PERIODIC ORBITS OF GALACTIC MOTION

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Abstract. Poincaré's surface of section method is used to find and classify the main periodic orbits in a two-dimensional galactic potential first introduced by Hénon and Heiles. The stability of these periodic orbits is studied. Numerical integration with Bulirsch–Stoer method is used.

1. Introduction

In the last two decades increasing attention has been given to the study of different models of galactic motion and to the motion of the stars in these galaxies. Among the types studied are the spiral, barred, tri-axial, and elliptical galaxies.

One of the main questions still to be answered is the following: Is the motion of stars in a galaxy chaotic or ordered? A theoretical answer to this quesiton remains unknown until now.

Hénon and Heiles (1964) and Contopoulos (1960) used a numerical approach to gain insight into this matter. Although their potentials were fairly simple without Coriolis forces nor singularities, it turned out that these potentials are very useful in theoretical investigations. The motion in any of these two systems represents the motion of a star in the rotating meridian plane of a galaxy in the neighbourhood of a circular orbit or in the equatorial plane of a galaxy with axial symmetry.

Davoust (1983) used Lindstedt's method to make an inventory of the family of periodic orbits for Contopoulos's problem.

The model of Hénon and Heiles was studied by many authors. Among these are Hénon (1969), Magnenat (1979), and Contopoulos (1970). Their studies concentrated on the question of ergodic zones of motion. Contopoulos also studied the stability of the periodic orbits using Liapunov characteristic exponent method.

Up till now no one tried to classify the main periodic orbits in Hénon and Heiles model.

It is the aim of the present work to study this model in order to find and classify the main periodic orbits using Poincaré's surface of section method. The stability of these periodic orbits is also studied.

The bifurcation from these main orbits will be addressed by the authors in a future article.

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2. Description of the Method

Hénon and Heiles introduced the axisymmetrical potential

$$U(x, y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + x^2y - \tfrac{1}{3}y^3 . \tag{1}$$

The euqations of motion become

$$\ddot{x} = -\frac{\partial U}{\partial x} = -x - 2xy , \qquad \ddot{y} = -\frac{\partial U}{\partial y} = -y - x^2 + y^2 , \tag{2}$$

where $(x, y, \dot{x}, \dot{y})$ are the coordinates in phase space.

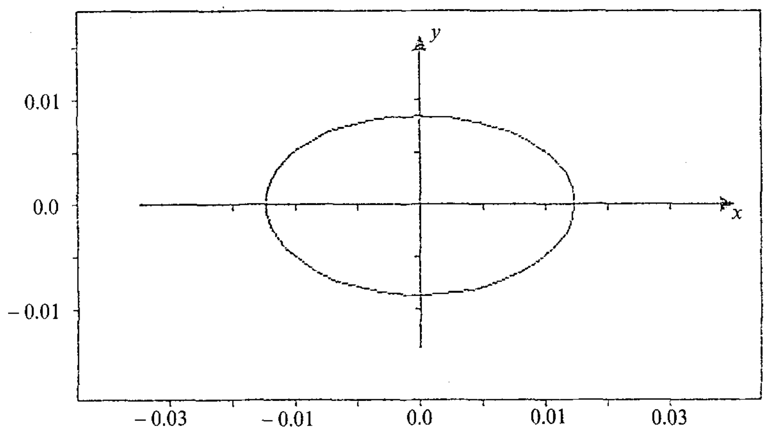

Fig. 1. Class A. $E = 0.01$, $y_0 = 0.01$, No. revs. 1.

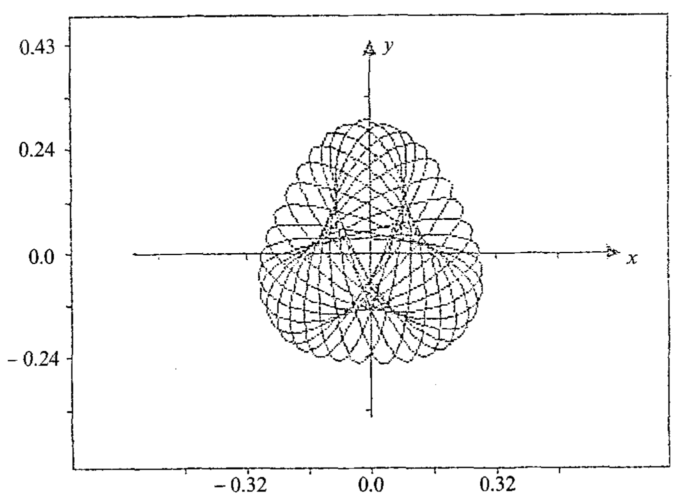

Fig. 2. Class C. $E = 0.08$, $y_0 = 0.3$, No. revs. 17.

The integral of energy for this system has the form

$$\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + x^2y - \tfrac{1}{3}y^3 + \tfrac{1}{2}(\dot{x}^2 + \dot{y}u^2) = E\,, \tag{3}$$

where E is the constant total energy.

The method of fixed point introduced by Poincaré will be used to find the periodic orbits for this system. In this method we do not consider the whole trajectory in phase space but only its consecutive crossings of a definite surface; in particular, with a plane.

The fixed point method has been applied successfully to the dynamics of rigid body in Kovalevskaya's case (cf. El-Sabaa, 1982). The solution of Equations (2) represent the trajectory in phase-space $(x, y, \dot{x}, \dot{y})$. Along this trajectory the value of the constant E

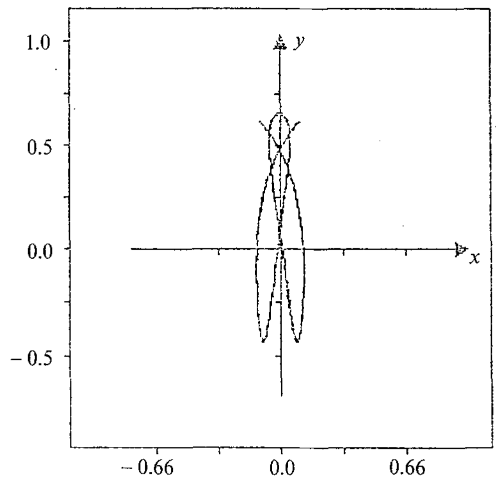

Fig. 3. Class D. $E = 0.07$, $y_0 = 0.65$, No. revs. 5.

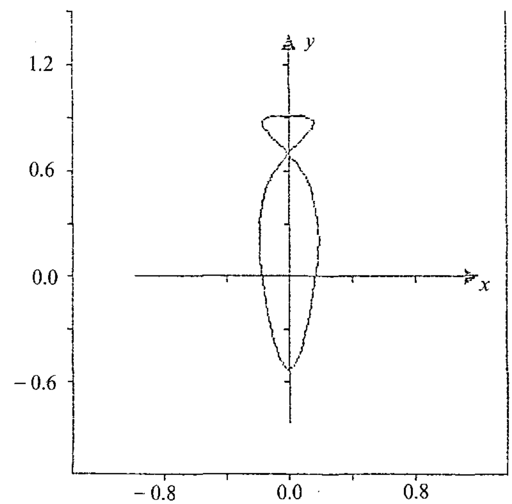

Fig. 4. Class E. $E = 0.05$, $y_0 = 0.91$, No. revs. 2.

is fixed. Thus for a given value of E the trajectory of the problem will be truncated in the three-dimensional space $(x, y, \dot{y})$. Let us examine consecutive crossings of this trajectory with the $y - \dot{y}$ plane in the positive direction, i.e., points of the trajectory which satisfy the conditions

$$x = 0, \qquad \dot{x} > 0.$$

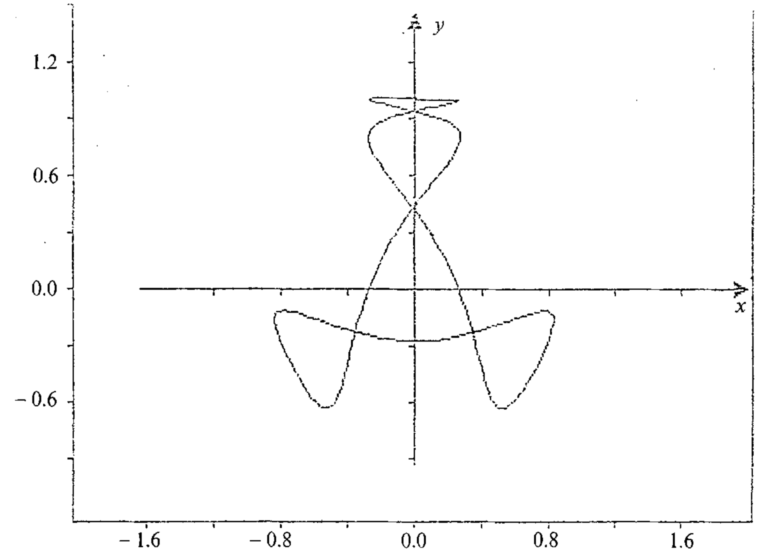

Fig. 5. Class F. $E = 0.06072$, $y_0 = 1$, No. revs. 3.

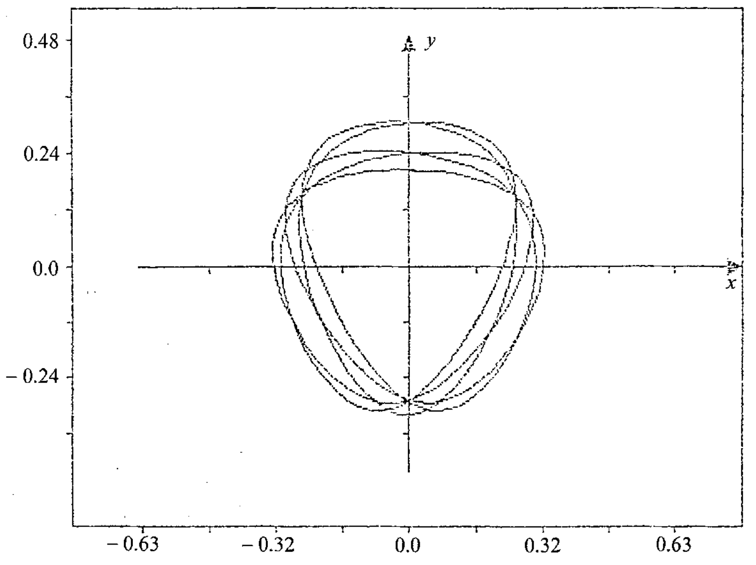

Fig. 6. Class G. $E = 0.2$, $y_0 = 0.2$, No. revs. 5.

The investigation of this trajectory is reduced to the study of the manifold of such intersection points in the plane $(y, \dot{y})$. If we define a mapping M^p such that

$$M^p: \quad (y_0, \dot{y}_0) \to (y, \dot{y})$$

that takes a point in the plane $(y, \dot{y})$ to another point in the same plane, then the fixed point of M^p for which $y = y_0 = 0$ are the symmetric periodic orbits of order p. So we

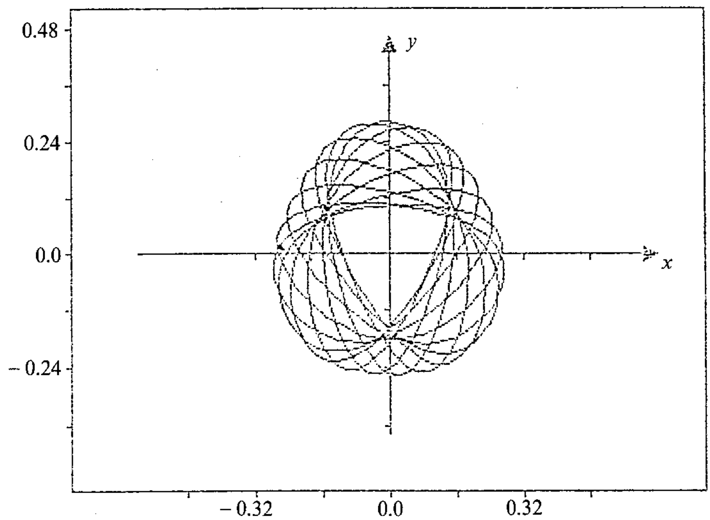

Fig. 7a. Class H. $E = 0.11$, $y_0 = 0.28$, No. revs. 11.

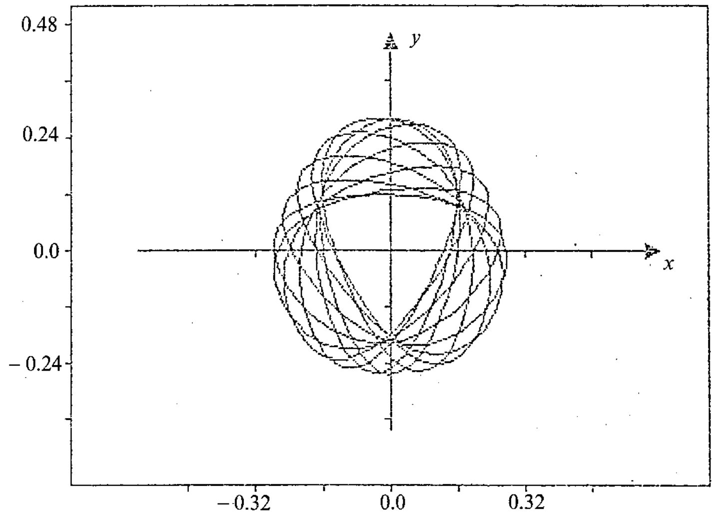

Fig. 7b. Class H. $E = 0.12$, $y_0 = 0.28$, No. revs. 10.

have the initial conditions

$$x_0 = 0\,, \qquad \dot{x}_0 = 0\,, \qquad y = y_0\,, \qquad \dot{y}_0 > [\tfrac{1}{2}(E - U(0, y_0))]^{1/2}\,.$$

Starting from the above initial conditions the trajectories were computed numerically. Two different numerical techniques of integration were used. First the Runge–Kutta fourth-order method and secondly the Bulirsch–Stoer method. Both methods yielded the same results through the second method was considerably faster.

A tolerance TOL = 10^{-3} was chosen. The trajectory was displayed on a graphic

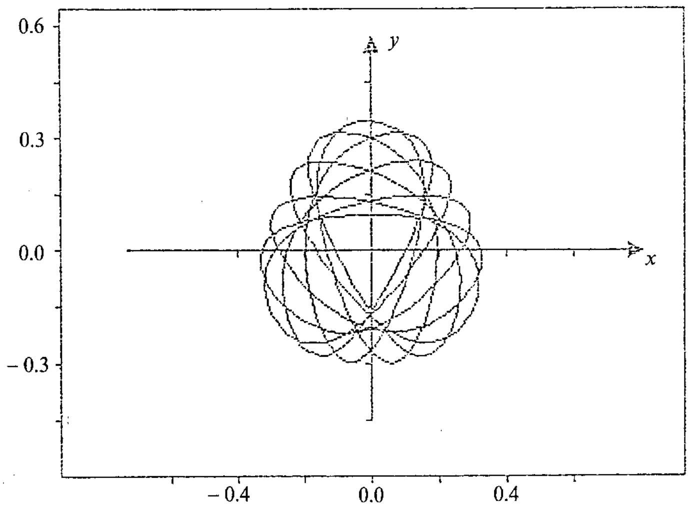

Fig. 7c. Class *H*. $E = 0.13$, $y_0 = 0.34$, No. revs. 8.

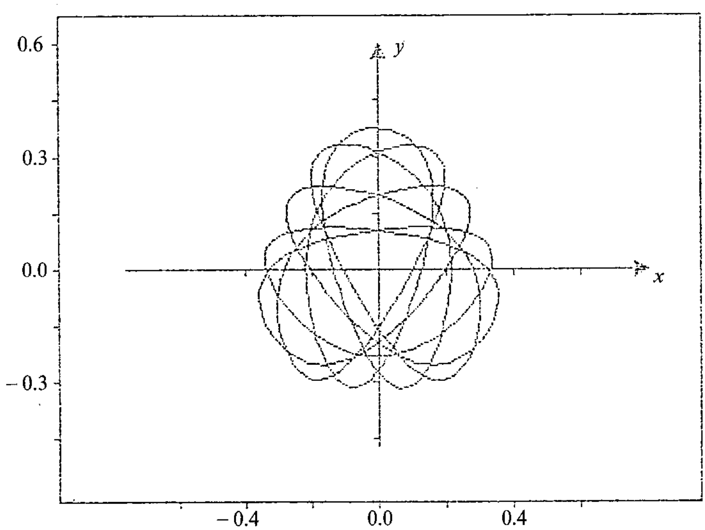

Fig. 7d. Class *H*. $E = 0.14$, $y_0 = 0.37$, No. revs. 7.

terminal using a Mega 2 ST computer. The orbit was considered as periodic with p revolutions if after p crossings of the y-axis moving in the positive x-direction the following two conditions were satisfied

(1) $|(y - y_0)/y_0| < \mathrm{TOL}$,

(2) $|\dot{y}/(\dot{x}^2 + \dot{y}^2)^{1/2}| < \mathrm{TOL}$.

The particle returns so to peak to its original position moving in its original direction (parallel to x-axis) within the specified tolerance.

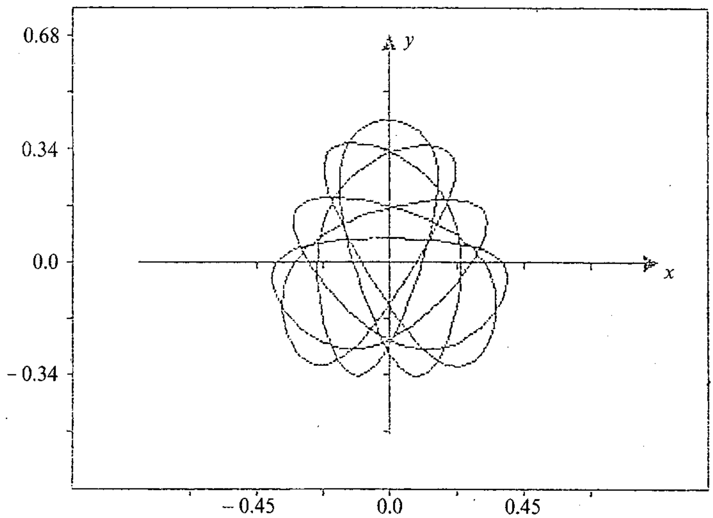

Fig. 7e. Class H. $E = 0.15$, $y_0 = 0.425$, No. revs. 6.

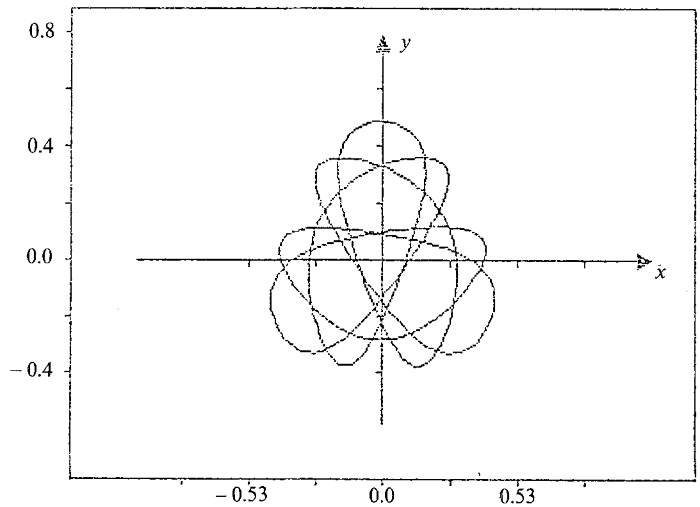

Fig. 7f. Class H. $E = 0.16$, $y_0 = 0.48$, No. revs. 5.

Nine classes of periodic orbits were classified accoridng to the configuration of the trajectory. They are labeled *A–I*.

Class *A* is a clas of ellipse-like simple periodic orbits shown in Figure 1. Orbits in class *B* start out as simple elliptic orbit but after one revolution the ellipse does not close properly but goes on to describe a nearby shifted ellipse and so on the particle returns to its original position after completing between 30 and 70 ellipses whose major axes make an increasing angle with the x-axis. Class *C*, Figure 2, resembles class *B* except that the outer boundary of the trajectory is not circular but more like a triangle with

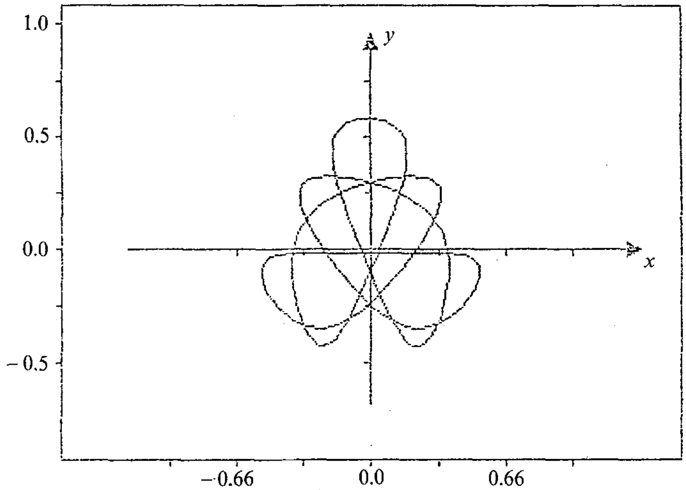

Fig. 7g. Class *H*. $E = 0.17$, $y_0 = 0.578$, No. revs. 4.

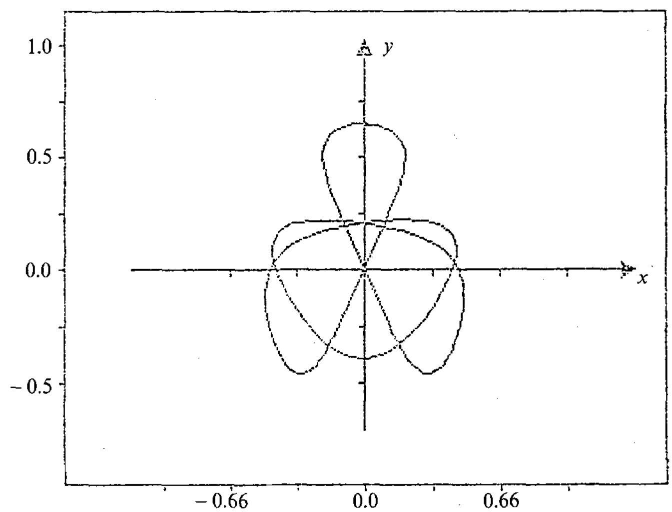

Fig. 7h. Class *H*. $E = 0.18$, $y_0 = 0.64$, No. revs. 3.

rounded edges. It should be noted also that there is a triangular region in the middle whose area depends on y_0.

Class D, Figure 3, is completely different. The particle starts as usual moving to the right then goes down and crosses the x and y axes make a curve upward and crosses the y-axes then stops in its tracks reverses its motion exactly until it reaches its original position in the opposite direction and then goes on to describe the other half of the trajectory. Classes E and F are shown in Figures 4 and 5, respectively, while class G is shown in Figure 6.

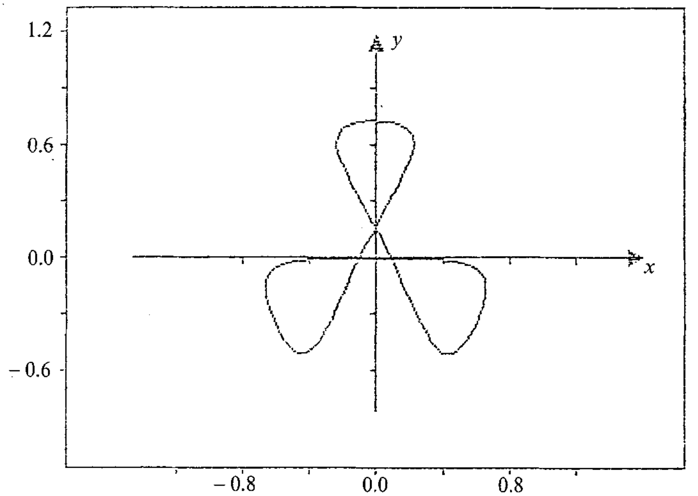

Fig. 7i. Class H. $E = 0.19$, $y_0 = 0.72$, No. revs. 2.

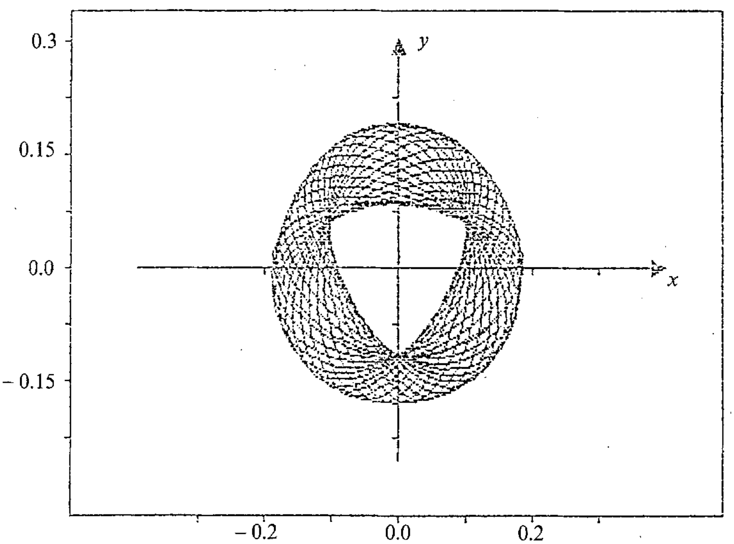

Fig. 8. Class I. $E = 0.01$, $y_0 = 0.081$, No. revs. 23.

Class H is shown in Figures 7(a)–7(i). It is worth noting that as E increases the number of revolutions decreases from 11 revolutions when $E = 0.11$ (Figure 7(a)) to 2 revolution when $E = 0.19$ (Figure 7(i)). The characteristic curve for this class is shown in Figure 9. Class J is shown in Figure 8.

The value of the energy E was taken as 0.01 at start and increased in steps of 0.01.

For $E = 0.01$ the admissible values of y_0 gave trajectories in classes A and B only. As E increases the other orbits started to occur in alphabetical order. The classes G–I occur only when $E > 0.1$, when $E > 0.2$ the trajectories are no longer periodic though Figure 6 shows a periodic orbit for $E = 0.2$ which is a value slightly larger than the limiting value found by Hénon and Heiles (1964).

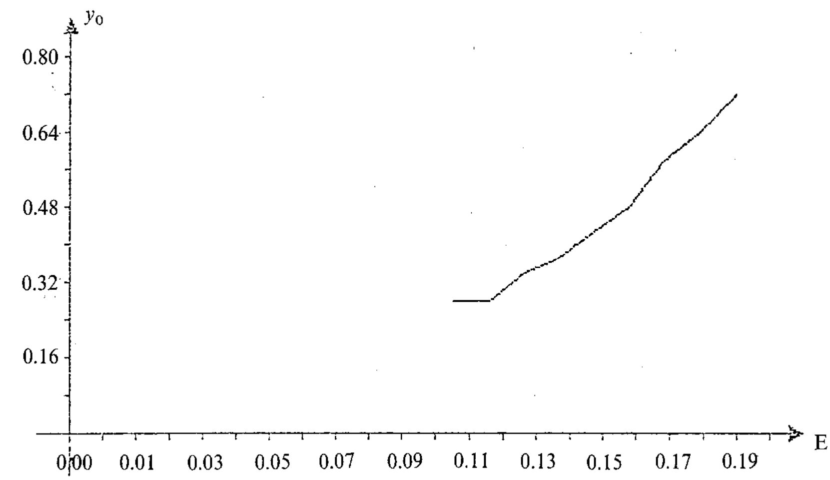

Fig. 9. Characteristic curve for class H.

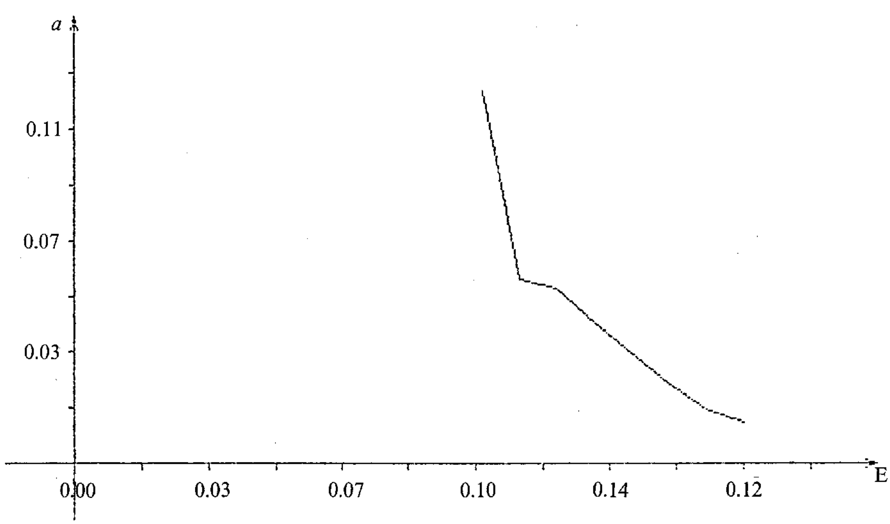

Fig. 10. Stability curve for class H.

The stability of periodic orbits may be studied by considering orbits near the periodic orbits with the same value of the energy constant. Define a transformation $G(t)$ by

$$y = G(t)y_0 ,$$

which transforms the point y_0 in phase space occupied by the particle at time $t_0 = 0$ into the point y occupied by the particle at time t. Birkhoff (1927) proved that this mapping preserves the area.

It can be shown (cf. Szebehely, 1967) that the stability condition for symmetric periodic orbits with P-revolutions is

$$a = \left|\frac{\Delta y_1}{\Delta y_0}\right| < 1 ,$$

where

$$\Delta y_0 = y_1 - y_0 , \qquad \Delta y_1 = y_2 - y_1 ;$$

y_0 is the initial value corresponding to a periodic orbit, y_1 is a value close to y_0 and y_2 is the value of y after p revolutions starting from the initial value y_1.

It was found that orbits in class A are highly stable those of classes E and F highly unstable.

A typical curve showing the relation between the total energy E and the quantity a for class H is shown in Figure 10.

3. Conclusions

Poincaré's fixed-point method proved very useful in detecting periodic orbits for the Hénon and Heiles model.

This method was chosen because of its success in predicting about 15 new periodic orbits for the restricted three-body problem.

Also this method itself is used in establishing the stability of periodic orbits without recourse to any other method such as Liapunov characteristic method which saves computational time.

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