International Journal of Theoretical Physics, Vol. 33, No. 7, 1994

Fuzzy Quantum Logics and Infinite-Valued Łukasiewicz Logic

Jarosław Pykacz[1]

Received December 28, 1993

A new, physically more plausible definition of a fuzzy quantum logic is proposed. It is shown that this definition coincides with the previously studied definition of a fuzzy quantum logic; therefore it defines objects which are traditional quantum logics with ordering sets of states. The new definition is expressed exclusively in terms of fuzzy set operations which are generated by connectives of multiple-valued logic studied by Łukasiewicz at the beginning of the 20th century. Therefore, the logic of quantum mechanics is recognized as a version of infinite-valued Łukasiewicz logic.

1. INTRODUCTION

The so-called quantum logic approach to the foundations of quantum mechanics (Beltrametti and Cassinelli, 1981, and references cited therein) is based on the following notion:

Definition 1. A *quantum logic* is an orthocomplemented σ-orthocomplete orthomodular poset, i.e., a partially ordered set $\mathscr{L}$ which contains the smallest element $\mathbf{0}$ and the greatest element $\mathbf{1}$, in which the orthocomplementation map $\perp: \mathscr{L} \to \mathscr{L}$ satisfying the conditions (i)–(iii) exists:

(i) $(a^\perp)^\perp = a$ (idempotency).

(ii) If $a \le b$, then $b^\perp \le a^\perp$ (order reversing).

(iii) The greatest lower bound (*meet*) $a \wedge a^\perp$ and the least upper bound (*join*) $a \vee a^\perp$ with respect to the given partial order exist in $\mathscr{L}$ for any $a \in \mathscr{L}$, and they satisfy the law of contradiction, $a \wedge a^\perp = \mathbf{0}$, and the excluded middle law, $a \vee a^\perp = \mathbf{1}$.

[1]Institute of Mathematics, University of Gdańsk, 80-952 Gdańsk, Poland.

0020-7748/94/0700-1403$07.00/0

Moreover, the σ-orthocompleteness condition holds:

(iv) If $a_i \leq a_j^{\perp}$ for $i \neq j$, then the join $\bigvee_i a_i$ exists in $\mathscr{L}$.

And so does orthomodular identity:

(v) If $a \leq b$, then $b = a \vee (a^{\perp} \wedge b) = a \vee (a \vee b^{\perp})^{\perp}$.

We warn the reader accustomed to the standard fuzzy set notation that $\wedge$ and $\vee$ throughout this paper denote meet and join with respect to the given partial order and that they in general *do not* denote Zadeh (1965) standard fuzzy operations.

Elements of a logic represent elementary statements about physical system. *Probability measures* on a logic $\mathscr{L}$, i.e., mappings $s\colon \mathscr{L} \to [0, 1]$ such that

(a) $s(\mathbf{1}) = 1$
(b) $s(\bigvee_i a_i) = \sum_i s(a_i)$ for any sequence $\{a_i\}$ of *pairwise orthogonal* elements, i.e., elements such that $a_i \leq a_j^{\perp}$ for $i \neq j$

represent states of a physical system and, therefore are often themselves called *states* on a logic $\mathscr{L}$. A set of states $\mathscr{S}$ on a logic $\mathscr{L}$ is called *ordering* iff

$$s(a) \leq s(b) \quad \text{for all} \quad s \in \mathscr{S} \qquad \text{implies} \qquad a \leq b$$

Let $\mathscr{U} \neq \varnothing$ be a fixed set called a *universe*. According to Zadeh (1965), the *fuzzy set A* in $\mathscr{U}$ is defined by its *membership function* $\mu_A : \mathscr{U} \to [0, 1]$ in such a way that for any $x \in \mathscr{U}$ the number $\mu_A(x) \in [0, 1]$ represents the *degree of membership* of x to the fuzzy set A. Many authors identify fuzzy sets with their membership functions and write $A(x)$ instead of $\mu_A(x)$. This convention is adopted throughout the rest of this paper.

2. FUZZY QUANTUM LOGICS

It was noticed by the author (Pykacz, 1987*a*,*b*, 1988, 1990, 1992) that, due to the theorem of Mączyński (1973, 1974), any quantum logic $\mathscr{L}$ with an ordering set of states $\mathscr{S}$ can be isomorphically represented in the form of a family $\mathbb{L}(\mathscr{S})$ of fuzzy subsets of $\mathscr{S}$ such that:

(f.1) $\mathbb{L}(\mathscr{S})$ contains the empty set $\varnothing$.
(f.2) $\mathbb{L}(\mathscr{S})$ is closed under the standard fuzzy set complementation, i.e.,

$$\text{if} \quad A \in \mathbb{L}(\mathscr{S}), \quad \text{then} \quad A' = 1 - A \in \mathbb{L}(\mathscr{S}) \tag{1}$$

(f.3) If $A_i \cap A_j = \varnothing$ for $i \neq j$, i.e., if sets $A_i, A_2, \ldots$ are *weakly disjoint* (Giles, 1976), then $\sum_i A_i \leq 1$ and $\bigcup_i A_i \in \mathbb{L}(\mathscr{S})$.

Remark 1. $\sum_i A_i$ denotes algebraic sum of (membership functions of) fuzzy sets $A_1, A_2, \ldots$. Notation $A \cap B$ denotes the *Giles intersection* (Giles, 1976) of fuzzy sets A and B:

$$(A \cap B)(x) = \max[A(x) + B(x) - 1, 0] \tag{2}$$

$A \cup B$ denotes the *Giles union* (Giles, 1976) of fuzzy sets A and B:

$$(A \cup B)(x) = \min[A(x) + B(x), 1] \tag{3}$$

Conversely, due to the same Mączyński Theorem, any family $\mathbb{L}(\mathcal{U})$ of fuzzy subsets of an arbitrary universe $\mathcal{U}$ which satisfies conditions (*f*.1), (*f*.2), and (*f*.3) is a quantum logic partially ordered by the fuzzy set inclusion and with the standard fuzzy set complementation (1) as an orthocomplementation. Therefore, in Pykacz (1987*b*, 1988, 1992) the following definition was adopted:

Definition 2. A *fuzzy quantum logic* (FQL) on a universe $\mathcal{U}$ is a family $\mathbb{L}(\mathcal{U})$ of fuzzy subsets of $\mathcal{U}$ which satisfies conditions (*f*.1), (*f*.2), and (*f*.3).

The strongest and, I dare say, the strangest assumption in the definition of a fuzzy quantum logic is the part of the condition (*f*.3) which says that the algebraic sum of membership functions of any sequence of pairwise weakly disjoint sets does not exceed 1. Weak disjointness of a pair of fuzzy sets $A \cap B = \varnothing$ is equivalent to the condition

$$A(x) + B(x) \leq 1 \qquad \text{for any} \quad x \in \mathcal{U} \tag{4}$$

Therefore, the above-mentioned assumption says that even for arbitrary long sequences of membership functions which pairwisely satisfy the inequality (4) the algebraic sum of all functions does not exceed 1. This condition seems to be neither natural nor easy to fulfill. However, careful examination of the proof of the Mączyński Theorem shows that the above-mentioned assumption is responsible for σ-orthocompleteness of the obtained structure and for the fact that $A' = 1 - A$ is an orthocomplementation, i.e., for the features that make it possible to use the obtained structure as a basis of reasonable generalized probability theory and for building models of physical systems.

From condition (iii) of the definition of a quantum logic it follows that if $a \leq a^{\perp}$ for some $a \in \mathcal{L}$, then $a = \mathbf{0}$ ($a \leq a^{\perp}$ implies $a \wedge a^{\perp} = a$, but $a \wedge a^{\perp} = \mathbf{0}$ for any $a \in \mathcal{L}$) and, similarly, if $a^{\perp} \leq a$ for some $a \in \mathcal{L}$, then $a = \mathbf{1}$. This result is very natural since if elements of a logic are interpreted as propositions about a physical system, and partial order and orthocomplementation play, respectively, the role of implication and negation, then $a \leq a^{\perp}$ would mean that a proposition implies its own negation.

Any fuzzy set E such that

$$E \subseteq E' \tag{5}$$

was called a *weakly empty set* by Piasecki (1985). Any fuzzy set U such that

$$U' \subseteq U \tag{6}$$

was called by him a *weak universe*. Since conditions $a \leq a^{\perp} \Rightarrow a = \mathbf{0}$ and $a^{\perp} \leq a \Rightarrow a = \mathbf{1}$ translated into the language of fuzzy quantum logics (Pykacz, 1992) mean that

$$\text{for any} \quad A \in \mathbb{L}(\mathcal{U}) \quad \text{if} \quad A \subseteq A', \quad \text{then} \quad A = \varnothing \tag{7'}$$

$$\text{for any} \quad A \in \mathbb{L}(\mathcal{U}) \quad \text{if} \quad A' \subseteq A, \quad \text{then} \quad A = \mathcal{U} \tag{7''}$$

we see that a fuzzy quantum logic does not contain any weakly empty set except $\varnothing$ and any weak universe except $\mathcal{U}$, or, equivalently, that it does not contain any nonempty set weakly disjoint with itself:

$$\text{for any} \quad A \in \mathbb{L}(\mathcal{U}) \quad \text{if} \quad A \cap A = \varnothing, \quad \text{then} \quad A = \varnothing \tag{8}$$

We shall show that if we adopt these natural conditions as an axiom, then we can obtain the structure of a σ-orthocomplete orthomodular poset, i.e., a quantum logic, without assuming that algebraic sums of membership functions of sequences of weakly disjoint sets do not exceed 1.

3. INTRINSICALLY HOMOGENOUS DEFINITION OF A FUZZY QUANTUM LOGIC

Let us study the following definition of a fuzzy quantum logic in which the part of the condition (f.3) of Definition 2 saying that $\sum_i A_i \leq 1$ is rejected, but any equivalent condition (7′), (7″), or (8) is adopted instead. We shall show in the sequel that objects defined by this definition are in fact identical with objects defined by Definition 2.

Definition 3. A *generalized fuzzy quantum logic* (GFQL) on a universe $\mathcal{U}$ is a family $\mathbb{G}(\mathcal{U})$ of fuzzy subsets of $\mathcal{U}$ such that:

(g.1) $\mathbb{G}(\mathcal{U})$ contains the empty set $\varnothing$.

(g.2) $\mathbb{G}(\mathcal{U})$ is closed under the standard fuzzy set complementation.

(g.3) $\mathbb{G}(\mathcal{U})$ is closed under Giles unions of pairwise weakly disjoint sets, i.e., if $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\bigcup_i A_i \in \mathbb{G}(\mathcal{U})$.

(g.4) $\mathbb{G}(\mathcal{U})$ does not contain any nonempty weakly empty set, i.e., in $\mathbb{G}(\mathcal{U})$ the condition (7′) or equivalent condition (7″) or (8) holds.

Let us note that $A \subseteq A'$ is equivalent to $A(x) \leq 1/2$ for all $x \in \mathcal{U}$, and $A' \subseteq A$ is equivalent to $A(x) \geq 1/2$ for all $x \in \mathcal{U}$.

We shall prove that any GFQL is a quantum logic in the traditional sense. Before we pass to the main results we list some useful identities fulfilled by Giles operations and the standard fuzzy complementation.

Lemma 1.

(*a*) $A \cup B = B \cup A$, $A \cap B = B \cap A$ (commutativity)
(b) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)
(c) $\varnothing \cup A = A$, $\mathscr{U} \cap A = A$ (neutral elements of union and intersection)
(d) $\mathscr{U} \cup A = \mathscr{U}$, $\varnothing \cap A = \varnothing$
(e) $A \cap A' = \varnothing$ (law of contradiction)
(f) $A \cup A' = \mathscr{U}$ (excluded middle law)
(g) $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$ (De Morgan laws)
(h) if $A \subseteq B$, then $B = A \cup (A' \cap B)$ ("orthomodular" identity)

All identities follow immediately from definitions of Giles operations and the standard fuzzy complementation.

Many identities of Lemma 1 were noticed already by Giles (1976). If $\mathbb{G}(\mathscr{U})$ is a GFQL and $A, B, C \in \mathbb{G}(\mathscr{U})$, then in the case of expressions which do not belong to $\mathbb{G}(\mathscr{U})$ by the definition of GFQL, the identities of Lemma 1 are fulfilled in $\mathbb{G}(\mathscr{U})$ when all necessary unions and intersections are defined in $\mathbb{G}(\mathscr{U})$. Let us show that all components of the "orthomodular" identity belong to $\mathbb{G}(\mathscr{U})$ for any $A, B \in \mathbb{G}(\mathscr{U})$, $A \subseteq B$.

Lemma 2. If $A, B \in \mathbb{G}(\mathscr{U})$ and $A \subseteq B$, then A and B', as well as A and $A' \cap B$, are weakly disjoint, which means that $A \cup B'$, $(A \cup B')' = A' \cap B$, $A \cup (A' \cap B)$, and $(A \cup (A' \cap B))' = A' \cap (A \cup B')$ belong to $\mathbb{G}(\mathscr{U})$. Moreover, $A' \cap B = B - A$ and $A \cup (A' \cap B) = A + (B - A)$, so the "orthomodular" identity takes the trivial form

$$\text{if} \quad A \subseteq B, \quad \text{then} \quad B = A + (B - A) \tag{9}$$

Proof. If $A \subseteq B$, then

$$(A \cap B')(x) = \max[A(x) + 1 - B(x) - 1, 0] = \max[A(x) - B(x), 0] = 0$$

i.e., $A \cap B' = \varnothing$. Therefore, by (*g*.3), $A \cup B' \in \mathbb{G}(\mathscr{U})$, which, by (*g*.2), implies that $(A \cup B')' = A' \cap B \in \mathbb{G}(\mathscr{U})$.

Weak disjointness of A and $A' \cap B$ follows from the identities (b), (d), and (e) of Lemma 1:

$$A \cap (A' \cap B) = (A \cap A') \cap B = \varnothing \cap B = \varnothing \tag{10}$$

Therefore, by (*g*.3), $A \cup (A' \cap B) \in \mathbb{G}(\mathscr{U})$.

If $A \subseteq B$, then

$$(A' \cap B)(x) = \max[1 - A(x) + B(x) - 1, 0]$$
$$= \max[B(x) - A(x), 0] = B(x) - A(x) \tag{11}$$

Therefore,

$$[A \cup (A' \cap B)](x) = \min[A(x) + B(x) - A(x), 1]$$
$$= A(x) + B(x) - A(x) = B(x) \quad \blacksquare \tag{12}$$

The following theorem was conjectured in the first version of the paper. However, the given proof was erroneous. The correct proof of (an even more general version of) this theorem was given later by Mesiar (1994).

Theorem 1. Let $\mathbb{G}(\mathscr{U})$ be a GFQL and let $A_1, A_2, \ldots, \in \mathbb{G}(\mathscr{U})$. If $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\bigvee_i A_i = \bigcup_i A_i \tag{13}$$

i.e., the join of any sequence of pairwise weakly disjoint sets exists in $\mathbb{G}(\mathscr{U})$ and coincides with the Giles union of these sets.

Proof. See Mesiar (1994).

We shall show now that the standard fuzzy set complementation is an orthocomplementation in any GFQL.

Theorem 2. Let $\mathbb{G}(\mathscr{U})$ be a GFQL and let $'$: $\mathbb{G}(\mathscr{U}) \to \mathbb{G}(\mathscr{U})$, $A' = 1 - A$ be the standard fuzzy complementation. Then it is an orthocomplementation with respect to the fuzzy set inclusion as partial order, i.e.:

(i) $A'' = A'$.
(ii) If $A \subseteq B$, then $B' \subseteq A'$.
(iii) For any $A \in \mathbb{G}(\mathscr{U})$, $A \vee A'$ and $A \wedge A'$ exist in $\mathbb{G}(\mathscr{U})$, and $A \vee A' = \mathscr{U}$, $A \wedge A' = \varnothing$.

Proof. Conditions (i) and (ii) follow from the very definition of the standard fuzzy set complementation.

The proof of condition (iii) follows from the condition (*g*.4) since, obviously, $\varnothing \subseteq A, A' \subseteq \mathscr{U}$, and if there exists $B \in \mathbb{G}(\mathscr{U})$ such that $B \subseteq A, A'$, then for any $x \in \mathscr{U}$

$$B(x) \leq \min[A(x), A'(x)] = \min[A(x), 1 - A(x)] \leq 1/2 \tag{14}$$

which means that such a B is a weakly empty set, i.e., by (*g*.4), $B = \varnothing$.

Similarly, if there exists $B \in \mathbb{G}(\mathscr{U})$ such that $A, A' \subseteq B$, then for any $x \in \mathscr{U}$

$$B(x) \geq \max[A(x), A'(x)] = \max[A(x), 1 - A(x)] \geq 1/2 \tag{15}$$

which means that B is a weak universe, i.e., by (g.4) $B = \mathscr{U}$. ■

Theorems 1 and 2 imply, since $A_i \cap A_j = \varnothing$ is equivalent to $A_i \subseteq A_j' = A_j^\perp$, i.e., to orthogonality of A_i and A_j, that $\mathbb{G}(\mathscr{U})$ is a σ-orthocomplete orthoposet and that for any $\mathbb{G}(\mathscr{U})$ the "orthomodular" identity (h) of Lemma 1 is actually an orthomodular identity in the traditional sense:

$$\text{if} \quad A, B \in \mathbb{G}(\mathscr{U}) \quad \text{and} \quad A \subseteq B, \quad \text{then} \quad B = A \vee (A' \wedge B) \tag{16}$$

All the above-mentioned results can be summarized as follows.

Corollary 1. Any GFQL $\mathbb{G}(\mathscr{U})$ is an orthocomplemented σ-orthocomplete orthomodular poset, i.e., it is a quantum logic in the traditional sense with respect to the standard fuzzy set inclusion as partial order and the standard fuzzy set complementation as orthocomplementation. Orthogonality of elements is equivalent to their weak disjointness and for an arbitrary sequence of pairwise weakly disjoint elements their join exists in $\mathbb{G}(\mathscr{U})$ and coincides with their Giles union.

As we mentioned at the end of Section 2, a fuzzy quantum logic does not contain any nonempty weakly empty set. Therefore, any fuzzy quantum logic is obviously a generalized fuzzy quantum logic. Let us recall that by the Mączyński Theorem any quantum logic $\mathscr{L}$ with an ordering set of states $\mathscr{S}$ is isomorphic to any FQL $\mathbb{L}(\mathscr{S})$, and conversely: any FQL $\mathbb{L}(\mathscr{U})$ is a quantum logic in the traditional sense with an ordering set of states (Pykacz, 1987*b*, 1988, 1990, 1992). However, there exist quantum logics without an ordering set of states, or even without any state at all (Greechie, 1971; Pták and Pulmannová, 1991). Therefore, the question of whether any GFQL is an FQL is not trivial, especially since the condition $\sum_i A_i \leq 1$ for any sequence of pairwise weakly disjoint sets seems to be more restrictive than the condition (g.4) of Definition 3. However, the following theorem shows that this condition is satisfied in any GFQL, i.e., that objects defined by Definitions 1 and 2 are identical, so the word "generalized" in the name "generalized fuzzy quantum logic" should be in fact omitted.

Theorem 3. Any GFQL is an FQL.

Proof. Let $\mathbb{G}(\mathscr{U})$ be a GFQL. We shall prove by induction that $\sum_i A_i \leq 1$ for any finite sequence of pairwise weakly disjoint elements of $\mathbb{G}(\mathscr{U})$.

For $n = 2$, $A_1 \cap A_2 = \varnothing$ is equivalent to $A_1 + A_2 \leq 1$ by the very definition of Giles intersection (2).

Let us assume that $\sum_{i=1}^{n} A_i \leq 1$ for any sequence of pairwise weakly disjoint sets of the length n. Let $\{A_i\}_{i=1}^{n+1}$ be any sequence of pairwise weakly disjoint sets of the length $n+1$. By the induction hypothesis we can write down $n+1$ inequalities:

$$\begin{aligned}
A_2 + A_3 + A_4 + \cdots + A_n + A_{n+1} &\leq 1 \\
A_1 + A_3 + A_4 + \cdots + A_n + A_{n+1} &\leq 1 \\
A_1 + A_2 + A_4 + \cdots + A_n + A_{n+1} &\leq 1 \\
&\vdots \\
A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_{n+1} &\leq 1 \\
A_1 + A_2 + A_3 + \cdots + A_{n-1} + A_n &\leq 1
\end{aligned} \tag{17}$$

After summing them up and dividing by n we obtain

$$\sum_{i=1}^{n+1} A_i \leq \frac{n+1}{n} \tag{18}$$

Let us denote $B_n = \bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i$ and $B_{n+1} = \bigcup_{i=1}^{n+1} A_i = B_n \cup A_{n+1}$, and calculate $(B'_{n+1} + A_{n+1})(x)$ There are two possibilities:

1. If $\sum_{i=1}^{n+1} A_i(x) = B_n(x) + A_{n+1}(x) > 1$, then

$$\begin{aligned}
(B'_{n+1} + A_{n+1})(x) &= [1 - (B_n \cup A_{n+1}) + A_{n+1}](x) \\
&= 1 - \min[B_n(x) + A_{n+1}(x), 1] + A_{n+1}(x) \\
&= 1 - 1 + A_{n+1}(x) = A_{n+1}(x) \leq 1
\end{aligned} \tag{19}$$

2. If $\sum_{i=1}^{n+1} A_i(x) = B_n(x) + A_{n+1}(x) \leq 1$, then

$$\begin{aligned}
(B'_{n+1} + A_{n+1})(x) &= 1 - \min[B_n(x) + A_{n+1}(x), 1] + A_{n+1}(x) \\
&= 1 - B_n(x) \leq 1
\end{aligned} \tag{20}$$

We see that in both cases $(B'_{n+1} + A_{n+1})(x) \leq 1$. This means that $B'_{n+1} \cap A_{n+1} = \varnothing$, so by the conditions ($g$.3) and ($g$.2) of Definition 3 both $B'_{n+1} \cup A_{n+1}$ and $(B'_{n+1} \cup A_{n+1})'$ belong to $\mathbb{G}(\mathscr{U})$. Let us calculate now $[(B'_{n+1} \cup A_{n+1})' + B'_n](x)$, considering the same two possibilities as before and taking into account that in both cases

$$B'_{n+1} \cup A_{n+1} = B'_{n+1} + A_{n+1} \tag{21}$$

so we can use (19) or (20), respectively.

1. If $\sum_{i=1}^{n+1} A_i(x) > 1$, then

$$\begin{aligned}[(B'_{n+1} \cup A_{n+1})' + B'_n](x) &= A'_{n+1}(x) + B'_n(x) \\ &= 1 - A_{n+1}(x) + 1 - B_n(x) \\ &= 2 - [B_n(x) + A_{n+1}(x)] \\ &= 2 - \sum_{i=1}^{n+1} A_i(x) < 1 \end{aligned} \tag{22}$$

2. If $\sum_{i=1}^{n+1} A_i(x) \le 1$, then

$$[(B'_{n+1} \cup A_{n+1})' + B'_n](x) = B''_n(x) + B'_n(x) = B_n(x) + B'_n(x) = 1 \tag{23}$$

Therefore, again in both cases $[(B'_{n+1} \cup A_{n+1})' + B'_n](x) \le 1$, which means that $(B'_{n+1} \cup A_{n+1})' \cup B'_n$ belongs to $\mathbb{G}(\mathscr{U})$. However, combining (18) with (22) and (23), we obtain for any $n \ge 2$ and any $x \in \mathscr{U}$

$$\frac{1}{2} \le 2 - \frac{n+1}{n} \le [(B'_{n+1} \cup A_{n+1})' \cup B'_n](x) \le 1 \tag{24}$$

This means that $(B'_{n+1} \cup A_{n+1})' \cup B'_n$ is a weak universe, so $[(B'_{n+1} \cup A_{n+1})' \cup B'_n]' = (B'_{n+1} \cup A_{n+1}) \cap B_n$ is a weakly empty set. Now, if we assume that there exists $x \in \mathscr{U}$ such that $\sum_{i=1}^{n+1} A_i(x) > 1$, we have, according to (21) and (19),

$$\begin{aligned}[(B'_{n+1} \cup A_{n+1}) \cap B_n](x) &= \max[(B'_{n+1} + A_{n+1})(x) + B_n(x) - 1, 0] \\ &= \max[A_{n+1}(x) + B_n(x) - 1, 0] \\ &= \max\left[\sum_{i=1}^{n+1} A_i(x) - 1, 0\right] \\ &= \sum_{i=1}^{n+1} A_i(x) - 1 \neq 0 \end{aligned} \tag{26}$$

so $(B'_{n+1} \cup A_{n+1}) \cap B_n$ is a nonempty weakly empty set, i.e., the condition (g.4) of Definition 3 is not satisfied and $\mathbb{G}(\mathscr{U})$ cannot be a GFQL.

Since for a countable sequence $\sum_i A_i$ is a pointwise limit of finite sums, we infer that for any sequence of pairwise weakly disjoint sets $\{A_i\} \subseteq \mathbb{G}(\mathscr{U})$ the assumption that there exists $x \in \mathscr{U}$ such that $\sum_i A_i(x) > 1$ inevitably implies that $\mathbb{G}(\mathscr{U})$ is not a GFQL, which finishes the proof. ■

4. (FUZZY) QUANTUM LOGICS AND INFINITE-VALUED ŁUKASIEWICZ LOGIC

Due to Theorem 3, Definitions 2 and 3 define the same class of objects: fuzzy quantum logics. However, Definition 3 is formulated entirely in terms of the standard fuzzy set complementation and Giles operations on fuzzy sets. It was pointed out by Giles (1976) that operations on fuzzy sets arise as immediate consequences of building set theory on the basis of Łukasiewicz infinite-valued logic. Indeed, if we denote $\tau(p)\in[0,1]$ the truth-value of the sentence "p" and follow Łukasiewicz (1970), who chose negation $\neg$ and implication $\rightarrow$ as basic connectives with the following rules of calculating their truth-values:

$$\tau(\neg p)=1-\tau(p) \tag{26}$$

$$\tau(p\rightarrow q)=\min[1-\tau(p)+\tau(q),1] \tag{27}$$

then, defining disjunction $p\cup q$ and conjunction $p\cap q$ as in classical logic,

$$\tau(p\cup q)=\tau(\neg p\rightarrow q) \tag{28}$$

$$\tau(p\cap q)=\tau[\neg(\neg p\cup\neg q)] \tag{29}$$

we obtain

$$\tau(p\cup q)=\min[\tau(p)+\tau(q),1] \tag{30}$$

$$\tau(p\cap q)=\max[\tau(p)+\tau(q)-1,0] \tag{31}$$

Many-valued Łukasiewicz logic is related to the fuzzy set theory exactly as classical logic is related to the ordinary set theory (Giles, 1976): if we apply two-valued logic to evaluate the truth-value of a sentence "*x belongs to A*," then the set A is crisp, while if we apply in this case many-valued logic, the set A is fuzzy with $A(x)=\tau(\text{"}x\in A\text{"})$. Of course disjunction (28), (30), and conjunction (29), (31) generate, respectively, Giles union (3) and intersection (2) of fuzzy sets, exactly as this happens in the case of two-valued classical logic and ordinary set theory.

In the vast literature on fuzzy sets much more often other operations of union and intersection, introduced already by Zadeh (1965), are utilized:

$$(A\cup_z B)(x)=\max[A(x),B(x)] \tag{32}$$

$$(A\cap_z B)(x)=\min[A(x),B(x)] \tag{33}$$

These operations on fuzzy sets can be treated as generated by the other pair of multiple-valued disjunction and conjunction (Łukasiewicz, 1970)

$$\tau(p\ or\ q)=\max[\tau(p),\tau(q)] \tag{34}$$

$$\tau(p\ and\ q)=\min[\tau(p),\tau(q)] \tag{35}$$

obtained from the negation (26) and implication (27) by assuming that

$$\tau(p\ or\ q)=\tau[(p\rightarrow q)\rightarrow q] \tag{36}$$

$$\tau(p\ and\ q)=\tau[\neg(\neg p\ or\ \neg q)] \tag{37}$$

Remark 2. It is possible to study infinite families of fuzzy set operations which give Giles operations (2), (3) or Zadeh operations (32), (33) as special cases (Dubois and Prade, 1985). However, recent results of Mesiar (1994) strongly indicate that Giles operations together with the standard fuzzy set complementation are the only (up to an isomorphism) pointwisely generated fuzzy set operations which can endow families of fuzzy sets with a structure of an orthocomplemented σ-orthocomplete orthomodular poset, i.e., a structure of a quantum logic.

Actually, Łukasiewicz (1970) in his papers on many-valued logic dealt mostly with disjunction (34) and conjunction (35). However, in one of his earliest papers (Łukasiewicz, 1913), which contains results of his studies undertaken already in 1909, we can find theorems and formulas which, after some calculations, yield expressions (30) and (31). Frink (1938) studied the algebra of Łukasiewicz logic endowed with both pairs of connectives: (30), (31), which he called arithmetic operations, and (34), (35) which he called logical operations. In the same paper he studied also the algebra of, at that time 2 years old, Birkhoff and von Neumann (1936) quantum logic and compared it briefly with the algebra of Łukasiewicz logic. Frink's comparison was rather superficial, but it clearly shows that quantum logic has more common features with Łukasiewicz logic endowed with arithmetic, rather than logical operations.

The results of the previous sections show that there are deeper links between fuzzy quantum logics (therefore, by the Mączyński Theorem, also traditional quantum logics with ordering sets of states) and infinite-valued Łukasiewicz logic endowed with operations (26)–(31):

With every fuzzy quantum logic $\mathbb{L}(\mathcal{U}) = \{\varnothing, \mathcal{U}, A, B, C, \ldots\}$ we can associate a family $L = \{p_{A,x}\}$, $A \in \mathbb{L}(\mathcal{U})$, $x \in \mathcal{U}$, consisting of propositions of the form $p_{A,x} =$ "x belongs to A" with $\tau(p_{A,x}) = A(x) \in [0, 1]$. If $\mathbb{L}(\mathcal{U})$ describes properties of a physical system Σ, in which case $\mathcal{U}$ is identified with the set of all states of Σ (Pykacz, 1987*a*,*b*, 1988, 1990, 1992), then $p_{A,x}$ can be identified with a proposition "the physical system Σ has a property A when it is in a state x." In such a case, according to the interpretation of truth-values in multiple-valued logics considered already by Łukasiewicz (1970), the truth-value $\tau(p_{A,x})$ is numerically equal to the *probability* that this proposition is true, i.e., to the probability that a test designed to check the property A gives the positive result when the system Σ is in the state x. Alternatively, according to the very idea of the fuzzy set theory, $\tau(p_{A,x})$ can be interpreted as the degree to which the physical system Σ has the property A when it is in the state x (Pykacz, 1992). Let us note that in the realm of many-valued logics we are always allowed to say: "the physical system Σ in a state x has a property A," even before this property is measured, and that the negation of this proposition is also allowed and true to the degree $\tau(\neg p_{A,x}) = 1 - \tau(p_{A,x})$. When a dichotomic test designed to check the property A is performed we are forced to say that the system Σ in the state x either *had* or *had not* the property A (note the past tense used in this proposition!).

Let us consider a subfamily of propositions $\{p_{A,x}\}_{x\in\mathscr{U}} \subset L$ generated by the same element A of a (fuzzy) quantum logic. We shall refer to such a subfamily as the *general proposition*, denote it p_A, and, if L describes the properties of a physical system Σ, read it: "the physical system Σ has the property A." Actually, a general proposition p_A is a propositional function since it becomes a proposition $p_{A,x}$ for any $x \in \mathscr{U}$, and only then can we assign to it a truth-value $\tau(p_{A,x})$. It follows from general features of a (fuzzy) quantum logic (Pykacz, 1992) that there are only two general propositions in L whose truth-value is the same in any state x: the always-true proposition $p_{\mathscr{U}}$ [$\tau(p_{\mathscr{U},x}) = 1$ for all $x \in \mathscr{U}$] and the always-false proposition $p_{\varnothing}$ [$\tau(p_{\varnothing,x}) = 0$ for all $x \in \mathscr{U}$]. In the case of a logic of propositions about a physical system Σ they can be read, respectively, $p_{\mathscr{U}}$ = "the system Σ exists" and $p_{\varnothing}$ = "the system Σ does not exist." According to the results of Pykacz (1992), there is no other general proposition in L whose truth-value would be a constant function on $\mathscr{U}$.

We say that two propositions p, q are *exclusive* if

$$\tau(p \cap q) = \max[\tau(p) + \tau(q) - 1, 0] = 0 \tag{38}$$

and that p *surely implies* q if

$$\tau(p \to q) = \min[1 - \tau(p) + \tau(q), 1] = 1 \tag{39}$$

We can generalize these notions, and also notions of negation, implication, disjunction, and conjunction, to general propositions $p_A, p_B \in L$ associated with a (fuzzy) quantum logic $\mathbb{L}(\mathscr{U})$ in an obvious way by assuming that formulas (26), (27), (30), (31), (38), and (39) are fulfilled for all $x \in \mathscr{U}$ in a pointwise manner.

The conditions (g.1)–(g.4) of Definition 3 can be now expressed in the language of many-valued Łukasiewicz logic pertaining to the family of propositions $L = \{p_{A,x}\}$, $A \in \mathbb{L}(\mathscr{U})$, $x \in \mathscr{U}$, associated with a fuzzy quantum logic $\mathbb{L}(\mathscr{U})$, and have the following meaning (by a small abuse of notation we write "p_A belongs to L" instead of "p_A is a subfamily of L" in the case of general propositions):

(g.1′) The always-true general proposition belongs to L.
(g.2′) If a proposition belongs to L, its negation also belongs to L.
(g.3′) The disjunction of pairwise exclusive general propositions belongs to L.
(g.4′) The always-false proposition is the only general proposition in L which surely implies its own negation.

The problem of whether many-valued Łukasiewicz operations (26)–(31) have any physical meaning in other cases than allowed by the above-mentioned conditions is interesting and worth further studies.

5. CONCLUSIONS

In the historical development of the foundations of quantum mechanics since Birkhoff and von Neumann (1936) a lot of attention has been paid to the order-theoretic structure of the set of elementary propositions about a physical system. It is often stressed that meets and joins express connectives "and" and "or" and therefore allow the building up of compound propositions. However, let us note that in fact in any computation in which there appear probability measures on a logic, not joins of arbitrary elements, but only joins of pairwise orthogonal elements are utilized. Theorem 1 says that in fuzzy quantum logics these joins exist and coincide with Giles unions. This suggests that the main interest should be shifted from order-theoretic operations of meet, join, and complementation to Giles union, intersection, and the standard fuzzy set complementation (if we work with fuzzy sets), or to Łukasiewicz disjunction (30), conjunction (31), and negation (26) if we prefer to work with infinite-valued Łukasiewicz logic. Therefore, the logic of a quantum mechanical system should be seen first of all as a family of propositions which belong to the domain of infinite-valued Łukasiewicz logic endowed with operations (26)–(31) and which satisfy conditions ($g.1'$)–($g.4'$), rather than being treated as a lattice-theoretic model of a family of closed subspaces of a Hilbert space.

ACKNOWLEDGMENTS

The author is greatly indebted to Prof. Beloslav Riečan and to the Slovak Academy of Sciences for the hospitality extended to him during his visits to Bratislava, where the successive versions of the paper were presented and discussed. Special thanks are due to Prof. Sylvia Pulmannová for pointing out an error in an early version of the proof of Theorem 1, and to Prof. Radko Mesiar for providing the author with his manuscript containing the correct proof of the generalized version of this theorem, and for many valuable discussions which led to Theorem 3.

The paper contains results of research supported by the University of Gdańsk grant BW/5100-5-0117-3.

REFERENCES

Beltrametti, E. G., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823–843.

Dubois, D., and Prade, H. (1985). *Information Science*, **36**, 85–121.
Frink, O. (1938). *American Mathematics Monthly*, **45**, 210–219.
Giles, R. (1976). *International Journal of Man-Machine Studies*, **8**, 313–327.
Greechie, R. J. (1971). *Journal of Combinatorial Theory*, **10**, 119–132.
Łukasiewicz, J. (1913). *Die logischen Grundlagen der Warscheinlichkeitsrechnung*, Kraków [English translation, Logical foundations of probability theory, in Łukasiewicz (1970), pp. 16–63].
Łukasiewicz, J. (1970). *Selected Works*, Reidel, Amsterdam.
Mączyński, M. J. (1973). *International Journal of Theoretical Physics*, **8**, 353–360.
Mączyński, M. J. (1974). *International Journal of Theoretical Physics*, **11**, 149–156.
Mesiar, R. (1994). *h*-Fuzzy quantum logics, *International Journal of Theoretical Physics*, this issue.
Piasecki, K. (1985). *Fuzzy Sets and Systems*, **17**, 271–284.
Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, and Veda, Bratislava.
Pykacz, J. (1987*a*). Quantum logics as families of fuzzy subsets of the set of physical states, in *Preprints of the Second IFSA Congress, Tokyo*, Vol. 2, pp. 437–440.
Pykacz, J. (1987*b*). *Bulletin pour les Sous-Ensembles Flous et leurs Applications*, **32**, 150–157.
Pykacz, J. (1988). *Bulletin pour les Sous-Ensembles Flous et leurs Applications*, **37**, 81–85.
Pykacz, J. (1990). *Bulletin pour les Sous-Ensembles Flous et leurs Applications*, **43**, 49–53.
Pykacz, J. (1992). *International Journal of Theoretical Physics*, **31**, 1767–1783.
Zadeh, L. A. (1965). *Information and Control*, **8**, 338–353.