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Exponential Dichotomy of Linear Impulsive Differential Equations in a Banach Space

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The dichotomy of linear impulsive equations in a Banach space is investigated.

1. INTRODUCTION

A number of processes in physics, chemistry, biology, control theory, and robotics during their evolutionary development are subject to the action of short-time forces in the form of impulses. In most cases the duration of the action of these forces is negligibly small, as a result of which one can assume that the forces act only at certain moments of time. The impulsive differential equations represent a mathematical model of such processes. The work of Mil'man and Myshkis (1960) marked the beginning of the mathematical theory of these equations, and the work of Bainov *et al.* (1988, 1989) marked the beginning of the mathematical theory of the same equations in abstract spaces. Samoilenko and Perestyuk (1987) published the first monograph dedicated to this subject.

In the present paper the exponential dichotomy of linear impulsive equations in a Banach space is investigated. The exponential dichotomy of linear differential equations in the finite-dimensional case without impulses was investigated by Palmer (1979*a*,*b*; 1984*a*,*b*) and Elaydi and Hajek (1988).

2. STATEMENT OF THE PROBLEM

Let X be a complex Banach space with a norm $\|\cdot\|$. We shall call the following system a linear impulsive differential equation in X:

$$\frac{dx}{dt} = A(t)x\big|_{t \neq t_n} \qquad (n = 1, 2, 3, \ldots) \tag{1}$$

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$$x(t_n+0) = Q_n x(t_n) \qquad (n=1, 2, 3, \dots) \tag{2}$$

where $T=\{t_n\}_{n=1}^{\infty}$ is a finite or infinite sequence of numbers with the property

$$0 < t_1 < t_2 < \cdots < t_n < \cdots, \qquad \lim_{n\to\infty} t_n = \infty \tag{3}$$

$A(t)$ for $t \in R_+$ $\{R_+ = [0, \infty)\}$ is a continuous operator function with values in the space $L(X)$ of the continuous linear operators acting in X; and $\{Q_n\}_{n=1}^{\infty}$ is a sequence of impulsive operators, $Q_n \in L(X)$ $(n = 1, 2, 3, \dots)$.

Definition 1. A solution of the impulsive equation (1), (2) we shall call a function which for $t \neq t_n$ satisfies equation (1), for $t = t_n$ satisfies condition (2), and is continuous from the left.

We say that the sequence $T=\{t_n\}_{n=1}^{\infty}$ satisfies condition A if the following condition holds:

A. There exists a positive number l and an integer λ such that not more than λ members of the sequence T lie in any interval of length l.

In particular, this condition is fulfilled if

$$\overline{\lim_{\omega\to\infty}} \sup_{0\le t<\infty} \frac{\kappa(t, t+\omega)}{\omega} < \infty \tag{4}$$

where $\kappa(a, b)$ is the number of the elements of T lying in the interval (a, b).

It is known (Bainov *et al.*, 1988, 1989) that for the impulsive equation (1), (2) there exists an evolutionary Cauchy operator associating with any element $\xi \in X$ a solution $x(t)$ of the impulsive equation which satisfies the initial condition $x(\tau) = \xi$ $(0 \le \tau \le t < \infty)$.

Lemma 1. Let the condition $A(t)$, $Q_n \in L(X)$ hold, where $t \in R_+$, $n = 1, 2, 3, \dots$. Then the evolutionary operator $U(t, \tau)$ $(0 \le \tau \le t < \infty)$ has the form

$$U(t,\tau) = \begin{cases} U_0(t,\tau), & t_n < \tau \le t \le t_{n+1} \\ U_0(t, t_n)\left[\prod_{j=n}^{k+1} Q_j U_0(t_j, t_{j-1})\right] Q_k U_0(t_k, \tau) \\ \qquad t_{k-1} < \tau \le t_k < t_n < t \le t_{n+1} \end{cases}$$

where $U_0(t, \tau)$ $(0 \le \tau \le t < \infty)$ is the evolutionary operator of the equation

$$\frac{dx}{dt} = A(t)x$$

Lemma 1 is proved by a straightforward verification.

The operator-valued function $U(t, \tau)$ satisfies the equalities

$$U(\tau, \tau) = I \qquad (0 \le \tau < \infty) \tag{5}$$

$$U(t, \tau) = U(t, s)U(s, \tau) \qquad (0 \le \tau \le s \le t < \infty) \tag{6}$$

Moreover, it is differentiable at the points $t \in (t_{j-1}, t_j]$ $(j = 1, 2, 3, \ldots)$ and $\tau \in [t_{j-1}, t_j)$ $(j = 2, 3, 4, \ldots)$ and

$$U'_t(t, \tau) = A(t)U(t, \tau), \qquad U'_\tau(t, \tau) = U(t, \tau)A(\tau) \tag{7}$$

At the points t_n $(n = 1, 2, 3, \ldots)$ the following equalities hold:

$$U(t_n + 0, \tau) = Q_n U(t_n, \tau) \qquad (0 \le \tau \le t_n < \infty, \quad n = 1, 2, 3, \ldots) \tag{8}$$

We say that condition B is satisfied if the following condition holds:

B. The operators Q_n have continuous inverse operators Q_n^{-1} $(n = 1, 2, 3, \ldots)$.

Lemma 2. Let the following conditions hold:

1. $A(t), Q_n \in L(X)$, where $t \in R_+$, $n = 1, 2, 3, \ldots$.
2. Condition B holds.

Then the evolutionary operator $U(t, \tau)$ for $0 \le t, \tau < \infty$, has the form

$$U(t, \tau) = \begin{cases} U_0(t, \tau), & t_n < t, \tau \le t_{n+1} \\ U_0(t, t_n)\left[\prod_{j=n}^{k+1} Q_j U_0(t_j, t_{j-1})\right] Q_k U_0(t_k, \tau) & \\ \quad t_{k-1} < \tau \le t_k < t_n < t \le t_{n+1} & \\ U_0(t, t_n)\left[\prod_{j=n}^{k-1} Q_j^{-1} U_0(t_j, t_{j+1})\right] Q_k^{-1} U_0(t_k, \tau) & \\ t_{n-1} < t \le t_n < t_k < \tau \le t_{k+1} & \end{cases}$$

Lemma 2 is proved by a straightforward verification.

If the conditions of Lemma 2 are satisfied, then the following equalities are valid:

$$U(t, \tau) = U^{-1}(\tau, t), \qquad U(t, \tau) = U(t, s)U(s, \tau) \qquad (0 \le \tau, s, t < \infty) \tag{9}$$

and

$$U(t_{n+0}, \tau) = Q_n U(t_n, \tau) \qquad (0 \le t_n, \tau < \infty) \tag{10}$$

3. MAIN RESULTS

3.1. Dependence between Exponential Dichotomy and Existence of a Bounded Solution of a Nonhomogeneous Differential Impulsive Equation

Definition 2. The differential impulsive equation (1), (2) is said to be exponentially dichotomous if the space X is split into a direct sum $X = X_1 \dot{+} X_2$ of two subspaces X_1 and X_2 so that the following inequalities are valid:

$$\|U(t)P_1 U^{-1}(\tau)\| \le M e^{-\delta(t-\tau)} \qquad (0 \le \tau < t < \infty) \tag{11}$$

$$\|U(t)P_2 U^{-1}(\tau)\| \le M e^{-\delta(\tau-t)} \qquad (0 \le t < \tau < \infty) \tag{12}$$

where $U(t)=U(t,0)$ $(0\leq t<\infty)$, P_1 and P_2 are projectors corresponding to the decomposition $X_1\dot{+}X_2$, and M and δ are positive constants.

Consider the nonhomogeneous impulsive differential equation

$$\frac{dx}{dt}=A(t)x+f(t)|_{t\neq t_n} \qquad (n=1,2,3,\ldots) \tag{13}$$

$$x(t_n+0)=Q_nx(t_n)+h_n \qquad (n=1,2,3,\ldots) \tag{14}$$

where $f(t)$ $(t\geq 0)$ is a given bounded function and

$$h=\{h_n\}_{n=1}^{\infty}\in m(X)$$

[$m(X)$ is the space of all sequences $\{a_n\}_{n=1}^{\infty}$ $(a_n\in X,\ n=1,2,3,\ldots)$ for which the sequence $\{\|a_n\|\}_{n=1}^{\infty}$ is bounded].

Later we shall use the space $C=C(R_+,X,T)$ of all bounded on R_+ functions with values in X which are continuous for $t\in(t_{n-1},t_n]$ $(n=1,2,3,\ldots)$ and have limits as $t\to t_{n+0}$ $(n=1,2,3,\ldots)$ with the usual algebraic operations and with the supremum norm $\|\cdot\|_C$. It is standard to verify that the space is a Banach one.

Theorem 1. Let the following conditions be fulfilled:

1. $A(t),\ Q_n\in L(X)$ for $t\in R_+$, $n=1,2,3,\ldots$.
2. Conditions A and B hold.
3. Equation (1) with condition (2) is exponentially dichotomous.

Then:

1. For any function $f\in C$ and $\{h_n\}\in m(X)$ there exists at least one bounded on R_+ solution of the nonhomogeneous impulsive equation (13), (14).
2. The linear operator $L: D(L)\to C\times m(X)$ defined by the formula

$$Lx(t)=\left(\frac{dx}{dt}-A(t)x,\ (x(t_n+0)-Q_nx(t_n))\right) \tag{15}$$

has continuous from the right inverse operators [$D(L)$ is the definition domain of the operator L].

Proof. We shall show that the operator $K\colon C\dot{+}m\to C$ defined by the formula

$$\begin{aligned}K(f,h)(t)=&\int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\,d\tau-\int_t^{\infty}U(t)P_2U^{-1}(\tau)f(\tau)\,d\tau\\&+\sum_{t_j<t}U(t)P_1U^{-1}(t_j+0)h_j-\sum_{t_j\geq t}U(t)P_2U^{-1}(t_k+0)h_j\end{aligned} \tag{16}$$

is continuous and satisfies the identity

$$LK=I \tag{17}$$

(I is the identity operator in the space C).

In order to prove the continuity of K, we shall estimate the norms of the addends in (16). By (11) and (12) we have

$$\left\| \int_0^t U(t)P_1U^{-1}(\tau)f(\tau)\,d\tau \right\| \le \int_0^t \|U(t)P_1U^{-1}(\tau)\|\,\|f(\tau)\|\,d\tau \le M e^{-\delta t}\int_0^t e^{\delta\tau}\,d\tau\,\|f\|_C \le \frac{M}{\delta}\|f\|_C \tag{18}$$

$$\left\| \int_t^\infty U(t)P_2U^{-1}(\tau)f(\tau)\,d\tau \right\| \le \int_t^\infty \|U(t)P_2U^{-1}(\tau)\|\,\|f(\tau)\|\,d\tau \le M e^{\delta t}\int_t^\infty e^{-\delta\tau}\,d\tau\,\|f\|_C = \frac{M}{\delta}\|f\|_C \tag{19}$$

and analogously

$$\left\| \sum_{t_j<t} U(t)P_1U^{-1}(t_j+0)h_j \right\| \le \sum_{t_j<t} \|U(t)P_1U^{-1}(t_j+0)\|\,\|h_j\| \le M\left(\sum_{t_j<t} e^{\delta(t_j-t)}\right)\|h\|_{m(X)} \le \frac{M\lambda}{1-e^{-\delta l}}\|h\|_{m(X)} \tag{20}$$

$$\left\| \sum_{t\le t_j} U(t)P_2U^{-1}(t_j)h_j \right\| \le \sum_{t\le t_j} \|U(t)P_2U^{-1}(t_j+0)\|\,\|h_j\| \le M\left(\sum_{t\le t_j} e^{\delta(t-t_j)}\right)\|h\|_{m(X)} \le \frac{M\lambda}{1-e^{-\delta l}}\|h\|_{m(X)} \tag{21}$$

From (16) and (18)-(21) it follows that the function $K(f, h)$ is bounded and satisfies the inequality

$$\|K(f,h)\|_C \le \frac{2M}{\delta}\|f\|_C + \frac{2M\lambda}{1-e^{-\delta l}}\|h\|_{m(X)} \tag{22}$$

The continuity of $K(f, h)$ for $t \ne t_n$ $(n = 1, 2, 3, \ldots)$ and the existence of the limit values $K(f, h)(t_n+0)$ $(n = 1, 2, 3, \ldots)$ are verified immediately.

It remains to verify equality (17). Differentiating equality (16) for $t \neq t_n$ and taking into account (7)-(10), we obtain

$$
\begin{aligned}
K(f, h)'(t) = {} & U(t)P_1U^{-1}(t)f(t) + U(t)P_2U^{-1}(t)f(t) \\
& + \int_0^t A(t)U(t)P_1U^{-1}(\tau)f(\tau)\,d\tau \\
& - \int_t^\infty A(t)U(t)P_2U^{-1}(\tau)f(\tau)\,d\tau \\
& + \sum_{t_j<t} A(t)U(t)P_1U^{-1}(t_j+0)h_j \\
& - \sum_{t_j\geq t} A(t)U(t)P_2U^{-1}(t_j+0)h_j \\
= {} & f(t) + A(t)(K(f, h)(t))
\end{aligned}
$$

Taking into account equality (8) for $n = 1, 2, 3, \ldots$, we obtain

$$
\begin{aligned}
K(f, h)(t_n+0) = {} & \int_0^{t_n} U(t_n+0)P_1U^{-1}(\tau)f(\tau)\,d\tau \\
& - \int_{t_n}^\infty U(t_n+0)P_2U^{-1}(\tau)f(\tau)\,d\tau \\
& + \sum_{t_j\leq t_n} U(t_n+0)P_1U^{-1}(t_j+0)h_j \\
& - \sum_{t_j>t_n} U(t_n+0)P_2U^{-1}(t_j+0)h_j \\
= {} & Q_n \int_0^{t_n} U(t_n)P_1U^{-1}(\tau)f(\tau)\,d\tau \\
& - Q_n \int_{t_n}^\infty U(t_n)P_2U^{-1}(\tau)f(\tau)\,d\tau \\
& + Q_n \sum_{t_j<t_n} U(t_n)P_1U^{-1}(t_j+0)h_j \\
& - Q_n \sum_{t_j\geq t_n} U(t_n)P_1U^{-1}(t_j+0)h_j \\
& + U(t_n+0)P_1U^{-1}(t_n+0)h_n \\
& + U(t_n+0)P_2U^{-1}(t_n+0)h_n \\
= {} & Q_nK(f, h)(t_n) + h_n
\end{aligned}
$$

Theorem 1 is proved. ■

Remark 1. It is not hard to verify that Theorem 1 still holds if the condition $\sup_{t\in R_+} \|f(t)\| < \infty$ is replaced by the weaker condition

$$\sup_{t\in R_+} \int_t^{t+1} \|f(\tau)\|\,d\tau < \infty$$

Remark 2. Theorem 1 still holds without condition A if we consider inhomogeneous equations with $h = 0$. In this case the values of the operator defined by (16) lie in the subspace $C_0 = C_0(R_+, X, T)$ of the space $C(R_+, X, T)$ which consists of the functions which satisfy the condition

$$x(t_n + 0) - Q_n x(t_n) = 0 \qquad (n = 1, 2, 3, \ldots) \tag{23}$$

Let L_0 be the restriction of the operator L to C_0. Then, if the function $A(t)$ is bounded, L_0 maps C_0 into C and if the impulsive equation (1) with condition (2) is exponentially dichotomous, then L_0 maps C_0 onto C and the operator K_0 defined by the equality

$$K_0 f = K(f, 0) \tag{24}$$

is right inverse for L_0.

Theorem 2. Let the following conditions be fulfilled:

1. $A(t), Q_n \in L(X)$ for $t \in R_+$, $n = 1, 2, 3, \ldots$.
2. The operator-valued function $A(t)$ is integrally bounded

$$\sup_{t \in R_+} \int_t^{t+1} \|A(s)\| \, ds < \infty \tag{25}$$

3. The sequences of operators $\{Q_n\}_{n=1}^{\infty}$ and $\{Q_n^{-1}\}_{n=1}^{\infty}$ are integrally bounded, i.e., for some $l > 0$ the following inequalities hold:

$$\sup_{t \le t < \infty} \sum_{t \le t_n \le t+l} \|Q_n\|, \qquad \sup_{0 \le t < \infty} \sum_{t \le t_n \le t+l} \|Q_n^{-1}\| < \infty \tag{26}$$

4. Conditions A and B hold.
5. The subspace

$$X_1 = \left\{ \xi \in X \colon \sup_{0 \le t < \infty} \|U(t)\xi\| < \infty \right\} \tag{27}$$

is complementary (there exists a subspace X_2 of X for which $X = X_1 \dotplus X_2$).

6. For any function $f \in C$ the nonhomogeneous equation (13) with condition (14) for $h = 0$ has at least one solution belonging to the subspace C_0.

Then the impulsive equation (1) with condition (2) is exponentially dichotomous.

Proof. Let P_1 and P_2 be the projectors corresponding to the decomposition $X = X_1 \dotplus X_2$. It is not hard to verify that if $x_1(t)$ and $x_2(t)$ are two solutions of the nonhomogeneous impulsive equation (13) with condition (14), then the difference $z(t) = x_1(t) - x_2(t)$ is a solution of the homogeneous impulsive equation (1) with condition (2). If the solutions $x_1(t)$ and $x_2(t)$ are bounded, then $z(t)$ is bounded, too, hence $z(0) \in X_1$. If $x(t)$ is a solution of (13), (14) lying in C_0, then $\tilde{x}(t)=x(t)-U(t)P_1x(0)$ is also a solution of (13), (14) lying in C_0 and with initial value $\tilde{x}(0)=P_2x(0)\in X_2$. From condition 6 of Theorem 2 it follows that for $f\in C$ the nonhomogeneous impulsive equation (13), (14) has a unique solution $\tilde{x}(t)$ lying in C_0 and for which the equality $P_1\tilde{x}(0)=0$ is valid. In this way an operator $\tilde{K}$ is defined mapping C into C_0 and associating with each element $f\in C$ a solution of the impulsive equation (13), (14). From Banach's closed graph theorem it follows that this operator is continuous, i.e., there exists a number K for which

$$\|\tilde{K}f\|_{C_0}\leq K\|f\|_C \tag{28}$$

We shall prove the validity of inequalities (11), (12). Let $x(t)$ be a bounded solution of the impulsive equation (1), (2) with initial value $x(0)\in X_1$. Set

$$y_a(t)=\left[\int_0^t \chi_a(\tau)\|x(\tau)\|^{-1}\,d\tau\right]x(t)$$

where

$$\chi_a(t)=\begin{cases}1, & 0\leq t\leq \tau+a\\ 1-(t-\tau-a), & \tau+a<t\leq \tau+a+1\\ 0, & \tau+a+1<t<\infty\end{cases}$$

It is not hard to verify that the function $y_a(t)$ is a solution of the nonhomogeneous impulsive equation (13), (14) for $f_a(t)=\chi_a(t)\|x(t)\|^{-1}x(t)$ and $h=0$. Since $f_a\in C$ and $\|f_a\|=1$, then (28) implies the validity of the inequality

$$\|y_a(t)\|\leq K$$

or

$$\int_0^t \chi_a(\tau)\|x(\tau)\|^{-1}\,d\tau\,\|x(t)\|\leq K$$

For $a\to\infty$ we obtain the inequality

$$\int_0^t \|x(\tau)\|^{-1}\,d\tau\,\|x(t)\|\leq K \tag{29}$$

From conditions 2–4 of Theorem 2 and Lemmas 1 and 2 there follows the existence of constants $\tilde{M}$ and $\tilde{N}$ for which any solution of the impulsive equation (1), (2) satisfies the inequality

$$\|x(t)\|\leq\|x(\tau)\|\tilde{M}e^{\tilde{N}(t-\tau)}\qquad(0\leq t,\ \tau<\infty) \tag{30}$$

From inequalities (29) and (30) for $t-\tau\geq 1$ we obtain the following inequalities:

$$\begin{aligned}\|x(t)\| &\leq \frac{K}{\int_0^t \|x(s)\|^{-1}\,ds}\\ &\leq K\,e^{1/K}\,e^{-1/K(t-\tau)}\frac{1}{\int_\tau^{\tau+1}\|x(s)\|^{-1}\,ds}\\ &\leq \tilde{M}K\,e^{1/K+\tilde{N}}\|x(\tau)\|\,e^{-1/K(t-\tau)}\end{aligned}\tag{31}$$

For $t-\tau\leq 1$ we obtain

$$\|x(t)\|\leq \tilde{M}\,e^{\tilde{N}+1/K}\,e^{-(1/K)(t-\tau)}\|x(\tau)\|\tag{32}$$

From inequalities (31) and (32) there follows the inequality

$$\|x(t)\|\leq M\,e^{-\delta(t-\tau)}\|x(\tau)\|\qquad (0\leq\tau\leq t<\infty)\tag{33}$$

where

$$M=\max\{\tilde{M}K\,e^{1/K+\tilde{N}},\,\tilde{M}\,e^{1/K+\tilde{N}}\},\qquad \delta=1/K$$

Analogously, we consider the case when the solution of the impulsive equation (1), (2) is with initial value $x(0)\in X_2$. In this case instead of $y_a(t)$ we consider the function

$$\tilde{y}_a(t)=\left[\int_t^\infty \chi_a(\tau)\|x(\tau)\|^{-1}\,d\tau\right]x(t)$$

We obtain the inequality

$$\|x(t)\|\leq M\,e^{\delta(t-\tau)}\|x(\tau)\|\qquad (0\leq t<\tau<\infty)\tag{34}$$

Inequalities (33) and (34) are equivalent, respectively, (11) and (12).

Theorem 2 is proved. ■

Remark 3. Conditions (25) and (26) are fulfilled if the function $A(t)$ $(t\geq 0)$ and the sequences $\{Q_n\}_{n-1}^\infty$ and $\{Q_n^{-1}\}_{n=1}^\infty$ are bounded.

3.2. Stability of the Notion of Exponential Dichotomy

Consider the perturbed impulsive differential equation

$$\frac{dx}{dt}=(A(t)+B(t))_x\big|_{t\neq t_n}\qquad (n=1,2,3,\ldots)\tag{35}$$

$$x(t_n+0)=(Q_n+R_n)x(t_n)\qquad (n=1,2,3,\ldots)\tag{36}$$

where $B(t)$ is a continuous operator-valued function with values in $L(X)$ and $(R_n\}_{n=1}^\infty$ is a sequence of elements of $L(X)$.

By $B(X)$ we shall denote the set of continuous operator-valued functions $Z \in L(X)$ for which

$$\sup_{0 \le a < b < \infty} \frac{1}{1+b-a} \int_a^b \|Z(s)\| \, ds < \infty$$

The set $B(X)$ is a Banach space with norm

$$\|Z\|_{B(X)} = \sup_{0 \le a < b < \infty} \frac{1}{1+b-a} \int_a^b \|Z(s)\| \, ds$$

By $M(X)$ we shall denote the Banach space of bounded sequences $Z = \{Z_n\}_{n=1}^{\infty}$ [$Z_n \in L(X)$, $n = 1, 2, 3, \ldots$] with norm

$$\|Z\|_{M(X)} = \sup_n \|Z_n\|$$

Lemma 3. Let the following conditions be fulfilled:

1. $A(t)$, $Q_n \in L(X)$ for $t \in R_+$, $n = 1, 2, 3, \ldots$
2. Conditions A and B hold.
3. Conditions (11) and (12) hold.
4. $B \in B(X)$.
5. $\{R_n\}_{n=1}^{\infty} \in M(X)$.

Then the operator S defined by the formula

$$\begin{aligned} Sz(t) = {} & \int_0^t U(t)P_1U^{-1}(\tau)B(\tau)z(\tau)\, d\tau \\ & - \int_t^{\infty} U(t)P_2U^{-1}(\tau)B(\tau)z(\tau)\, d\tau \\ & + \sum_{t_j < t} U(t)P_1U^{-1}(t_j+0)R_jz(t_j) - \sum_{t_j \ge t} U(t)P_2U^{-1}(t_j+0)z(t_j) \end{aligned} \quad (37)$$

is a continuous linear operator $C \to C$; moreover, the following estimate is valid:

$$\|S\|_{L(X)} \le 2M[(1+\delta^{-1})\theta + \lambda(1-e^{-l})^{-1}\eta] \quad (38)$$

where the number θ is defined by the inequality

$$\frac{1}{1+b-a} \int_a^b \|B(s)\| \, ds \le \theta \qquad (0 \le a < b < \infty) \quad (39)$$

and η by the inequality

$$\sup_j \|R_j\| \le \eta \quad (40)$$

Proof. It suffices to show that for $z \in C$ the right-hand side of (37) makes sense and the following inequality holds:

$$\|Sz(t)\| \leq 2M\left(\frac{\theta(1+\delta)}{\delta}+\frac{\lambda\eta}{1-e^{-l}}\right)\|z\|_C \tag{41}$$

Let $z \in C$. Then from (11) and (12) there follows the estimate

$$\|Sz(t)\| \leq M\left(\int_0^\infty e^{-\delta|t-\tau|}\|B(\tau)\|\,d\tau+\sum_j e^{-\delta|t-t_j|}\|R_j\|\right)\|z\|_C \tag{42}$$

Inequality (37) implies the estimate

$$\int_0^\infty e^{-\delta|t-\tau|}\|B(\tau)\|\,d\tau \leq 2\left(1+\frac{1}{\sigma}\right)\theta \tag{43}$$

and (38) implies the estimate

$$\sum_{j=1}^\infty e^{-\delta|t-t_j|}\|R_j\|_{L(X)} \leq \frac{2\lambda\eta}{1-e^{-l}} \tag{44}$$

From inequalities (42)-(44) there follows the validity of inequality (41), therefore of (40) as well.

Lemma 3 is proved. ■

Theorem 3. Let the conditions of Lemma 3 hold.

Then for sufficiently small values of $\|B\|_{B(X)}$ and $\|\{R_n\}_{n=1}^\infty\|_{M(X)}$ the perturbed impulsive equation (35), (36) is exponentially dichotomous.

Proof. From Lemma 3 it follows that for small values of θ and η the norm of the operator S can be made smaller than 1, so that the operator $I-S$ is invertible. Set

$$\Gamma(t)\xi=(I-S)^{-1}U(t)\xi \qquad (\xi \in X_1) \tag{45}$$

It is not hard to see that for $\xi \in X_1$ the function $x(t)=\Gamma(t)\xi$ is a bounded solution of the perturbed impulsive equation (35), (36). From the definition (37) of the operator S there follows the relation

$$Sz(0) \in X_2 \qquad (z \in C) \tag{46}$$

That is why the operator

$$Rx=S(\Gamma(\cdot)x)(0) \tag{47}$$

acts from X_1 into X_2 and is continuous and, moreover,

$$\|R\|_{L(X)} \leq \frac{\|S\|M}{1-\|S\|} \tag{48}$$

The operator $I - P_2RP_1$ is continuous in the space X and has a continuous inverse $(I - P_2RP_1)^{-1} = I + P_2RP_1$. That is why it maps the subspace X_1 onto a closed subspace X_1. This subspace is complementary since the operator

$$\tilde{P}_1 = P_1 - P_2RP_1 \tag{49}$$

because of

$$\tilde{P}_1 = (I - P_2RP_1)P_1(I - P_2RP_1)^{-1}$$

is a projector onto $\tilde{X}_1$. The complementary projector

$$\tilde{P}_2 = P_2(I + RP_1) \tag{50}$$

maps X into the subspace $\tilde{X}_2$ and $X = \tilde{X}_1 \dot{+} \tilde{X}_2$.

We shall show that the decomposition $X = X_1 \dot{+} X_2$ induces an exponential dichotomy for the perturbed impulsive equation (35), (36). For this purpose we shall prove the validity of the following inequalities:

$$\|\tilde{U}(t)\tilde{P}_1\tilde{U}^{-1}(\tau)\| \le \tilde{M}_1\, e^{-\tilde{\delta}_1(t-\tau)} \qquad (0 \le \tau < t < \infty) \tag{51}$$

$$\|\tilde{U}(t)\tilde{P}_2\tilde{U}^{-1}(\tau)\| \le \tilde{M}_1\, e^{\tilde{\delta}_1(t-\tau)} \qquad (0 \le t < \tau < \infty) \tag{52}$$

where $\tilde{U}(t)$ is the Cauchy operator for the perturbed impulsive equation (35), (36) and $\tilde{M}$ and $\tilde{\delta}_1$ are positive constants, or, which is the same, the validity of the inequalities

$$\|x(t)\| \le \tilde{M}\, e^{-\tilde{\delta}(t-\tau)}\|x(\tau)\| \qquad (x(0) \in \tilde{X}_1,\quad 0 \le \tau \le t < \infty) \tag{53}$$

$$\|x(t)\| \le \tilde{M}\, e^{\tilde{\delta}(t-\tau)}\|x(\tau)\| \qquad (x(0) \in \tilde{X}_2,\quad 0 \le t \le \tau < \infty) \tag{54}$$

for all solutions of (35), (36).

Let $x(t)$ be a solution with initial condition $x(0) \in X_1$. It is not hard to check that the function $x(t)$ satisfies the linear integral equation

$$\begin{aligned} x(t) = U(t)\xi &+ \int_0^t U(t)P_1U^{-1}(s)B(s)x(s)\, ds \\ &- \int_t^\infty U(t)P_2U^{-1}(s)B(s)x(s)\, ds \\ &+ \sum_{t_j < t} U(t)P_1U^{-1}(t_j + 0)R_jx(t_j) \\ &- \sum_{t_j \ge t} U(t)P_2U^{-1}(t_j + 0)R_jx(t_j) \end{aligned} \tag{55}$$

for some $\xi \in X_1$. Setting $t = \tau$ in equality (55) for ξ we get

$$\xi = U^{-1}(\tau)x(\tau) - \int_0^\tau P_1 U^{-1}(s)B(s)x(s)\,ds + \int_\tau^\infty P_2 U^{-1}(s)B(s)x(s)\,ds - \sum_{t_j<\tau} P_1 U^{-1}(t_j+0)R_j x(t_j) + \sum_{t_j\geq\tau} P_2 U^{-1}(t_j+0)R_j x(t_j)$$

i.e.,

$$\xi = P_1 U^{-1}(\tau)x(\tau) - \int_0^\tau P_1 U^{-1}(s)B(s)x(s)\,ds - \sum_{t_j<\tau} P_1 U^{-1}(t_j+0)R_j x(t_j)$$

We substitute this value of ξ into (55) and for $t > \tau$ obtain the following linear equation:

$$x(t) = U(t)P_1 U^{-1}(\tau)x(\tau) + \int_\tau^t U(t)P_1 U^{-1}(s)B(s)x(s)\,ds - \int_t^\infty U(t)P_2 U^{-1}(s)B(s)x(s)\,ds + \sum_{\tau\leq t_j<t} U(t)P_1 U^{-1}(t_j+0)R_j x(t_j) - \sum_{t_j\geq t} U(t)P_2 U^{-1}(t_j+0)R_j x(t_j) \qquad (56)$$

In view of inequalities (11) and (12), for $\|x(t)\|$ we obtain the estimate

$$\|x(t)\| \leq M e^{-\delta(t-\tau)}\|x(\tau)\| + \int_\tau^\infty M e^{-\delta|t-s|}\|B(s)\|\,\|x(s)\|\,ds + \sum_{t_j\geq\tau} M e^{-\delta|t-t_j|}\|R_j\|\,\|x(t_j)\| \qquad (57)$$

Set

$$z(t) = e^{\tilde{\delta}(t-\tau)}\|x(t)\| \qquad (58)$$

where $\tilde{\delta} \in (0, \delta)$, substitute into inequality (57), and obtain

$$z(t) \leq M\{\exp[(\tilde{\delta}-\delta)(t-\tau)]\}z(\tau) + M\int_\tau^\infty \{\exp[-\delta|t-s| + \tilde{\delta}(t-s)]\}\,\|B(s)\|z(s)\,ds + M\sum_{t_j\geq t} \{\exp[-\delta|t-t_j| + \tilde{\delta}(t-t_j)]\}\,\|R_j\|z(t_j) \qquad (59)$$

From inequality (39) for the integral in (59) there follows the estimate

$$\int_\tau^\infty \{\exp[-\delta|t-s| + \tilde{\delta}(t-s)]\}\,\|B(s)\|\,ds \leq 2\left(1 + \frac{\delta}{(\delta+\tilde{\delta})(\delta-\tilde{\delta})}\right)\theta$$

From inequality (40) for the sum in (59) there follows the estimate

$$\sum_{t_j \geq t} \{\exp[-\delta|t-t_j|+\delta(t-t_j)]\}\,\|R_j\|$$

$$\leq \left(\frac{1}{1-\exp[-(\delta-\tilde{\delta})l]}+\frac{1}{1-\exp[-(\delta+\tilde{\delta})l]}\right)\lambda\eta$$

From inequality (59) for sufficiently small values of θ and η there follows the inequality

$$z(t) \leq C(\theta, \eta)z(\tau)$$

where $C(\theta, \eta)$ is a constant.

In view of (58) we obtain the inequality

$$\|x(t)\| \leq C(\theta, \eta)M\{\exp[-\tilde{\delta}(t-\tau)]\}\,\|x(\tau)\|\,d\tau \qquad (0 \leq \tau < t < \infty)$$

which coincides with (51).

Let $X(t)$ be a solution of the perturbed impulsive equation (35), (36) with initial value $x(0) \in \tilde{X}_2$. Then

$$x(t) = U(t)x(0) + \int_0^t U(t)P_1U^{-1}(s)B(s)x(s)\,ds + \sum_{t_j<t} U(t)P_1U^{-1}(t_j+0)R_jx(t_j) \tag{60}$$

We set $t=\tau$ in (60) and solve the equation with respect to $x(0)$:

$$x(0) = U^{-1}(\tau)x(\tau) - \int_0^\tau U^{-1}(s)B(s)x(s)\,ds - \sum_{t_j<\tau} U^{-1}(t_j+0)R_jx(t_j)$$

Since $P_2x(0) = x(0)$, then

$$x(0) = P_2U^{-1}(\tau)x(\tau) - \int_0^\tau P_2U^{-1}(s)B(s)x(s)\,ds - \sum_{t_j<\tau} P_2U^{-1}(t_j+0)R_jx(t_j)$$

We substitute this expression for $x(0)$ into (60) and for $t \leq \tau$ we obtain the following linear equation:

$$\begin{aligned} x(t) = {} & U(t)P_2U^{-1}(\tau)x(\tau) - \int_t^\tau U(t)P_2U^{-1}(s)B(s)x(s)\,ds \\ & + \int_0^t U(t)P_1U^{-1}(s)B(s)x(s)\,ds - \sum_{t \leq t_j < \tau} U(t)P_2U^{-1}(t_j+0)R_jx(t_j) \\ & + \sum_{t_j<t} U(t)P_1U^{-1}(t_j+0)R_jx(t_j) \end{aligned} \tag{61}$$

Analogously to the evaluation of the norm of the solution of equation (56), for the norm of the solution of equation (61) we obtain the following inequality:

$$\|x(t)\| \le C(\theta, \eta) M e^{\tilde{\delta}(t-\tau)} \|x(\tau)\|$$

which is equivalent to inequality (52).

Theorem 3 is proved.

4. EXAMPLES OF EXPONENTIALLY DICHOTOMOUS EQUATIONS

Everywhere in this section we assume that condition B holds.

Example 1. Let $A(t) = A$ $(t \in R_+)$, $Q_n = Q$ $(n = 1, 2, 3, \ldots)$, $AQ = QA$, and let Q have a logarithm ln Q. Let, moreover, the points $\{t_n\}_{n=1}^{\infty}$ form an arithmetical progression, i.e., $t_n = nh$, $n = 1, 2, 3, \ldots, h > 0$. The impulsive equation (1), (2) in this case has the form

$$\frac{dx}{dt} = Ax\big|_{t \ne nh} \qquad (n = 1, 2, 3, \ldots) \tag{62}$$

$$x(t_n + 0) = Qx(t_n) \qquad (n = 1, 2, 3, \ldots) \tag{63}$$

We shall show that (62), (63) is exponentially dichotomous if the spectrum of the operator $A + h \ln Q$ does not intersect the imaginary axis.

In fact, in this case the operator-valued function $U(t)$ has the form $U(t) = e^{At} Q^{[t/h]}$ and the spectrum of the operator $A + h \ln Q$ splits into two parts σ_1 and σ_2, where σ_1 lies in the left half-plane and σ_2 in the right one. Let $X = X_1 \dot{+} X_2$ be the corresponding decomposition of the space X as a direct sum and let P_1 and P_2 be the corresponding projectors. It is not hard to check that

$$\|U(t) P_1 U^{-1}(\tau)\| \le M_1 e^{\mu_1(t-\tau)} \qquad (0 \le \tau < t < \infty) \tag{64}$$

$$\|U(t) P_2 U^{-1}(\tau)\| \le M_2 e^{\mu_2(t-\tau)} \qquad (0 \le t < \tau < \infty) \tag{65}$$

where μ_1 is an arbitrary number of $(\sup \sigma_1, 0)$, μ_2 is an arbitrary number of $(0, \inf \sigma_2)$, and M_1 and M_2 are constants.

Example 2. Let the operator-valued function $A(t)$ $(0 \le t < \infty)$ and the operators Q_n $(n = 1, 2, 3, \ldots)$ commute with one another, i.e.,

$$A(t)A(s) = A(s)A(t) \qquad (0 \le t, s < \infty) \tag{66}$$

$$A(t)Q_n = Q_n A(t) \qquad (0 \le t < \infty, \quad n = 1, 2, 3, \ldots) \tag{67}$$

From equality (66) it follows that the evolutionary operator of the equation

$$\frac{d}{dt} = A(t)x$$

has the form

$$U_0(t, \tau) = \exp\left[\int_\tau^t A(s)\, ds\right] \qquad (0 \le \tau \le t < \infty)$$

It is not hard to check that the operator $U(t)$ in this case has the form

$$U(t) = \left\{\exp\left[\int_0^t A(s)\, ds\right]\right\} Q_{n(t)} \cdots Q_1$$

where $n(t) = \min\{n: t < t_n\}$.

In this case the impulsive equation (1), (2) is exponentially dichotomous if there exist projectors P_1 and P_2 which commute with $A(t)$ $(0 \le t < \infty)$ and Q_n $(n = 1, 2, 3, \ldots)$ and for which for some positive constants M and δ the following inequalities hold:

$$\left\| \left\{\exp\left[\int_\tau^t A(s)\, ds\right]\right\} Q_{n(t)} \cdots Q_{n(\tau)} P_1 \right\|$$

$$\le M \exp[-\delta(t-\tau)] \qquad (0 \le \tau < t < \infty) \tag{68}$$

$$\left\| \left\{\exp\left[\int_\tau^t A(s)\, ds\right]\right\} Q_{n(t)}^{-1} \cdots Q_{n(\tau)}^{-1} P_2 \right\|$$

$$\le M \exp[-\delta(\tau - t)] \qquad (0 \le t < \tau < \infty) \tag{69}$$

Example 3. Let the operator-valued function $A(t)$, the sequence $\{t_n\}_{n=1}^{\infty}$, and the sequence $\{Q_n\}_{n=1}^{\infty}$ be periodic, i.e., there exist $\omega > 0$ and a positive integer l for which the following equalities are valid:

$$A(t+\omega) = A(t) \qquad (0 \le t < \infty)$$

$$t_n + \omega = t_{n+l} \qquad (n = 1, 2, 3, \ldots)$$

$$Q_{n+l} = Q_n \qquad (n = 1, 2, 3, \ldots)$$

Then the operator-valued function $U(t)$ satisfies the functional equation

$$U(t+\omega) = U(t)U(\omega) \tag{70}$$

and the main properties of the impulsive equation (1), (2) are determined by the operator of monodromy $U(\omega)$.

Equation (1), (2) is exponentially dichotomous if the spectrum of the operator $U(\omega)$ does not intersect the unit circle. Indeed, in this case the space X is split into a direct sum $X = X_1 \dotplus X_2$ of two subspaces invariant with respect to $U(\omega)$. The spectrum of the restriction of $U(\omega)$ to X_1 lies inside the unit circle and the spectrum of the restriction of $U(\omega)$ to X_2 lies outside the unit circle. If P_1 and P_2 are the corresponding projectors, then it is not hard to see that there exist positive constants M and δ for which inequalities (11), (12) are valid.

Example 4. Let the space X be a direct sum of subspaces X_1 and X_2 invariant with respect to $A(t)$ $(0 \le t < \infty)$ and Q_n $(n = 1, 2, 3, \ldots)$ and let the evolutionary operator $U_0(t)$ of the ordinary differential equation

$$dx/dt = A(t)x \tag{71}$$

satisfy the inequalities

$$\| U_0(t) P_1 U_0^{-1}(\tau) \| \le e^{-m_1(t-\tau)} \qquad (0 \le \tau < t < \infty) \tag{72}$$

$$\| U_0(t) P_2 U_p^{-1}(\tau) \| \le e^{m_2(t-\tau)} \qquad (0 \le t < \tau < \infty) \tag{73}$$

where P_1 and P_2 are the corresponding projectors and $m_1, m_2 > 0$.

We note that the above conditions are fulfilled if, for instance, X is a Hilbert space and the spectrum of the restriction of $A(t)$ to X_2 lies in the interval (m_2, ∞).

Let the sequences $\{Q_n\}_{n=1}^{\infty}$ and $\{Q_n^{-1}\}_{n=1}^{\infty}$ be bounded and the following relations hold:

$$\overline{\lim_{t-\tau,\, \tau \to \infty}} \frac{\sum_{\tau \le t_n < t} \ln \| Q_n P_1 \|}{t - \tau} < -m_1 \tag{74}$$

$$\overline{\lim_{\tau - t,\, t \to \infty}} \frac{\sum_{t < t_n \le \tau} \ln \| Q_n^{-1} P_2 \|}{\tau - t} < m_2 \tag{75}$$

In this case the impulsive equation (1), (2) is exponentially dichotomous. Indeed, the evolutionary operators $U(t)$ and $U_0(t)$ for $t \in [0, \infty)$ are related by the equality

$$U(t) = U_0(t) U_0^{-1}(t_{n-1}) Q_{n-1} U_0(t_{n-1}) U_0^{-1}(t_{n-2}) Q_{n-2} \cdots Q_1 U_0(t_1)$$
$$(t_{n-1} < t \le t_n), \tag{76}$$

whence we obtain the inequality

$$\| U(t) P_1 U^{-1}(\tau) \| \le e^{-m_1(t-\tau)} \prod_{\tau \le t_n < t} \| Q_n P_1 \|$$

From inequality (74) there follows the existence of numbers $n_1 < m_1$ and N_1 such that

$$\sum_{\tau \le t_n < t} \ln \| Q_n P_1 \| \le N_1 + n_1(t - \tau) \qquad (0 \le \tau < t < \infty)$$

which implies

$$\| U(t) P_1 U^{-1}(\tau) \| \le e^{N_1} e^{(-m_1 + n_1)(t-\tau)} \qquad (0 \le \tau < t < \infty)$$

In this way inequality (11) is satisfied with $M = e^{N_1}$ and $\delta = m_1 - n_1$. The validity of inequality (12) is analogously proved.

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