BIRKHOFF-TYPE THEOREM IN THE SCALE-COVARIANT THEORY OF GRAVITATION

(Letter to the Editor)

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Abstract. It is shown that an analog of Birkhoff's theorem of general relativity exists in the scale-covariant theory of gravitation when the gauge function which occurs in the theory is independent of time.

1. Introduction

In recent years there has been considerable interest in alternative theories of gravitation. The most important among them being scalar-tensor theories proposed by Brans and Dicke (1961), Sen and Dunn (1971), Ross (1972), Dunn (1974), Nordtvedt (1970), and Barber (1982). In the Brans–Dicke theory there exists a variable gravitation parameter G. Another theory which admits a variable G is the scale-covariant theory of Canuto *et al.* (1977a), which is available alternative to general relativity (Wesson, 1980; Will, 1984). Within the framework of the scale-covariant theory the cosmological 'constant' Λ appears as a variable parameter. On the other hand, Einsteins's general relativity does not admit the possibility of variable G or variable Λ. The field equations in the scale-covariant theory are (Canuto *et al.*, 1977)

$$R_{ij} - \tfrac{1}{2}g_{ij}R + f_{ij}(\phi) = -8\pi G(\phi)T_{ij}(\phi) + \Lambda(\phi)g_{ij}\,, \tag{1}$$

where

$$\phi^2 f_{ij} = 2\phi\phi_{i,j} - 4\phi_i\phi_j - g_{ij}(\phi\phi^k_{;k} - \phi^k\phi_k)\,; \tag{2}$$

in which ϕ is a scalar function (the gauge function) satisfying $0 < \phi < \infty$. In these equations R_{ij} is the Ricci tensor; R, the Ricci scalar; g_{ij}, the metric tensor; Λ, the cosmological 'constant'; G, the gravitational 'constant'; and T_{ij}, the energy-momentum tensor. A semi-colon denotes covariant derivative and ϕ_i denotes the ordinary derivatives with respect to x^i. A particular feature of the theory is that no independent equation for ϕ exist. Canuto *et al.* (1977b) and Beesham (1986a, b, c) have investigated several aspects of the scale-covariant theory.

In this paper, we have shown that a theorem analogous to Birkhoff's theorem in general relativity is true for the scale-covariant theory proposed by Canuto *et al.* (1977a) when the scalar function (the gauge function) introduced in the theory is independent of time.

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2. Birkhoff's Theorem in Scale-Covariant Theory

Birkhoff (1927) has shown that every spherically-symmetric solution of the Einstein vacuum field equations is static. This fact is known as Birkhoff's theorem. Reddy (1973), Krori and Nandy (1978, 1980), Duttachoudhury and Bhattacharya (1980), and Singh (1986) have shown that an analog of Birkhoff's theorem of general relativity exists in the various scalar-tensor theories of gravitation when the scalar field introduced in the theories is independent of time. On similar lines we show here that Birkhoff's theorem holds in the present theory of gravitation when the gauge function is independent of time.

We consider the spherically-symmetric metric in the form

$$ds^2 = e^{\nu}\,dt^2 - e^{\lambda}\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\Phi^2)\,, \tag{3}$$

where

$$\lambda = \lambda(r, t)\,, \quad \nu = \nu(r, t)\,;$$

with the scalar function $\phi = \phi(r, t)$.

The field equations (1) for the metric (3) are, in vacuum, as

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} - \Lambda - e^{-\lambda}\left[\frac{\phi''}{\phi} - \frac{\phi'}{\phi}\left(\frac{\lambda' + \nu'}{2} - \frac{2}{r}\right) - \frac{3\phi'^2}{\phi^2}\right] - - e^{-\nu}\left[\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}}{\phi}\left(\frac{\dot{\lambda} + \dot{\nu}}{2}\right) - \frac{\dot{\phi}^2}{\phi^2}\right] = 0\,, \tag{4}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{\lambda'\nu'}{4}\right) - e^{-\nu}\left(\frac{\ddot{\lambda}^2}{4} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) - \Lambda + + e^{-\lambda}\left[\frac{\phi''}{\phi} + \frac{\phi'}{\phi}\left(\frac{\nu' - \lambda'}{2}\right) - \frac{\phi'^2}{\phi^2}\right] - e^{-\nu}\left[\frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{\lambda} - \dot{\nu}}{2}\right) - \frac{\dot{\phi}^2}{\phi^2}\right] = 0\,, \tag{5}$$

$$e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} - \Lambda + e^{-\lambda}\left[\frac{\phi''}{\phi} - \frac{\phi'}{\phi}\left(\frac{\lambda' + \nu'}{2} - \frac{2}{r}\right) - \frac{\phi'^2}{\phi^2}\right] + + e^{-\nu}\left[\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}}{\phi}\left(\frac{\dot{\lambda} + \dot{\nu}}{2}\right) - \frac{3\dot{\phi}^2}{\phi^2}\right] = 0\,, \tag{6}$$

$$e^{-\lambda}\,\frac{\dot{\lambda}}{r} - e^{-\lambda}\left[\frac{2\dot{\phi}'}{\phi} - \frac{\dot{\lambda}\phi'}{\phi} - \frac{\nu'\dot{\phi}}{\phi} - \frac{4\phi'\dot{\phi}}{\phi^2}\right] = 0\,; \tag{7}$$

where primes denote partial differentiation with respect to r and over dots denote partial differentiation with respect to t.

When the gauge function ϕ is independent of t – that is,

$$\dot{\phi} = 0 \tag{8}$$

from (7), we have

$$\dot{\lambda}\left(\frac{1}{r} + \frac{\phi'}{\phi}\right) = 0\,;$$

which implies that either

$$\dot{\lambda} = 0 \tag{9}$$

or

$$\phi = \phi_0/r\,, \qquad \phi_0 = \text{const.} \tag{10}$$

When $\dot{\phi} = 0$, we have from (4) and (6)

$$1 + \frac{r}{2}\,(\nu' - \lambda') + r^2\,\frac{\phi'^2}{\phi^2} = e^{+\lambda}(1 + r^2\Lambda)\,. \tag{11}$$

By use of (8) and (11) in (4) we get

$$\frac{r}{2}\,(\nu' - \lambda') + 2 + e^{\lambda}(1 - r^2\Lambda) = 0\,; \tag{12}$$

and, lastly, from (10), (11), and (12) we get

$$e^{\lambda} = 0\,. \tag{13}$$

Therefore, in this case, no solution arises. The only possibility is then

$$\dot{\lambda} = 0\,.$$

Now a differentiation of (11) with respect to t along with the use of (8) and (9) gives

$$\dot{\nu}' = 0\,. \tag{14}$$

From this, we have

$$\nu = f(r) + g(t)\,, \tag{15}$$

where f and g are arbitrary functions of r and t, respectively. We can now use the transformation (cf. Das, 1960)

$$\mathrm{d}t' = e^{g(t)/2}\,\mathrm{d}t\,.$$

This turns into a function of r only. This together with (9) reduces the metric (3) to the static case.

Thus we have shown that Birkhoff's theorem of general relativity is true in the scale-covariant theory of gravitation when the gauge function in the theory is independent of time.

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