\frac{1}{27}k^3H^{-6} + \tfrac{1}{4}c_1^2H^{-4} \tag{2.14}$$

If $c_1^2 < (4/27)H^{-2}$, $k = 1$, then

$$\Delta > 0 \tag{2.15}$$

The three roots are real. If

$$c_1^2 > \frac{4}{27}H^{-2}, \qquad k = 1 \tag{2.16}$$

$$c_1^2 > -\frac{4}{27}H^{-2}, \qquad k = -1 \tag{2.16'}$$

$$c_1^2 > 0, \qquad k = 0 \tag{2.16''}$$

then $\Delta < 0$.

One of the roots is real and the two remaining ones are a complex-conjugate pair. We shall show that the GUT time t_G $(= H^{-1})$ is 1.0×10^{-35} sec.

In the case of (2.15) including (2.6) and assuming $c_1 \simeq c_1 BH$ $(BH = \cosh Ht|_{t\to t_G} \simeq 1)$ a solution is obtained,

$$B \simeq H^{-1}(1 - c_1 H)^{1/2} \cosh Ht \tag{2.17}$$

In the case of (2.16), (2.16′), and (2.16″), including (2.10) and (2.10′), and assuming $kB \simeq kH^{-1}$, we get

$$B \simeq H^{-1}(Hc_1 - k)^{1/3} \sinh^{2/3}(\tfrac{3}{2}Ht), \qquad cH_1 > 1 \tag{2.18}$$

$$B \simeq H^{-1}(-Hc_1 + k)^{1/3} \cosh^{2/3}(\tfrac{3}{2}Ht), \qquad c_1 H < -1 \tag{2.18′}$$

In the case of (2.13), assuming $c_1 \simeq kB$, the solution is

$$B \simeq H^{-1}\, e^{Ht} \tag{2.19}$$

When c_1 takes a variety of values, the algebraic solution cannot be obtained; then we give solutions of equation (2.5) in the five forms (2.6), (2.6′), (2.10), (2.10′), and (2.10″), which embody the characteristic features of the Kantowski–Sachs universe models.

3. FEATURES OF AN INFLATIONARY ERA

The characteristic feature of an inflationary era is that space-time expands exponentially owing to the repulsive gravitation of a dominating vacuum energy. This vacuum energy is due to Higgs fields that produce a large cosmological constant Λ at the GUT time $t_G = H^{-1} = 1.0 \times 10^{-35}$ sec. From the line elements (2.8), (2.8′), (2.10), and (2.10′), at this point of time there is a transition from an anisotropic Kantowski–Sachs universe to an isotropic de Sitter universe with an exponentially expanding scale factor.

The expansion Σ $(\Sigma \equiv \dot{A}/A + 2\dot{B}/B)$ is

$$\Sigma = H(\text{ctgh}\, Ht + 2\, \text{tgh}\, Ht) \qquad \text{for (2.8)} \tag{3.1}$$

$$\Sigma = H(\text{tgh}\, Ht + 2\, \text{ctgh}\, Ht) \qquad \text{for (2.8′)} \tag{3.2}$$

$$\Sigma = \tfrac{3}{2}H(\text{ctgh}\, \tfrac{3}{2}Ht + \text{tgh}\, \tfrac{3}{2}Ht) \qquad \text{for (2.12), (2.12′)} \tag{3.3}$$

which tend to $3H$. The shear σ $[\sigma = 3^{-1/2}(\dot{A}/A - \dot{B}/B)]$ is

$$\sigma = 3^{-1/2}H(\text{ctgh}\, Ht - \text{tgh}\, Ht) \qquad \text{for (2.8)} \tag{3.4}$$

$$\sigma = 3^{-1/3}H(\text{tgh}\, Ht - \text{ctgh}\, Ht) \qquad \text{for (2.8′)} \tag{3.5}$$

$$\sigma = 3^{-1/3}\tfrac{3}{2}H(\text{tgh}\, \tfrac{3}{2}Ht - \text{ctgh}\, \tfrac{3}{2}\, Ht) \qquad \text{for (2.12)} \tag{3.6}$$

$$\sigma = 3^{-1/3}\tfrac{3}{2}H(\text{ctgh}\, \tfrac{3}{2}Ht - \text{tgh}\, \tfrac{3}{2}Ht) \qquad \text{for (2.12′)} \tag{3.7}$$

which decay exponentially to zero when $Ht \gg 1$.

The GUT inflationary era (Guth, 1981) lasts from t_G to $t_0 = 1.3 \times 10^{-33}$ sec. During this era, the shear diminishes by factor 10^{-56}.

4. CONCLUSIONS

The Kantowski–Sachs cosmological models emerge from a pointlike singularity for (2.12), from a stringlike singularity for (2.8′), and from a membranelike singularity for (2.8) and (2.12′), respectively, and develop toward an isotropic de Sitter universe. For (2.10″) the model initially is a de Sitter universe.

If the cosmological constant Λ is of the same magnitude as that induced by a GUT phase transition, the universe will evolve extremely rapidly from a shear-dominated state before t_G to a vacuum dominated isotropic state after t_G. The shear diverges initially, for (2.8) and (2.12′), in an infinitely thin universe with extension in its own plane. From $t=0$ until $t=t_G$ the thickness of the universe increases approximately linearly with time, while its extension in the other two directions remains approximately constant. For (2.8′) in an infinitely thin universe with extension in its own wire, from $t=0$ until $t=t_G$ the width of the universe increases approximately linearly with time, while its extension in the other direction remains approximately constant. For (2.12) in an infinitely thin universe with extension in its own point from $t=0$ until $t=t_G$, the thickness of the universe increases in all directions (increase in the r direction is faster than that in the θ, ϕ directions). For $t > t_G$ the universe expands exponentially in all directions, and the shear decays exponentially toward zero. Though c_1 takes different values, the characteristic feature of an inflationary era described by KS models is consistent.

REFERENCES

Grøn, Ø. (1986). *Journal of Mathematical Physics*, **27** (5).
Guth, A. (1981). *Physical Review D*, **23**, 347.
Kantowski, R. (1966). Some relativistic cosmological models, PhD Thesis, University of Texas.
Kantowski, R. (1969). *Astrophysical Journal*, **155**, 89.
Kantowski, R., and Sachs, R. K. (1966). *Journal of Mathematical Physics*, **7**, 443.
MacCallum, M. A. H. (1971). *Mathematical Physics*, **20**, 57.
MacCallum, M. A. H., Stewart, J. M., and Schmidt, B. G. (1970). *Communications in Mathematical Physics*, **30**, 153.
Weber, E. (1984). *Journal of Mathematical Physics*, **26**, 3279.
Weber, E. (1985). *Journal of Mathematical Physics*, **26**, 1308.