AREND HEYTING

INTUITIONISTIC VIEWS ON THE NATURE OF MATHEMATICS

1. INTRODUCTION

One of the questions which philosophers ask about mathematics is: Why are mathematical theorems so certain? Whence does mathematics take its evidence, its indubitable truth? The answer of intuitionists to these questions is: The basic notions of mathematics are so extremely simple, even trivial, that doubts about their properties do not rise at all. Intuitionism is not a philosophical system on the same level with realism, idealism, or existentialism. The only philosophical thesis of mathematical intuitionism is that no philosophy is needed to understand mathematics. On the contrary, every philosophy is conceptually much more complicated than mathematics.

Logic in the usual sense does depend upon philosophical questions. One of its basic notions is that of a proposition being true. But what is a proposition? Does it coincide with the sentence by which it is expressed or is it something behind the sentence, some meaning? If so, what is the relation between the proposition and the sentence? And what does it mean that the proposition is true? Does this notion presuppose the existence of an external world in which it is true? If the proposition is the same as the sentence analogous questions can be asked. I am not going to answer them; they have been solved in a hundred different ways, none of them quite convincing, and all of them showing that logic is complicated and therefore unsuitable as a basis for mathematics. I shall come back to the relations of logic to mathematics later in this talk.

We look for a basis of mathematics which is directly given and which we can immediately understand without philosophical subtleties. The first that presents itself is the process of counting. However, counting establishes a correspondence between material or non-material objects and the natural numbers, so it can only be understood if both an external world (or at least some sort of objects) and abstract numbers are given. It is still too complicated to serve as a basis for mathematics. An analysis

Synthese **27** (1974) 79–91.

of the process of counting will lead us to simpler, more immediate notions. We can count all sorts of things, but they have one property in common, namely that they can be isolated. Isolating an object, focusing our attention on it, is a fundamental function of our mind. No thinking is possible without it. In isolating objects the mind is active. Our perception at a given moment is not given as a collection of entities; it is a whole in which we isolate entities by a more or less conscious mental act.

It seems as if we have got no further, for we are still counting material objects. In reality, what we isolate mentally are not objects, but perceptions. I can fix my attention on a certain impression, in most cases visual. In practice this impression is immediately associated with innummerable memories, impressions, and images to form the notion of an object in the general sense of the word. But for counting it is inessential what there is isolated, it is the mental act of isolating that matters. The entity conceived in the human mind is the starting point of all thinking, and in particular of mathematics. When we think, we think in entities. This does not mean that all our mental life consists of thinking in entities. On the contrary, the more intensely we live, the less we think in isolated entities. Under the influence of strong emotions the world seems a whole, loaded with emotion. Only after the emotions are soothed we map out aims and ways to attain them.

Instead of 'fixing my attention on a perception' I shall say 'creating an entity', but we must be well aware that the verb 'to create' has here another meaning than in 'creating a work of art'. After being created a painting exists in an external world, but this is not the case for the mentally created entity.

Mentally creating an entity is an act which everybody performs at almost any moment when he is awake. We can ask philosophical questions about it, for instance: How is it possible that we can think in entities? But we do it without answering such questions, like we are conscious without knowing how consciousness is possible and like we live without knowing how it is possible that living creatures exist. It is undeniable that conceiving an entity is an act of the individual mind. For the moment I leave the question of objectivity aside; it belongs to philosophy. In its simplest form mathematics remains confined to one mind; we must discuss later how it can be communicated.

2. Arithmetic

Mathematics would be of little use if it stopped after creating one entity; this act can be repeated. Again we can ask philosophical questions: How is it possible that an entity that has been created, maintains its identity and how can it be distinguished from another entity?

But again this is reasoning post factum: Anybody can experience for himself that he is able to fix his attention on a perception, and then on another perception, keeping the first in his memory. This is the basis of counting. It does not matter what is counted, but the process of counting itself, the mental activity, is essential. By creating an entity, another, still another, etc., we construct mentally natural numbers. It is clear that in constructing the number five, say, the nature of the entities which constitute that number is completely irrelevant. As soon as numerals were introduced, people have learnt to abstract from the content of the perceptions which are isolated and to consider them as pure entities.

We have now constructed each number individually. We are not yet able to make statements about every natural number. Such a statement is usually formulated by means of a generalizing quantifier: For every natural number n, $A(n)$ holds. But a better formulation is: Let n be a natural number, then $A(n)$ holds. More explicitly: suppose that we have constructed a natural number n, then we can prove $A(n)$. We see that it contains the notion of a hypothetical construction. This notion is fundamental in mathematics. Almost every theorem can be brought in the form: Suppose that construction A has been performed, then we can also perform construction B. The proof of such a theorem consists in a construction which, joined to the construction A, yields the construction B.

Let me give an example. I wish to prove the theorem: If n is any natural number, then there exists a prime number greater than n. The proof is: calculate $n!+1$. Factorize this number. Each of its prime factors will be greater than n. The proof is a general method of construction, to be applied to a hypothetical construction.

So far we have needed the notions of a *natural number*, of a *hypothetical construction* of a natural number and of a *general method* of construction to be applied to a hypothetical construction.

These notions are sufficient for arithmetic. Let us consider in particular the principle of complete induction:

$$A(1)\ \&\ \bigwedge x(A(x) \to A(x+1)) \to \bigwedge x A(x).$$

Suppose that we have proved $A(0)$ and that we have a general method M which allows us for any natural number x, to deduce a proof of $A(x+1)$ from a hypothetical proof of $A(x)$. Let n be any natural number. In order to prove $A(n)$ we construct the number n, and at each step from x to $x+1$ we apply M to obtain $A(x+1)$. The result will be a proof of $A(n)$.

Now I must warn for the misunderstanding that we need a general principle of complete induction; we only need the application to every particular case, and this is evident in every case. For instance, I wish to prove that $\sum_1^n k = (n(n+1))/2$. This is true for $n=1$. Suppose it is proved for x. (Hypothetical construction.)

$$\sum_1^{x+1} k = \frac{x(x+1)}{2} + x + 1 =$$
$$= \frac{(x+1)(x+2)}{2} \quad \text{(General method)}.$$

Let n be any natural number. I can prove $A(x)$ successively for $x=1, \ldots, n$. The latter is a direct application of the definition of a natural number.

It can be argued that no other notions than those which I have mentioned are needed in arithmetic. Arithmetical propositions are formed out of primitive relations $a=b$ and $a<b$ by means of the connectives &, $\vee$, $\to$, $\neg$ and the quantifiers $\bigwedge$, $\bigvee$. Now a proof of $a=b$ consists in the simultaneous construction of a and of b in such a way that each time that an entity is added to a, the same is done for b. An analogous explanation can be given for $a<b$. Of course the logical constants must be interpreted in terms of constructions. I shall discuss this later; here a few remarks will be useful. The interpretation of $A \to B$ is implicit in what I have said: A proof of $A \to B$ consists of a general method which converts every proof of A into a proof of B. A proof of $\neg$ A consists in a method which would convert a supposed proof of A into a contradiction. I am inclined to say that it must be taken as primitive. We see clearly that it is impossible that $1=2$, but the notion of impossibility is not reducible to

the other notions which I have mentioned. It is good to avoid negations where it is possible. Bishop's work shows that the most important parts of analysis can be built up positively [Bishop, 1967]. A proof of $\bigwedge x\, A(x)$ consists in a general method which converts the construction of a natural number x into a proof of $A(x)$. Finally, a proof of $\bigvee x\, A(x)$ is the combination of the construction of a natural number x and a proof of $A(x)$.

The only new fundamental notion is that of a *contradiction.*

3. The continuum

So much about arithmetic. The next step is that of introduction of real numbers, which gives rise to great difficulties for the constructivist. A real number is defined by means of an infinite sequence of natural numbers. Here the infinite is much more essential than in arithmetic, where it occurs only in the form 'after each natural number there is a next one'.

In analysis we make statements about every real number, that is, about every infinite sequence of natural numbers. The difficulty is that we have no clear notion of a hypothetical sequence; there is no general method for the construction of sequences like there is for the construction of natural numbers. One solution was made possible by the theory of recursive functions; recursive analysis has become an important field of research. But the notion of a recursive function was introduced in the 30's, whilst Brouwer's work on real numbers falls between 1907 and 1927. Moreover, as is well known, the recursive real numbers do not exhaust the continuum; the set of recursive real numbers is denumerable while the continuum is not. Brouwer tried to find a constructive notion which is as near as possible to that of the usual continuum. He struggled with this problem all his life. In his thesis of 1907 he introduced the continuum as a primitive notion. Man has an intuition of a continuum (the intuition of time) on which he can construct a dense, denumerably infinite scale. A point on the continuum is defined by a convergent sequence of points of the scale. If we restrict ourselves to sequences determined by a law (predetermined sequences), we do not obtain every point of the continuum. In non-constructive mathematics there is no difficulty. One simply defines the set of all convergent sequences, whether defined by a law or not. But for the constructivist only the predetermined sequences exist as individuals. Brouwer found the way out by introducing the notion of a choice sequence. A convergent sequence of rationals can be obtained by choosing its members one after the other: $r_1, r_2, \ldots$; the convergence can be secured for instance by the restriction that $|r_{n+1} - r_n| < 2^{-n}$ for every n. We have here a simple example of a *spread.* A spread is defined by a rule which determines the restrictions on the choices.

From 1918 on Brouwer no longer mentions the continuum as a primitive notion. He can do without it because the spread defined above represents it completely, as far as its mathematical properties go.

The notion of a spread is not problematic. It is defined by a restriction on finite sequences. But the choice sequence as an element of the spread is an important new fundamental notion, which gives rise to several questions. A first question is, how free a choice sequence must be. It has been tried to define a choice sequence as a lawless sequence, in which every choice must be completely free. However, these lawless sequences have unpleasent properties; they are hermits, incapable of intercourse with each other. [Troelstra, 1969]. The only relation possible between two lawless sequences is that of complete identity, for if they are not the same, they are completely independent. Thus, in order to do mathematics with choice sequences, restrictions on the freedom of choices must be allowed. Brouwer did this from the beginning. We had an example in the spread which represents the continuum. It is reasonable to allow restrictions to be made during the process of choosing. For instance, I start without restrictions and I choose $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{8}$. At this moment I can make the restriction that every further member will be $\frac{3}{8}$. Another possibility is to leave the possibility open to choose always $\frac{3}{8}$, or from $n=k$ on always $\frac{3}{8}+2^{-k}$. In the latter case we do not know whether the sequence will define the number $\frac{3}{8}$ or some number which is a little greater than $\frac{3}{8}$. At any moment a choice sequence α consists of a finite segment together with certain restrictions on its continuation. As the proof for a property of α must be given in a finite time, it can depend only on these data. This fact is known as Brouwer's continuity principle. It makes us understand some peculiar theorems on the continuum. For instance, every function that is defined everywhere on a closed interval, is uniformly continuous. Let f be the function; we wish to calculate $f(a) = b$. Here a is defined by a choice sequence $a_1, a_2, \ldots$ of rationals, b by a sequence $b_1, b_2, \ldots$; b_n must be determined by a finite sequence $a_1, \ldots, a_m$, but then all sequences beginning with $a_1, \ldots, a_m$ will give the same b_n. This means that a certain approximation to b is determined by a certain approximation to a. This is not a proof of the theorem. I only made it plausible.

It is possible to do constructive mathematics without choice sequences, but these are interesting for several reasons.

(1) A notion of the continuum which corresponds to the usual notions is only possible by using choice sequences.

(2) The reasonings on choice sequences are interesting in themselves, such as for instance the continuity theorem, and they lead to interesting results.

(3) The exact formulation of the notion and the basic properties of choice sequences lead to interesting questions, which have attracted much attention in the last 10 years. It is remarkable, that formalization is the most important method used in this work; this has strongly influenced the relation between intuitionism and formalism, about which I shall speak later on.

Another argument in favour of choice sequences is, that we can calculate with them. For instance, if $\{a_n\}$ and $\{b_n\}$ are convergent choice sequences which define the real numbers a and b, then $\{a_n+b_n\}$ will be a convergent sequence defining $a+b$.

4. Set theory

A few words must be said about set theory. It is a widely spread opinion that intuitionists admit only decidable sets, such as that of the even numbers or the prime numbers, but this is not the intuitionistic point of view; it is much too narrow. There is no objection to admitting any property of mathematical entities as the definition of a set. Brouwer calls such a set a species, but this is only a question of terminology. For instance, I can speak about the species S of digits which occur infinitely often in the decimal expansion of π. Though I cannot show an element of S, I know that S cannot be empty. Thus, if NE is the species of non-empty species of natural numbers, then $S \in NE$.

The theory of species is strictly predicative in this sense that the elements of a species must be defined independently of the species itself. We start with natural numbers; the next level is formed by choice sequences of natural numbers and by spreads which can be considered as species of choice sequences. Species of natural numbers and spreads are species of type 0. A species like NE is of type 1, and so on. Quantification over species is admissible, but only restricted to the elements of a given spread or species.

5. Logic

After this brief sketch of the fundamental ideas of intuitionistic mathematics I shall now speak about its relations to logic, to philosophy and to language. The word 'logic' is used for different notions; accordingly a logical law admits different interpretations.

Let us consider the syllogism

(1) Socrates is a man.
(2) Every man is mortal.
(3) Socrates is mortal.

(I) It can be considered as a rule of language.

(1) *A* is a *B*.
(2) Every *B* is a *C*.
(3) *A* is a *C*.

When I agree with (1) and (2), I am expected to agree also with (3).
(II) It can be considered as a statement about the world: If (1) is true and (2) is true, then (3) is true.
(III) It can be considered as a mathematical theorem. If the entity *A* belongs to the species *B*, and *B* is part of the species *C*, then *A* belongs to *C*.

$$\frac{\begin{array}{l} A \in B \\ B \subset C \end{array}}{A \in C.}$$

It is clear that none of these interpretations can be used for the foundation of mathematics. On the contrary, each of them presupposes mathematics. (I) and (II) belong to applied mathematics, for the theory of language as well as any theory about the real world is applied mathematics. (III) is clearly a theorem from set theory, which itself is a rather advanced part of mathematics.

More generally, logic can be considered as a part of linguistics or as a philosophic theory about the world; in both cases it belongs to applied mathematics. In pure mathematics only the third interpretation comes up for discussion. Logical theorems are mathematical theorems. Logic is not the foundation of mathematics, on the contrary, it is conceptually a complicated and sophisticated part of mathematics.

If mathematics consists of mental constructions then every mathematical proposition must be an assertion about mental constructions. More exactly: every mathematical proposition is of the form: A construction with the following properties has been performed: .... In logic we consider the case that the construction is built up out of simpler constructions by means of the logical constants. I have already spoken about the interpretation of the logical constants, but some additional remarks will be useful. For conjuntion there is no difficulty. As to disjunction, we can assert $A \vee B$ when we have performed one of the constructions A or B; but be careful, it is nonsense to say that I have performed A or B without knowing which of the two. When I can assert $A \vee B$, I am always able either to assert A or to assert B, or both. Implication is interpreted as follows: I may assert $A \rightarrow B$ when I am able to convert any proof of A into a proof of B. In other words, I must possess a general method of construction which, applied to a proof of A, yields a proof of B.

I have spoken about the reduction of negation to the basic notion of a contradiction.

Does the law of excluded middle $A \vee \neg A$ hold with these interpretations? When we assert it, this means that for any proposition A we can either prove A or derive a contradiction from a supposed proof of A. Obviously we are not able to do this for every proposition A, so the law of excluded middle cannot be proved. If we do not know whether A is true or not, we better make no assertion about it.

It has been proposed to give a weaker interpretation to $A \vee B$, namely: A and B cannot both be false. Then $A \vee B$ would be the same as $\neg (\neg A \,\&$ $\&\neg B)$. For this interpretation the law of excluded middle becomes $\neg (\neg A \& \neg \neg A)$, so it is a special case of the law of contradiction. Though this interpretation is tenable, there are serious objections against it. I have already made the remark that negation ought to be avoided where it is possible. It is important that we can decide for every algebraic number whether it is rational or not; without the strong disjunction it would be impossible to express this property. The weak interpretation $\neg (\neg A \,\&$

$\& \neg\ \neg A)$ gives only the trivial result that an algebraic number cannot be irrational and not irrational at the same time. Intuitionistic logic makes finer distinctions possible, which classical two-valued logic is unable to express.

The interpretation of the existential quantifier $\bigvee$ is analogous to that of disjunction. I can assert $\bigvee x\, A(x)$ when I have constructed an element x and proved that $A(x)$ holds for it. The weak interpretation would be $\neg \bigwedge x \neg\ A(x)$, but the two notions are different and strong existence is by far the more important. We had an example in the proposition: The digit x occurs infinitely often in the decimal expansion of π. For this proposition $A(x)$ it is easy to prove $\neg \bigwedge x \neg A(x)$, but we are not able to prove $\bigvee x\, A(x)$, for if a is any digit, it is still possible that $\neg A(a)$.

I shall not give many such examples but I must say a few words about their use, because it is a persistent misunderstanding that they are an essential part of intuitionistic mathematics. Their function is the same as that of similar examples in classical mathematics. For instance, an example of a continuous function which is nowhere differentiable is useful as a warning against mistakes, but it is not an essential part of analysis.

It is clear that the generalizing quantifier can only be used when the range of the variable is given by some species. A theorem can hold for every natural number, for every real number, for every species of natural numbers, etc., but not for everything. The case where the variable ranges over a spread is interesting, for an element x of a spread S is a choice sequence, so when $A(x)$ holds for every element x of S, it must be known for every x after a finite segment of x has been chosen. In other words, the assertion $\bigwedge x A(x)$, where x ranges over a spread, is very strong.

6. Mathematics and language

So much about logic. Let us now ask how mathematics can be communicated. In my opinion there is no essential difference between the use of language for this and for other purposes. We use language to influence the thoughts and actions of other people. When a mathematician writes a paper or a book, he intends to suggest mathematical constructions to other people; when he makes notes to aid his memory, his future self plays the part of another person. Like any other use of language, the communication of mathematics is not immune for misunderstanding.

There is strong evidence for the hypothesis that the construction of small natural numbers is the same for all men, but for the communication about more complicated structures even a strong effort for clearness cannot warrant complete understanding.

In this respect intuitionism is exactly the opposite of formalism. It is not my task to describe the standpoint of formalism, but comparing both directions may contribute to the clarification of each of them. I take the liberty to consider the most radical kind of formalism which is best suited to be compared with intuitionism. The formalist considers every intuitive mathematical reasoning as inexact. He studies the language in which such reasonings are expressed and tries to formalize them. The result is a formal system, consisting of a finite number of symbols and a finite number of rules for combining them into formulas. From the intuitionistic point of view, this process belongs to applied mathematics, and the result is a very simple mathematical system. This formal system can be applied in science and in industry; its function is comparable to that of a machine in a factory.

Of course there is no objection against the activity of formalists, also it is undeniable that scientists and engineers are more interested in mathematical formulas themselves than in their abstract interpretation. There is no conflict between intuitionism and formalism when each keeps to its own subject, intuitionism to mental constructions, formalism to the construction of a formal system, motivated by its internal beauty or by its utility for science and industry. They clash when formalists contend that their systems express mathematical thought. Intuitionists make two objections against this contention. In the first place, as I have argued just now, mental constructions cannot be rendered exactly by means of language; secondly the usual interpretation of the formal system is untenable as a mental construction.

In the history of formalist research much work has been done on consistency proofs. From the point of view which I am sketching their importance is mainly practical. An inconsistent system, in which every formula is derivable, cannot be very useful. The pretension that a consistency proof would afford an interpretation of the formal system is completely unfounded.

Yet there is a possible application of formal methods to intuitionistic mathematics. It is the best method for investigating the assumptions which are made in a given proof. In recent years it has been successfully applied to the proofs in Brouwer's theory of choice sequences. The formalization of intuitionistic logic served another purpose, namely to express the logical theorems in a language which is understood by traditional mathematicians. The metamathematical work on the formal system of intuitionistic logic, however interesting in itself, has little to do with intuitionistic mathematics.

7. Applied mathematics

From what I have said at the beginning of this talk it will be clear that all conscious thinking can be considered as applied mathematics. Of course we do not start by constructing a mathematical system which we apply afterwards to our impressions. In daily life the creation of an entity goes together with the impression which it represents and with the complex of memories and expectations which is connected with this impression. We cannot say which comes first, for the whole complex arises together at the moment when I fix my attention on it. Only afterwards can I analyze it more or less into its components. A reasoning of the sort we make in daily life consists of a finite number of such entities and relations between them.

There is a gradual development from this natural activity in everyday life to the most abstract science constructed by intense collaboration of groups of scientists. Science tries to organize domains of experience which so far seemed quite apart from each other in a wider structure which embraces them together. In this respect there is no fundamental difference between science in the narrow sense (science of nature) and the humanities like history or psychology. The differences are gradual: the mathematical systems used in modern physics are enormously more refined than those that are at the basis of history, but the work of the historian also consists in establishing relations between the facts that he has isolated in the continuous stream of events. The essential difference between different sciences does not consist in the methods by which they try to order their material, but in the manner in which they obtain their raw material. For example, introspection is useless in physics, but it is a legitimate method in history, because it allows us to find relations between the circumstances in which a man is placed and his reactions. But this registration of facts belongs to the pre-scientific phase: measuring black spots on a photographic negative, having an interview, imaging one's reactions, that is not yet science.

Every science goes far beyond what is directly perceived. It constructs a mathematical system into which the facts and their relations can be fitted. The geometrical structure which we impose on the material world contains much more than what we can actually observe. Analogous remarks can be made about history: historians try to adjust the facts which they have learnt from the documents to a wider structure of hypothetical events.

It will now be clear that linguistics and logic, the latter in so far as it is not considered as a chapter of mathematics, belong to applied mathematics. Scientific philosophy also belongs to applied mathematics, but many important works on philosophy are not scientific. I do not mean this in a depreciating sense. Many philosophical books are works of art; they belong to literature or to poetry rather than to science; often they are very poetical and belong to the best of literature. But it is wrong that they pretend to belong to science and the scientific garb does not become them.

University of Amsterdam

BIBLIOGRAPHY

E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York 1967.

L. E. J. Brouwer, *Over de Grondslagen der Wiskunde*, Amsterdam 1907.

L. E. J. Brouwer, 'De Onbetrouwbaarheid der Logische Principes', *Tijdschrift voor Wijsbegeerte* **2** (1908) 152–158.

(Translations of these two papers will be contained in the collected works of L. E. J. Brouwer, to appear in 1974.)

L. E. J. Brouwer, 'Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten, Erster Teil, Allgemeine Mengenlehre', *Verhandelingen Nederl. Akad. Wetenschappen*, 1e Sectie, **12**, No. 5 (1918).

A. S. Troelstra, 'Informal Theory of Choice Sequences', *Studia Logica* **25** (1969) 31–54