THERMODYNAMICS OF A MODEL OF NONADIABATIC SPHERICAL GRAVITATIONAL COLLAPSE

TH. GRAMMENOS
Nuclear and Particle Physics Division, Physics Department,
University of Athens Panepistemiopolis-Ilissia, Athens, Greece

(Received 13 September, 1993)

Abstract. In this work, a thermodynamic treatment of a Friedmann-like model of nonadiabatic spherical gravitational collapse is presented. The calculations have been performed according to Eckart's theory of dissipative relativistic fluids, while the diffusion approximation has been adopted for the radiation transport. The conclusions deduced are in agreement with the predictions of the theory of late stellar evolution.

1. Introduction

The study of gravitational collapse of a body representing a realistic astronomical matter distribution is of great importance for astrophysics. Hence, the introduction of physically tenable conditions (concerning mostly several forms of dissipation) into idealized problems of collapse has been attempted by several authors in the past (Misner and Sharp, 1965). In a recent paper (Kolassis *et al.*, 1988a), Kolassis, Santos and Tsoubelis (hereinafter KST) studied the collapse of a spherical body consisting of a shear-free fluid in the presence of radial heat dissipation. Chan *et al.* (1989), and Grammenos and Kolassis (1992) presented some further results concerning the dynamic evolution of the above model without any reference to its thermodynamics.

A completely proper thermodynamical investigation of a collapsing dissipative fluid remains still an open subject, since various simple approaches to the relativistic thermodynamics of dissipative fluids show some pathological behavior in several aspects. Therefore their application is limited in a way. Beyond that, the detailed behavior of a dissipative relativistic fluid depends strongly on the thermodynamic properties of the fluid which are, in general, inadequately known for realistic matter.

The most comprehensive and attractive alternative of a series of new (second order) theories is the Israel–Stewart class of theories (Israel, 1976), but despite the superiority of these theories, many calculations are still performed in the Eckart (Eckart, 1940), or Landau–Lifshitz (Landau and Lifshitz, 1979) frame. We believe that the unstable and noncausal behavior associated with these (first order) theories occurs in a nonphysical domain (Hiscock and Salmonson, 1991) and if, in addition, one assumes small deviations from thermal equilibrium, the application of anyone of the first order theories is justified. Moreover, the results obtained in this way are not in contradiction with the predictions of the theory of late stellar evolution (Zel'dovich *et al.*, 1971).

Astrophysics and Space Science **211:** 31–40, 1994.

In Section 2 of this paper we give a rather brief review of the KST model, presenting only the main equations for the sake of completeness. Finally, in Section 3 we give a thermodynamical treatment of the collapsing fluid in this model, following Eckart's theory of dissipative relativistic fluids and assuming the diffusion approximation scheme for the radiation transport.

2. The Friedmann-Like Model

The spacetime exterior to the sphere is described by the outgoing Vaidya metric (Vaidya, 1953) due to the heat flow:

$$\mathrm{d}s_+^2 = -\left(1 - \frac{2m(v)}{\mathfrak{r}}\right)\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}\mathfrak{r} + \mathfrak{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{2.1}$$

with $x^i = (v, \mathfrak{r}, \theta, \varphi)$ and $m(v)$, representing the (Newtonian) mass of the sphere as measured by an observer at infinity, is a function of the retarded time coordinate v.

The energy-momentum tensor $T_{\alpha\beta}$, related to (2.1) via Einstein's equations, is that of pure radiation. In geometrical units ($c = G = 1$) we have

$$T^+_{\alpha\beta} = -\frac{1}{4\pi\mathfrak{r}^2}\frac{\mathrm{d}m}{\mathrm{d}v}\delta^0_\alpha\delta^0_\beta. \tag{2.2}$$

The quantity $L = -\mathrm{d}m/\mathrm{d}v$ represents the total luminosity perceived by an observer at rest at spatial infinity (Lindquist *et al.*, 1965) and must therefore be positive. This implies that $m(v)$ is a nonincreasing function of v, which in turn denotes that the sphere is losing mass due to the outgoing radiation.

The shear free motion of the fluid in the sphere's interior allows the use of isotropic and comoving coordinates (t, r, θ, φ) in which the line element of the interior reads

$$\mathrm{d}s_-^2 = -A^2\,\mathrm{d}t^2 + B^2[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)], \tag{2.3}$$

where A, B are assumed to be positive functions of r, t.

The corresponding energy-momentum tensor is

$$T^-_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + p g_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha, \tag{2.4}$$

where μ is the energy density in the rest frame of the fluid, p is the isotropic pressure, $u^\alpha = (1/A)\delta^\alpha_0$ is the fluid's four-velocity, and $q^\alpha = q\delta^\alpha_1$ the energy flux vector which can be interpreted as heat flux. The heat flux vector q^α must be radial due to the spherical symmetry (Kolassis *et al.*, 1988b).

The Einstein field equations for the metric (2.4) are

$$8\pi\mu = -\frac{1}{B^2}\left[\frac{2B''}{B} - \left(\frac{B'}{B}\right)^2 + \frac{4B'}{rB}\right] + 3\left(\frac{\dot{B}}{AB}\right)^2, \tag{2.5}$$

$$8\pi p = \frac{1}{B^2}\left[\left(\frac{B'}{B}\right)^2 + \frac{2B'}{rB} + \frac{2A'B'}{AB} + \frac{2A'}{rA}\right] + $$
$$+ \frac{1}{A^2}\left[-2\frac{\ddot{B}}{B} - \left(\frac{\dot{B}}{B}\right)^2 + \frac{2\dot{A}\dot{B}}{AB}\right], \tag{2.6}$$

$$8\pi qB = -\frac{2}{AB}\left[\frac{A'\dot{B}}{AB} - \left(\frac{\dot{B}}{B}\right)'\right] \tag{2.7}$$

plus an equation due to the isotropy of the pressure

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{2B'}{B} + \frac{1}{r}\right)\left(\frac{A'}{A} + \frac{B'}{B}\right). \tag{2.8}$$

The dot and prime mean differentiation with respect to t and r, respectively.

The metrics (2.1) and (2.3) are matched on a spherical hypersurface Σ defined by

$$r_\Sigma = b = \text{constant}, \tag{2.9}$$

while the junction conditions yield

$$\mathfrak{r}_\Sigma = (rB)_\Sigma, \tag{2.10}$$
$$p_\Sigma = (qB)_\Sigma, \tag{2.11}$$
$$[r(rB)']_\Sigma = \left[\frac{\mathfrak{r}\dot{v}}{A}\left(1 - \frac{2m}{\mathfrak{r}}\right) + \frac{\mathfrak{r}\dot{\mathfrak{r}}}{A}\right]_\Sigma, \tag{2.12}$$
$$m(v) = \left[\frac{r^3B\dot{B}^2}{2A^2} - r^2B' - \frac{r^3B'^2}{2B}\right]_\Sigma. \tag{2.13}$$

With the assumption that the fluid trajectories are geodesics we have

$$A = 1, \tag{2.14}$$

while a particular solution for B, satisfying the field equations is given by

$$B = \frac{M}{2b}\left(\frac{1 - b^2\lambda}{1 - r^2\lambda}\right)u^2, \tag{2.15}$$

$$u \equiv \left(\frac{6t}{M}\right)^{1/3}, \tag{2.16a}$$
$$\lambda \equiv ae^u, \tag{2.16b}$$

with $a \geq 0$, M constants.

If the heat flux vanishes, i.e. $a = 0$, one obtains dust in a spatially flat ($k = 0$) Friedmann–Robertson–Walker interior with a Schwarzschild exterior:

$$B = \frac{3}{b}\left(\frac{M}{6}\right)^{1/3} t^{2/3}, \tag{2.17a}$$

$$8\pi\mu = \frac{4}{3}t^{-2}, \tag{2.17b}$$

$$p = 0, \tag{2.17c}$$

$$q = 0, \tag{2.17d}$$

$$m(v) = M. \tag{2.17e}$$

The total luminosity at infinity is given by

$$L = -\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{2b^2\lambda}{(1-b^2\lambda)^2}\left(\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda}\right)^2 . \tag{2.18}$$

As a consequence of the junction conditions one obtains

$$\dot{v} = \left(\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda}\right)^{-1}, \tag{2.19}$$

which implies that the instant of the (apparent) horizon formation $u_H = f(ab^2)$ is the root of the equation

$$\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda} = 0. \tag{2.20}$$

It has been shown in Grammenos and Kolassis (1992) that all energy conditions are satisfied during the whole period of collapse until the horizon formation, i.e. the model is physically acceptable, for the range

$$0 \le ab^2 < 0.85. \tag{2.21}$$

3. Thermodynamical Considerations

The collapsing fluid has to satisfy the following relations (Treciokas and Ellis, 1971):

(a) The equation of conservation of matter (baryon conservation),

$$(\rho u^{\mu})_{;\mu} = 0, \tag{3.1}$$

where ρ is the baryon number density as measured in the rest frame of u^{μ}.

(b) The temperature $T \geq 0$ and the specific entropy (entropy per baryon) s are assumed to obey the Gibbs relation,

$$T\,\mathrm{d}s = \mathrm{d}\Pi + p d\frac{1}{\rho}, \tag{3.2}$$

where the internal energy density Π is defined by

$$\mu = \rho(1+\Pi), \tag{3.3}$$

μ being the energy density of the fluid.

At this point one must notice that the Gibbs relation holds certainly near thermal equilibrium , i.e. at the early stages of the collapse. For large deviations from thermal equilibrium which is the case at the late stages of the collapse, one must be extremely careful with the use of relation (3.2).

(c) The second law of Thermodynamics,

$$s^{\mu}_{;\mu} \geq 0 \tag{3.4}$$

according to which we have a positive entropy flux production. The entropy flux vector s^{μ} is defined by

$$s^{\mu} = \rho s u^{\mu} + \frac{q^{\mu}}{T}. \tag{3.5}$$

The second law together with the conservation law and the axiom of local equilibrium (Gariel, 1986), which states that the specific entropy is a function of the energy density μ and the particle density n only, lead to a linear law for the heat flux q, the relativistic Fourier law (Eckart, 1940):

$$q^{\mu} = -\kappa h^{\mu\nu}(T_{,\nu} + T u_{\nu;\alpha} u^{\alpha}), \tag{3.6}$$

where κ is the thermal conductivity which has to satisfy

$$\kappa \geq 0 \tag{3.7}$$

and

$$h^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu} \tag{3.8}$$

is the projection tensor.

The linearity in the gradients of temperature and velocity is thus justified only in the approximation of small deviations from thermal equilibrium. As we mentioned in the introduction, more complicated (non-linear in the deviation from equilibrium) theories of dissipative fluids have been proposed (Israel, 1976; Carter, 1989), but we shall assume that Equations (3.1)–(3.6) are sufficient for our purpose.

It follows directly that inequality (3.4) is ensured by Equations (3.6) and (3.7), so that it will not be considered further.

Considering at first the metric coefficients of (2.3) A, B as unspecified, and since we have heat flux only in the radial direction, the only non-vanishing component of q^{μ} is the $\mu = 1$ component given by

$$q = -\frac{\kappa}{AB^2}(AT)'. \tag{3.9}$$

Assuming now that we have radiative transfer in the diffusion approximation (Frank-Kamenetskii, 1962; Kippenhahn and Weigert, 1991), we can set

$$\kappa = \gamma T^n, \tag{3.10}$$

where $\gamma > 0$, $n > 0$ are constants. Substituting Equation (2.7) for q into Equation (3.9), we obtain the following differential equation for the temperature:

$$T^n T' + \frac{A'}{A}T^{n+1} - \frac{1}{4\pi\gamma A}\left[\frac{A'\dot{B}}{AB} - \left(\frac{\dot{B}}{B}\right)'\right] = 0. \tag{3.11}$$

Now, returning to the particular solution (2.14)–(2.15), we get from Equation (3.9)

$$q = -\kappa T' \frac{4b^2}{M^2u^4}\left(\frac{1 - r^2\lambda}{1 - b^2\lambda}\right)^2. \tag{3.12}$$

Substituting (3.10) into (3.12) and integrating yields

$$T^{n+1} = -\frac{n+1}{2\pi\gamma M}\left(\frac{\lambda}{u^2(1 - r^2\lambda)}\right) + F(t), \tag{3.13}$$

with $F(t)$ an arbitrary function of time, to be determined later on. We must point out that for q obtained by (2.7) via (2.14)–(2.15), we have

$$\partial_r q > 0, \qquad \partial_u q > 0, \qquad \partial_{(ab^2)} q > 0, \tag{3.14}$$

so q is a monotonically increasing function of r, u and ab^2.

We choose $n = 3$ which represents the case of radiation interacting with matter (Frank-Kamenetskii, 1962), while γ reads (Kippenhahn and Weigert, 1991)

$$\gamma = \frac{4\epsilon c}{3K\rho}, \tag{3.15}$$

where ϵ is the radiation density constant for photons and K is the Rosseland mean opacity. Thus, we have from Equation (3.10)

$$\kappa = \frac{4\epsilon c}{3K\rho}T^3, \tag{3.16}$$

while Equation (3.13) becomes

$$T^4 = -\frac{2}{\pi\gamma M}\left(\frac{\lambda}{u^2(1 - r^2\lambda)}\right) + F(t). \tag{3.17}$$

Now, the effective surface temperature of a star is given by (Misner, 1969):

$$T_\Sigma^4 = \left(\frac{1}{4\pi\delta r_\Sigma^2}\right) L, \tag{3.18}$$

with δ a positive constant (Frank-Kamenetskii, 1962) and L given by Equation (2.18). With the help of Equations (2.10), (3.17), and (3.18) we determine the arbitrary function $F(t)$ to be

$$F(t) = \frac{2}{\pi\gamma M}\left(\frac{\lambda}{u^2(1-b^2\lambda)}\right) + \frac{2b^2\lambda}{\pi\delta M^2(1-b^2\lambda)^2u^4}\left[\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda}\right]^2 . \tag{3.19}$$

Finally, we have from Equation (3.17) utilizing (3.19)

$$T^4 = \frac{2}{\pi\gamma M}\frac{\lambda^2(b^2-r^2)}{u^2(1-b^2\lambda)(1-r^2\lambda)} + \frac{2b^2\lambda}{\pi\delta M^2(1-b^2\lambda)^2u^4}\left[\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda}\right]^2 , \tag{3.20}$$

while on the surface Σ,

$$T_\Sigma^4 = \frac{2b^2\lambda}{\pi\delta M^2(1-b^2\lambda)^2u^4}\left[\frac{2}{u} + \frac{1+b^2\lambda}{1-b^2\lambda}\right]^2 , \tag{3.21}$$

Differentiation of (3.20) and (3.21) with respect to ab^2 shows that T as well as T_Σ are monotonically increasing with ab^2. Furthermore, $T' < 0$, so the temperature decreases outwards. Considering now the early stages of the collapse, i.e. $u \to -\infty$, it follows that $T \to 0$, as well as $T_\Sigma \to 0$. The surface temperature $T_\Sigma \to 0$ also at the stage of the horizon formation $u \to u_H$, but not the general temperature:

$$T^4(u_H) = \frac{2}{\pi\gamma M u_H^2}\left[\frac{\lambda^2(b^2-r^2)}{(1-r^2\lambda)(1-b^2\lambda)}\right]_H , \tag{3.22}$$

where we have used (2.20)

The baryon conservation law, Equation (3.1), can be integrated for arbitrary $A(r,t)$, $B(r,t)$ giving

$$\rho(r,t) = \frac{\rho_0(r)}{B^3}. \tag{3.23}$$

For the particular solution (2.15), the baryon density reads

$$\rho = \rho_0(r)\left(\frac{1-r^2\lambda}{1-b^2\lambda}\right)^3 \frac{1}{u^6}, \tag{3.24}$$

where $\rho_0(r) > 0$.

From the differentiation of (3.24) with respect to ab^2 we conclude that ρ is also an increasinq function of ab^2. On the other hand, we find

$$\rho' = \frac{\rho_0'(r)}{u^6}\left(\frac{1-r^2\lambda}{1-b^2\lambda}\right)^3 - \frac{6\rho_0(r)}{u^6}\frac{(1-r^2\lambda)^2}{(1-b^2\lambda)^3}r\lambda. \tag{3.25a}$$

Assuming a density distribution of the form (Majumdar and Tomozawa, 1992)

$$\rho_0(r) = Cr^{-w}, \tag{3.25b}$$

where w and C positive constants, it follows that $\rho_0' < 0$, so ρ_0 decreases outwards. Substituting now the above Ansatz for ρ_0 into (3.25a) we find

$$\rho' < 0 \tag{3.26}$$

so that ρ decreases outwards. We also find that

$$\dot{\rho} > 0 \tag{3.27}$$

so that ρ increases monotonically with the time u. Initially $(u \to -\infty)$ ρ vanishes, while at the instant of the horizon formation u_H, ρ becomes

$$\rho(u_H) = \rho_0(r)\left(\frac{1-r^2\lambda}{1-b^2\lambda}\right)^3_H \frac{1}{u_H^6}. \tag{3.28}$$

The total baryon number N, enclosed by a sphere of arbitrary radius r_Σ is given by (Misner, 1969)

$$N = \int_0^{r_\Sigma} \rho u^0 (-g)^{1/2}\, \mathrm{d}^3x. \tag{3.29}$$

Substituting ρ from (3.23), $-g$ from (2.3) and insertinq $u^0 = 1/A$, we find

$$\frac{\partial N}{\partial t} = 0, \tag{3.30}$$

i.e., N is time-independent for any $A(r,t)$ and $B(r,t)$. Furthermore, N (i.e. $\rho_0(r)$) is fixed by the initial conditions. Substituting now the Ansatz (3.25b) for ρ_0 into (3.29) we get for w

$$0 < w < 3, \tag{3.31}$$

while the constant C reads

$$C = N\frac{2b^3}{M^3\pi}\left(\frac{3-w}{r_\Sigma^{3-w}}\right). \tag{3.32}$$

On the surface $r_\Sigma = b$, we have

$$C = \frac{N}{M^3\pi} 2b^w(3 - w). \tag{3.33}$$

From the Gibbs relation, Equation (3.2), one obtains the following set of differential equations for the specific entropy:

$$\rho T s' = [\mu' - (\mu + p)\frac{\rho'}{\rho}], \tag{3.34a}$$

$$\rho T \dot{s} = [\dot{\mu} - (\mu + p)\frac{\dot{\rho}}{\rho}]. \tag{3.34b}$$

Substituting into (3.34a) the corresponding equations for μ', p and ρ' which one obtains from (2.5), (2.6), and (3.23) with the particular solution (2.14)–(2.15), yields

$$\begin{aligned}\rho T s' = {} & \frac{6r\lambda}{\pi M^2 u^4 (1 - r^2\lambda)^2}\left[\frac{2}{u} - \frac{\lambda(b^2 - r^2)}{(1 - b^2\lambda)(1 - r^2\lambda)}\right] + \\ & + \frac{1}{\pi M^2 u^4}\left(\frac{\rho_0'}{\rho_0} - \frac{6r\lambda}{1 - r^2\lambda}\right) \times \\ & \times \left[-\frac{3}{2}\left(\frac{2}{u} - \frac{\lambda(b^2 - r^2)}{(1 - b^2\lambda)(1 - r^2\lambda)}\right)^2 + \frac{4b^2\lambda}{(1 - b^2\lambda)^2} - \right. \\ & \left. - \frac{\lambda(b^2 - \lambda^2)}{2(1 - b^2\lambda)(1 - r^2\lambda)}\left(\frac{8}{u} + \frac{5}{1 - r^2\lambda} - \frac{1}{1 - b^2\lambda} - 2\right)\right].\end{aligned} \tag{3.35}$$

The algebraic investigation of (3.35) has shown that

$$s' > 0, \tag{3.36}$$

i.e. the specific entropy increases outwards during the whole period of collapse until the horizon formation, in accordance with the theory of stellar evolution (Zel'dovich *et al.*, 1971).

Due to the complexity of (3.34b), it has not been possible to deduce a conclusion concerning $\dot{s}$ ($\dot{s} < 0$ could be expected).

Acknowledgements

The author would like to thank Dr. Th. Christodoulakis and Dr. Ch. Kolassis for many interesting discussions.

References

Carter, B.: 1989, in: N. Dadhich *et al.* (eds.), *A Random Walk in Relativity and Cosmology*, Wiley Eastern. Carter, B.: 1989, in: A.M. Anile (ed.), *Relativistic Fluid Dynamics*, Springer.

Chan, R. *et al.*: 1989, *Astrophys. J.* **342**, 976.
Eckart, C.: 1940, *Phys. Rev.* **58**, 919.
Frank-Kamenetskii, D.A.: 1962, *Physical Processes in Stellar Interiors*, Israel Program for Scient. Translations, Jerusalem, Chapt. 2 and 5).
Gariel, J.: 1986, *Nuovo Cim.* **94B**, 2, 119.
Grammenos, T. and Kolassis, C.: 1992, *Phys. Lett.* **A169**, 5.
Hiscock, W.A. and Salmonson, J.: 1991, *Phys. Rev.* **D43**, 3249.
Israel, W.: 1976, *Ann. Phys. (N.Y.)* **100**, 310. Israel, W. and Stewart, J.M.: 1979, *Ann. Phys. (N.Y.)* **118**, 341. Stewart, J.M.: 1977, *Proc. Roy. Soc. London* **357A**, 59.
Kippenhahn, R. and Weigert, A.: 1991, *Stellar Structure and Evolution*, Springer, Chap. 5.
Kolassis, C. *et al.*: 1988a, *Astrophys. J.* **327**, 755.
Kolassis, C. *et al.*: 1988b, *Class. Quant. Grav.* **5**, 1329.
Landau, L. and Lifshitz, E.M.: 1979, *Fluid Mechanics*, Pergamon Press, Chap. 15.
Lindquist, R.W. *et al.*: 1965, *Phys. Rev.* **137**, 1364.
Majumdar, A. and Tomozawa, Y.: 1992, *Nuov. Cim.* **B107**, 923.
Misner, C.W.: 1969, in: M. Chretien, S. Deser and J. Goldstein (eds.), *Astrophysics and Gen. Relativity* **I**, Gordon and Breach Science Publ.
Misner, C.W. and Sharp, D.H.: 1965, *Phys.Lett.* **15**, 279. Glass, E.N.: 1981, *Phys. Lett.* **A86**, 351. Glass, E.N.: 1990, *J. Math. Phys.* **31**, 1974. Herrera, L. *et al.*: (1987), *Phys. Rev.* **D36**, 2986. Lake, K.: 1982, *Phys. Rev.* **D26**, 518. Kramer, D.: (1992), *J. Math. Phys.* **33**, 1458. Oliveira, A.K.G. De and Santos, N.O.: 1987, *Astrophys. J.* **312**, 640. Oliveira, A.K.G. De *et al.*: 1985, *Monthly Not. Roy. Astr. Soc.* **216**, 1001. Oliveira, A.K.G. De *et al.*: 1988, *Monthly Not. Roy. Astr. Soc.* **231**, 1011. Kolassis, C. and Santos, N.O.: 1989, *Phys. Lett.* **A141**, 243.
Schwarzschild, M.: 1958, *Structure and Evolution of the Stars*, Dover, Chap. 1.
Treciokas, R. and Ellis, G.F.R.: 1971, *Comm. Math. Phys.* **23**, 1.
Vaidya, P.C.: 1953, *Nature* **171**, 260.
Zel'dovich, Yu.B. and Novikov, I.D.: 1971, *Stars and Relativity*, Univ. of Chicago Press, Chaps. 10 and 11.