THE PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM

Comments on the "Out-of-Plane" Equilibrium Points

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Abstract. The photogravitational restricted three-body problem is reviewed and the case of the "out-of-plane" equilibrium points is analysed. It is found that, when the motion of an infinitesimal body is determined only by the gravitational forces and effects of the radiation pressure, there are no "out-of-plane" stable equilibrium points.

1. Introduction

The existence of the equilibrium points, situated somewhere out of the orbital plane, has already been subject of many papers, from Radzievsky (1953) to Ragos and Zagouras (1994). Nevertheless, occasionally, in the corresponding literature we may read about doubtful conclusions or even contradictory opinions (e.g. Schuerman 1980; Ragos and Zagouras, 1994; Todoran, 1993).

In order to emphasize the circumstances in which some contradictory results have been obtained, we find that a new review of the problem could be of practical importance. Such a subject will be approached below.

2. General Equations

In the Euclidean space consider two stars S_1 and S_2, which are moving along circular orbits around the common mass-center, their masses being m_1 and m_2. From the gravitational point of view, these two bodies are assumed to be mass-points.

In addition, let us choose an infinitesimal body S which is attracted by the two stars, but does not attract them. Therefore, the motion of the infinitesimal body is determined by the *gravitational forces* of the two finite bodies S_1 and S_2. Nevertheless, in some peculiar cases, in order to explain the corresponding motion additional forces must be introduced and some supplementary effect taken into account. Among such forces, the most discussed could be the force of the radiation-pressure, when the gravitational role of the two masses m_1 and m_2 is reduced by the factors $1 - \beta_1$ and $1 - \beta_2$, respectively. Here, $\beta = F_R/F_G$ (F_R = radiation-pressure force and F_G = gravitational force), (e.g. Todoran and Roman, 1993).

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Consider now a rotating barycentric coordinate system (X, Y, Z) and the two finite bodies always situated on the X-axis, while the XY-plane coincides with the orbital plane. Under such conditions, with the notations used by Ragos and Zagouras (1988), the motion of the infinitesimal body is characterized by the following differential equations

$$\ddot{X} - 2\dot{Y} = X - \frac{Q_1}{r_1^3}(X + \mu) - \frac{Q_2}{r_2^3}(X + \mu - 1) = \frac{\partial U}{\partial X}, \tag{1}$$

$$\ddot{Y} + 2\dot{X} = Y\left(1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right) = \frac{\partial U}{\partial Y}, \tag{2}$$

$$\ddot{Z} = Z\left(-\frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right) = \frac{\partial U}{\partial Z}, \tag{3}$$

where the potential function U is defined by

$$U = \frac{1}{2}(X^2 + Y^2) + \frac{Q_1}{r_1} + \frac{Q_2}{r_2} \tag{4}$$

with

$$r_1^2 = (X + \mu)^2 + Y^2 + Z^2, \qquad r_2^2 = (X + \mu - 1)^2 + Y^2 + Z^2 \tag{5}$$

and

$$Q_1 = (1 - \beta_1)(1 - \mu), \qquad Q_2 = (1 - \beta_2)\mu. \tag{6}$$

Here, the masses of the secondary and primary stars are

$$\mu = \frac{m_2}{m_1 + m_2}, \qquad 1 - \mu = \frac{m_1}{m_1 + m_2}. \tag{7}$$

Now, as it is known, in the restricted three-body problem, the equilibrium points are double points situated somewhere on the equipotential surfaces of zero relative velocity, their positions being determined by the conditions

$$\frac{\partial U}{\partial X} = \frac{\partial U}{\partial Y} = \frac{\partial U}{\partial Z} = 0.$$

In such conditions, from Equations (1)–(3) we may write

$$X\left[1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right] - \left[\frac{Q_1\mu}{r_1^3} - \frac{Q_2(1-\mu)}{r_2^3}\right] = 0, \tag{8}$$

$$Y\left(1 - \frac{Q_1}{r_1^3} - \frac{Q_1}{r_1^3}\right) = 0, \tag{9}$$

$$Z\left(-\frac{Q_1}{r_1^3} - \frac{Q_1}{r_1^3}\right) = 0, \tag{10}$$

As it is easy to see, Equations (1)–(3) define the three-dimensional motion of the infinitesimal body, but if we assume $X \neq 0, Y \neq 0, Z = 0$, willing or not, we are obliged to accept the fact that the corresponding particle remains in the plane in which S_1 and S_2 revolve. In such a case, the motion of the infinitesimal body will be defined only by Equations (1) and (2), while Equation (3) is not taken into consideration.

Moreover, Szebehely (1967, p. 30) has written:

"Within the framework of Newtonian gravitational forces, if the initial position and velocity vectors of the third body are in the plane of motion of m_1 and m_2 then the motion of the third body will be confined to this plane since there are no forces taking it out of this plane".

Therefore, if it is assumed that the infinitesimal body remains in the orbital plane, we have a peculiar case of the restricted three-body problem. Such a case was studied by Schuerman (1980) who has written:

"Thus, although radiation pressure produces a large effect on the location of the L_4 and L_5 points, it has a very small effect on restricting the values of μ necessary for stability .... A similar analysis can be carried out for the equilibrium points L_1, L_2, and L_3 for which $Y = 0$.... It is found that, as in the classical case, these points are unstable and no equilibrium points exist for $\beta_1 > 1$ or $\beta_2 > 1$".

3. Equilibrium Points Assumed in the XZ-plane ($Z \neq 0$)

As it was before mentioned, the three-dimensional motion of the infinitesimal body is governed by Equations (1)–(3). Nevertheless, when the corresponding body crosses the XZ-plane we have $X_0 \neq 0$, $Y_0 = 0$ and $Z_0 \neq 0$. In addition, if the crossing point is assumed as being also an equilibrium point, we have to use Equations (8) and (10), whence we may write

$$X_0 = \frac{Q_1\mu}{r_{01}^3} - \frac{Q_2(1-\mu)}{r_{02}^3}, \tag{11}$$

$$\frac{Q_1}{r_{01}^3} + \frac{Q_2}{r_{02}^3} = 0, \tag{12}$$

with

$$r_{01}^2 = (X_0+\mu)^2 + Z_0^2, \qquad r_{02}^2 = (X_0+\mu-1)^2 + Z_0^2. \tag{13}$$

Moreover, Equation (12) could be valid only when, with the physical parameters of the two finite bodies, we may write

$$(1-\beta_1)(1-\beta_2) < 0, \text{ that is } Q_1Q_2 < 0, \tag{14}$$

which represents the "necessary condition" as it is considered by Ragos and Zagouras (1994) who have written:

"The necessary condition for the existence of ⟨ out-of-plane ⟩ equilibrium positions, $q_1 q_2 < 0$, is proved to make sense both from the theoretical and the physical point of view. There may exist, at most, two pairs of such points named L_6, L_7 and L_8, L_9".

Now, as it is easy to see, from Equations (11) and (12) it follows that

$$X_0 - \frac{Q_1}{r_{01}^3} = 0, \qquad X_0 + \frac{Q_2}{r_{02}^3} = 0, \tag{15}$$

whence it is evident that for $1 - \beta_1 < 0$ and $1 - \beta_2 > 0$ ($Q_1 < 0; Q_2 > 0$) we can always use only $X_0 < 0$. This remark is very important for the next step of the problem.

In order to find the coordinates (X_0, Z_0) of the equilibrium points, assumed somewhere in the XZ-plane we have to solve Equations (13) and (15) so we obtain

$$Z_0^2 = \frac{Q_1^{2/3} - X_0^{2/3}(X_0 + \mu)^2}{X_0^{2/3}}, \qquad Z_0^2 = \frac{Q_2^{2/3} - X_0^{2/3}(X_0 + \mu - 1)^2}{X_0^{2/3}}, \tag{16}$$

$$X_0 = \left[\frac{Q_1^{2/3} - Q_2^{2/3}}{2X_0 + 2\mu - 1}\right]^{3/2}. \tag{17}$$

Here we have to use a series of successive approximations with $X_0 < 0$ and the corresponding solution will be found in the range of the real numbers only if it is satisfied the condition

$$Q_1 > -Q_2, \tag{18}$$

which is a supplementary condition, to that mentioned by Ragos and Zagouras (1994) [see Equation (14)]. On the other hand, if we have in mind what Ragos and Zagouras (1994) have written: *There may exist, at most, two pairs of such points named* L_6, L_7 *and* L_8, L_9, we have to emphasize the fact that we could find, at most, one pair of points L_6, L_7, because Equation (17) has only one root.

4. The Stability of the Equilibrium Points

Generally, the stability of the equilibrium points is determined by investigation of the motion resulting from a small displacement of the infinitesimal body from its equilibrium position $(X_0, 0, Z_0)$.

Here we find very useful to emphasize the fact that at an equilibrium point the resultant force of the all effective forces is equal to zero and the infinitesimal body becomes *motionless*. That is why, it is of great importance for us to know what kind of forces could determine a small displacement of the infinitesimal body from the corresponding equilibrium position.

As it was before mentioned, in the photogravitational restricted three-body problem, we have taken into account only forces caused by gravitation and radiation-pressure of the two finite bodies S_1 and S_2 The equilibrium positions participate in the motion of the coordinate system, therefore the centrifugal forces cannot be used in order to cause a displacement of the infinitesimal body from the corresponding point.

The presence of another kind of forces will destroy the initial conditions of the corresponding problem.

Therefore, in order to obtain a small displacement, of the infinitesimal body from its equilibrium position, we have to give a push to the corresponding particle. But, in our binary system, such a push could be determined only by a change in the gravitational force or, what is more likely, such a push could be caused by some variations in the luminosities of the two stars.

Consequently, in our isolated binary system (S_1, S_2) the required push is possible only in the XZ-plane, where are confined the three points: S_1, S_2 and $(X_0, 0, Z_0)$.

Moreover, if the "initial" position and velocity vectors of the third body are in the XZ-plane, the corresponding motion will be confined to this plane, because there are no forces taking it out of this plane. That is why, Equations (1) and (3) will be linearized by using the substitutions:

$$X = X_0 + x\,\mathrm{e}^{\lambda t}, \qquad Z = Z_0 + z\,\mathrm{e}^{\lambda t},$$

where x, z and λ are constants. In these conditions, Equations (1) and (3) lead to

$$\begin{aligned} &(\lambda^2 - U_{XX})x - U_{XZ}z = 0, \\ &-U_{XZ}x + (\lambda^2 - U_{ZZ})z = 0, \end{aligned} \tag{19}$$

where we have

$$U_{XX} = 1 + 3\left[\frac{Q_1}{r_{01}^5}(X_0+\mu)^2 + \frac{Q_2}{r_{02}^5}(X_0+\mu-1)^2\right],$$

$$U_{XZ} = 3Z_0\left[\frac{Q_1}{r_{01}^5}(X_0+\mu) + \frac{Q_2}{r_{02}^5}(X_0+\mu-1)\right],$$

$$U_{ZZ} = 3Z_0^2\left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5}\right]$$

Now, from Equations (19), for the characteristic equation, we may write

$$\lambda^4 - \lambda^2 + 3Z_0^2\left[\frac{Q_1}{r_{01}^5} + \frac{Q_2}{r_{02}^5}\right] + 9Z_0^2\frac{Q_1 Q_2}{r_{01}^5 r_{02}^5} = 0. \tag{20}$$

Here we have to take into consideration Equations (15) and (13), so that

$$Q_1 = X_0 r_{01}^3, \qquad Q_2 = -X_0 r_{02}^3, \qquad r_{02}^2 - r_{01}^2 = 1 - 2X_0 - 2\mu$$

and Equation (20) becomes

$$\lambda^4 - \lambda^2 + C = 0 \tag{21}$$

with

$$C = 3\frac{X_0 Z_0^2}{r_{01}^2 r_{02}^2}(1 - 5X_0 - 2\mu).$$

Here, for $1 - \beta_1 < 0$ and $1 - \beta_2 > 0$ we have $X_0 < 0$ and it follows that $C < 0$. Therefore, Equation (21) has always, at least, one root real and positive. In such a case we may quote what Szebehely (1967, Chapter 5) has written:

"When some or all of the characteristic roots have positive real parts the equilibrium point is unstable. This is true also when some of the roots are multiple".

5. Concluding Remarks

(i) From the above presented theoretical considerations, it follows that, when the binary system (S_1, S_2) is isolated in space and the motion of the infinitesimal body is governed only by gravitational forces and effects of the radiation-pressure, the displacement from the equilibrium position $(X_0, 0, Z_0)$ may be caused only by those forces which are operative in the XZ-plane. This means that, after the corresponding disturbance, the infinitesimal body remains in the XZ-plane and, consequently, the corresponding "triangular points" $(X_0, 0, Z_0)$ are unstable.

So we have to repeat our old remark (Todoran, 1993): In the restricted three-body problem, there are no stable equilibrium points, at least as far as the effect of the radiation-pressure is concerned. Therefore, our old remark is not so wrong as it is appreciated by Drs. Ragos and Zagouras (1994).

(ii) The above obtained result is of a great importance for the study of double star evolution because, in the above mentioned conditions, the particles, ejected from the two finite bodies, cannot be "trapped" in the neighbourhood of the "new" triangular points, as long as these points are unstable.

(iii) From technical point of view, the above obtained result is important because it put in evidence the fact that the XZ-plane is not suited for a space colonization.

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