ISENTROPIC STARS IN GENERAL RELATIVITY

A. KOVETZ
Tel-Aviv University, Ramat-Aviv, Israel

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Abstract. In an investigation of the evolution of homogeneous, isentropic, stars through stages of diminishing entropy, Rakavy and Shaviv (1968) have recently found that stars of mass less than M_c (Chandrasekhar's limiting mass for white dwarfs) evolve into white dwarfs, while stars of mass greater than M_c approach a (singular) state of minimum entropy. An elementary explanation of these results is given and qualitative effects of general relativity are discussed. It is found that stars which are lighter than the Oppenheimer and Volkoff (1939) limit become white dwarfs, while heavier stars must become dynamically unstable at a finite stage in their evolution.

1. Introduction

Recently, Rakavy and Shaviv (1968) have suggested that homogeneous, isentropic, configurations should serve as useful models for advanced stages of stellar evolution. Taking into account the effects of radiation, pair creation and quantum degeneracy, they have followed the evolution of such models through stages of diminishing entropy. It was found that stars of mass less than $M_c = 5.75\,\overline{(Z/A)}^2 M_\odot$ (Chandrasekhar's limiting mass for white dwarfs with the chemical composition considered) evolved into white dwarfs, while stars heavier than M_c evolved into singular states, becoming point masses of infinite density and temperature and unique (finite) entropy.

An elementary explanation of this remarkable behaviour of isentropic models is given in Section 2. This is used, in Section 3, for discussing the effects of general relativity on the evolution and stability of isentropic stars.

2. The Limiting Mass-Entropy Relation

The behaviour of homogeneous, isentropic, stars can be explained by observing, firstly, that the adiabatic relationship between pressure p, density ϱ and specific entropy s (entropy per unit mass) is of the form $p = K_1(s)\,\varrho^{5/3}$ for low densities and $p = K_2(s)\,\varrho^{4/3}$ for high densities (for intermediate densities the exponent may be less then $\frac{4}{3}$ in some 'regions of dynamical instability'); secondly the inequality (Landau and Lifshitz, 1958)

$$\frac{\partial s(p, \varrho)}{\partial p} = \frac{c_v}{T}\frac{\partial T(p, \varrho)}{\partial p} > 0, \tag{1}$$

where T is the temperature and c_v is the specific heat at constant volume.

From the theory of polytropes (Chandrasekhar, 1939) we know that in a polytrope of mass M the central pressure is connected with the central density by the formula

$$P_c = (4\pi)^{1/3} G B_n M^{2/3} \varrho_c^{4/3}\,, \tag{2}$$

where G is the gravitational constant, $B_n^{-1} = (n+1)\,[-\xi_1^2 \theta_n'(\xi_1)]^{2/3}$ and θ_n is the Lane-Emden function with first zero at ξ_1. For $n=0$, 1,5. 3.0, 5.0 we have $B_n = 0.347$, 0.206, 0.156, 0.116 respectively. In a (ϱ, p) diagram (Figure 1) we now draw the curve

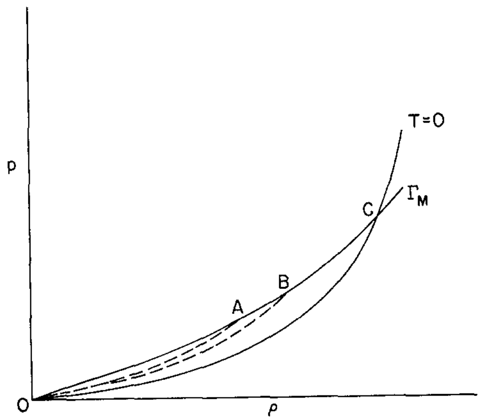

Fig. 1. For low masses the line of centres Γ_M crosses the degeneracy curve $T=0$. The broken curves OA and OB are structure lines.

Γ_M given by (2) with $B_n = B_{1.5} = 0.206$ for low densities and $B_n = B_3 = 0.156$ for high densities. The details of the passage from $B_{1.5}$ to B_3 are not important for the considerations which follow; in fact, we can think of Γ_M as having a small width corresponding to the spread in the values of B_n. According to (2), the centre of an isentropic star of mass M must lie on Γ_M. If M is sufficient small, Γ_M will cross the degeneracy curve corresponding to $T=0$ because that curve starts out as $p \propto \varrho^{5/3}$, and we have the situation depicted in Figure 1. The broken line OA corresponds to the structure line of an isentropic star with central conditions at A; it lies below Γ_M because for small densities $p = K_1(s)\,\varrho^{5/3}$. According to the inequality (1), a diminution of entropy will result in the *lower* structure line OB. Proceeding in this way, the centre will move along Γ_M until $s=0$ and the structure line coincides with the segment OC of the degeneracy curve: the star becomes a white dwarf. Moreover, since p_c along Γ_M is total pressure (including the contribution of radiation), the vertical difference between Γ_M and the degeneracy curve gives an indication of the temperature. As the density of the star increases towards that of a white dwarf, the star therefore heats up to a maximum temperature (corresponding to the maximal distance between Γ_M and $T=0$) and then cools off to zero temperature. This behaviour has been found for $M<M_c$ by Rakavy and Shaviv (1968) from detailed calculations.

Clearly, the foregoing discussion depends on Γ_M crossing the $T=0$ curve, which is true for masses less than the critical mass M_c for which Γ_M touches the degeneracy curve asymptotically. M_c is therefore determined by $K_2(0) = (4\pi)^{1/3}\; G B_3 M_c^{2/3}$ (the value of B_n is now certain), which is the formula for Chandrasekhar's limiting mass.

Consider now the case $M > M_c$. Then Γ_M does not cross the degeneracy curve (Figure 2) and it is clear that following a sequence of diminishing entropies we arrive at a configuration of infinite density and pressure, but of a finite minimal entropy determined by $K_2(s_{\min}) = (4\pi)^{1/3}\, G B_3 M^{2/3}$. Rakavy and Shaviv (1968) have solved

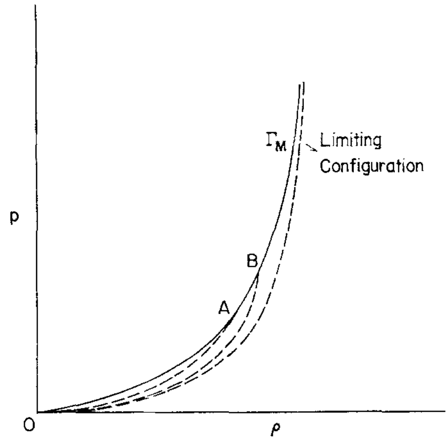

Fig. 2. The limiting configuration obtained when the structure line touches the line of centres Γ_M asymptotically.

this equation numerically for $s_{\min}(M)$, taking $Z/A = 0.5$ and ignoring the contribution of the ions to the pressure and entropy. In the next section we shall see that general relativity modifies the situation in such a way that the limiting configurations cease to be singular.

3. Effects of General Relativity

In general relativity we use the energy density e instead of the density ϱ. The behaviour of the adiabatic relationship between p and e as e increases is now from $p = K_1(s)\, e^{5/3}$ to $p = K_2(s)\, e^{4/3}$ to $p \to (\frac{1}{3})\, e$. The inequality (1), with ϱ replaced by e, still holds (Kovetz, 1968). The Γ_M curve describing the central conditions in the (e, p) diagram of isentropic stars of mass M will be the same as before when e is small, but for large e it will curve more sharply upwards as a result of the self-inforcement of the pressure in general-relativistic hydrostatic equilibrium (Kovetz, 1968).

We can confirm this behaviour of Γ_M by considering the polytropic relativistic fluid spheres, calculated by Tooper (1964). For $p = K e^{1+1/n}$ we have

$$p_c = (4\pi)^{1/3} G C_n M^{2/3} e_c^{4/3}\,, \tag{3}$$

where $C_n^{-1} = (n+1)\, v^{2/3}(\xi_1)$ and $v(\xi)$ is determined by the equations

$$\xi^2 \theta'(\xi) \frac{1 - 2\sigma(n+1)\, v/\xi}{1 + \sigma\theta} + v + \sigma\xi\theta \frac{\mathrm{d}v}{\mathrm{d}\xi} = 0, \qquad \theta(0) = 1\,, \tag{4}$$

$$v'(\xi) - \xi^2 \theta^n = 0\,, \qquad v(0) = 0\,, \tag{5}$$

with $\sigma = p_c/e_c$; ξ_1 is the first zero of $\theta(\xi)$. C_n in Equation (3) is thus a function of *both* n and $\sigma = p_c/e_c$. It decreases with increasing n and increases with increasing σ. Hence p_c increases along Γ_M more rapidly than $e_c^{4/3}$.

The situation is shown in Figure 3. The existence of a critical mass M_c can still be inferred, and isentropic stars of smaller mass must become white dwarfs. But owing to the sharper up-curving of Γ_M the critical mass will be smaller than the Chandrasekhar mass. In fact, if ion pressure is neglected, we have (Oppenheimer and Volkoff,

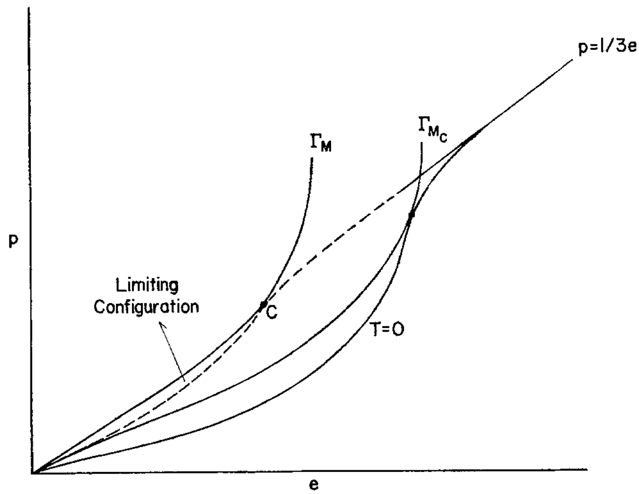

Fig. 3. In the general relativistic situation the structure lines, and the degeneracy curve $T=0$, approach $p=\frac{1}{3}e$ asymptotically. Points of tangency with lines of centres correspond to configuration with finite densities and radii.

1939) $M_c = 0.72\,\overline{(Z/A)}^2\,M_\odot$. Moreover, the limiting mass corresponds to a configuration of finite density and pressure with radius larger than the Schwarzschild radius $R_s = 2\,GM/c^2$ (p_c becomes infinite when $R \leqslant 1.125\,R_s$; Kovetz, 1968). Similar results apply to isentropic stars of mass larger than M_c. For these, a minimal entropy still exists; it is larger than the one obtained on Newton's theory of gravitation, and the limiting configuration is non-singular. We can therefore conclude that the maximal red shift $z = \Delta\lambda/\lambda$, which depends on the maximal value of R_s/R, is finite for any isentropic star. An examination of Oppenheimer and Volkoff's (1939) results for cold configurations, and of Tooper's (1964) calculations for polytropic models in general relativity shows that for all masses the maximal red shift is about ten percent.

A further conclusion concerns the stability of the limiting configurations. An isentropic star of mass $M > M_c$ must inevitably reach the limiting state represented by C in Figure 3, unless it encounters some instability before that. At C, however, a slight adiabatic compression will displace the central conditions upwards along the (broken) adiabatic curve while the pressure due to gravity will lie on Γ_M and will therefore be higher. This will cause further compression, and we conclude that at C the star is dynamically unstable. Since the red shift at C is only about 0.1, there is no question of a time dilatation which might cause such instability to appear as slow evolution to a distant observer.

To summarise, we have the following situation with respect to homogeneous, isentropic, stars: there exists a critical mass M_c such that stars of smaller mass evolve into cold configurations, while stars of heavier mass reach a limiting state with finite entropy and radius, at which they become dynamically unstable (unless they encounter such instability before that).

The foregoing ultimate instability of supercritical masses is not really confined to isentropic stars. It has been shown recently that the maximal central pressure (and the maximal central temperature) in a configuration of mass M and central energy density e_c is obtained when the configuration is isentropic (Kovetz, 1969). Furthermore, the minimal radius is also realised in this state. This implies that the point representing central conditions in Figure 3 cannot lie above Γ_M; it is generally between Γ_M and $T=0$, and in an isentropic star it lies on Γ_M. During the evolution the point will move towards higher densities and pressures, always keeping between Γ_M and $T=0$. But it must also remain below the line $p=(\frac{1}{3})\ e$. It is clear that for any possible track a point will be reached beyond which the pressure is less than what is required for sustaining the star against gravity, and this will mark the onset of dynamical instability.

The foregoing conclusions imply that observational evidence for an upper limit of about 10% for the gravitational red shift should yield a test for Einstein's field equations $G_{ij}+\kappa T_{ij}=0$ in non-empty space.

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