A SPATIALLY-FLAT COSMOLOGICAL MODEL

MARK ISRAELIT
University of Haifa, Oranim, Department of Physics, Tivon, Israel
and
Universität Konstanz, Fakultät für Physik, Konstanz, Germany

and

NATHAN ROSEN
Department of Physics, Technion, Haifa, Israel

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Abstract. Recently a homogeneous cosmological model free from singularities was proposed, based on the general relativity theory. It described a closed universe ($k = +1$), initially filled with prematter, characterized by a density ρ equal to the Planck density and a pressure $P = -\rho$, and undergoing oscillations. In the present work the case of a similar, but spatially flat, universe ($k = 0$) is investigated. In this case there is an initial geometric singularity (the scale factor $R = 0$), but not a physical one, since the initial density is finite. This universe begins its existence at a time $t = -\infty$ and, after going through the prematter and radiation-dominated eras, reaches the matter-dominated state and continues to expand indefinitely.

1. Introduction

Recently a cosmological model was proposed (Israelit and Rosen, 1989) which had no singularities and which oscillated in time. It described a closed universe ($k = 1$, where k is the curvature parameter in the Robertson-Walker metric), characterized by the fact that at the beginning of the expansion phase it was filled with "prematter" for which

$$P = -\rho, \tag{1}$$

with P the pressure and ρ the density. With prematter initially present there are also models with $k = 0$ (spatially flat) and $k = -1$ (open). In the above work these models were rejected because in each of them there was a time at which the scale parameter R of the Robertson-Walker metric vanished, and this was regarded as a singularity. However, one can argue that, when $R = 0$, the density ρ in these models is finite, and not infinite as in the case of the "big bang" models and therefore one does not have a physical singularity. Based on this standpoint, the present work investigates the model with $k = 0$. The motivation for this work is that there are a number of theoretical arguments in support of the assumption $k = 0$ (Primack, Seckel and Sadoulet, 1988).

2. Outline of Model

Since the present work is similar to the earlier one (Israelit and Rosen 1989), to be referred to as (I), it will be less detailed. We consider here a spatially flat

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homogeneous and isotropic universe with a geometry determined by the Einstein gravitational equations and described by a Robertson-Walker line-element

$$ds^2 = dt^2 - R^2(t)(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2), \tag{2}$$

with coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and scale parameter $R(t)$. The Einstein equations lead to the relations

$$\dot{R}^2 = (8\pi/3)\rho R^2, \tag{3}$$

$$\ddot{R} = -4\pi R\left(\frac{1}{3}\rho + P\right), \tag{4}$$

where ρ and P are functions of t, and a dot denotes a t-derivative. There is also the energy-conservation relation, which assures the consistency of the field equations

$$\dot{\rho} + 3(\dot{R}/R)(\rho + P) = 0, \tag{5}$$

with ρ and P related by an Eq. of state.

It is assumed that initially the density is of the order of the Planck density

$$\rho_p = c^3/\hbar G = 3.83 \times 10^{65}\ \mathrm{cm}^{-2}, \tag{6}$$

and that at this density one has prematter with the Eq. of state (1). For the sake of definiteness it will be assumed that the initial density is *equal* to ρ_p. The prematter period is followed by a radiation-dominated period for which the equation of state is that of isotropic radiation,

$$P = \frac{1}{3}\rho; \tag{7}$$

and this is followed by the present, matter-dominated period for which one can take

$$P = 0. \tag{8}$$

3. Prematter-Radiation Periods

To describe the transition from prematter to radiation we will take the same equation of state as in (I),

$$P = \frac{1}{3}\rho - \frac{4}{3}\rho^2/\rho_p, \tag{9}$$

so that for $\rho = \rho_p$ we get Eq. (1) and for $\rho \ll \rho_p$ Eq. (7). It should be stressed that Eq. (9) is not a fundamental equation. It has been chosen because it is simple and provides a transition between the two equations of state, but there are many other possible choices. Putting Eq. (9) into Eq. (5), one finds

$$\rho = \frac{a^4 \rho_p}{a^4 + R^4}, \tag{10}$$

where a is an arbitrary constant. From Eq. (9) and (10) one gets

$$P = \frac{a^4 \rho_p (R^4/3 - a^4)}{(a^4 + R^4)^2} \tag{11}$$

Let us take $a = 1 \times 10^{-3}$ cm as in (I). Then, for $R \ll a$, $\rho = -P = \rho_p$, corresponding to prematter. In this case Eq. (3) takes the form

$$\dot{R}^2 = \frac{R^2}{R_I^2}, \tag{12}$$

where

$$R_I = (3/8\pi\rho_p)^{1/2} = 5.58 \times 10^{-34} \text{ cm}, \tag{13}$$

so that the solution corresponding to an expanding universe is given by

$$R = R_I e^{t/R_I}; \tag{14}$$

where we have chosen the constant of integration so that, at $t = 0$, $R = R_I \ll a$. At $t = -\infty$ we have $R = 0$, but the density $\rho = \rho_p$, is finite, so that one can say that there is no singularity.

Let us now consider the more general situation described by Eq. (10). Putting the latter into Eq. (3), we get

$$\dot{R}^2 = \beta R^2/(a^4 + R^4), \tag{15}$$

with

$$\beta = (8\pi/3)a^4\rho_p = a^4/R_I^2 = 3.21 \times 10^{54} \text{ cm}^2, \tag{16}$$

For $R \gg a$ one is dealing with the radiation-dominated period. Eq. (10) then gives

$$\rho = a^4\rho_p/R^4. \tag{17}$$

If one puts this into Eq. (3), one gets

$$R^2\dot{R}^2 = \beta, \tag{18}$$

which has the solution

$$R^2 = 2\beta^{1/2}(t - t_0) \quad (t_0 = \text{constant}), \tag{19}$$

so that Eq. (17) gives

$$\rho = \frac{3}{32\pi(t - t_0)^2}. \tag{20}$$

In the general case the solution of Eq. (15) can be written

$$t = t_1 + R_1 \int_{R_1/a}^{R/a} (1 + x^4)^{1/2} x^{-1} \, \mathrm{d}x, \tag{21}$$

where t_1 and R_1 are any values of t and R which satisfy Eq. (14), and $R_1 \ll a$. Carrying out the integration and making use of Eq. (14) for t_1 and R_1, one finds

$$\frac{t}{R_1} = \frac{1}{2}\left(1 + \frac{R^4}{a^4}\right)^{1/2} + \frac{1}{4}\ln\frac{[1 + (R^4/a^4)]^{1/2} - 1}{[1 + (R^4/a^4)]^{1/2} + 1} - \frac{1}{2}\left(1 + \ln\frac{R_1^2}{2a^2}\right). \tag{22}$$

From Eq. (22) we get t for various values of R and from Eq. (10) the corresponding values of ρ. For $R \ll a$ Eq. (22) goes over into Eq. (14).

For $R \gg a$ one gets Eq. (19) with the constant t_0 negligibly small.

4. Temperature and Entropy in the Early Universe

To investigate the temperature T and the entropy S of the universe during the prematter-radiation periods, we proceed as in (I), following Weinberg (1972). We take $\rho = \rho(T)$, $P = P(T)$, and write

$$\mathrm{d}S(V, T) = \frac{1}{T}[\mathrm{d}(\rho V) + P\,\mathrm{d}V]. \tag{23}$$

Here V is the volume of some part of the universe having a fixed boundary in the space of r, θ, ϕ. From Eq. (2) we can write

$$V = V_0 R^3, \tag{24}$$

where V_0 is the volume determined by the spatial coordinates. For example, we can take a sphere with $r \leq r_0$ (r_0 = constant), so that

$$V_0 = \frac{4}{3}\pi r_0^3. \tag{25}$$

For $\mathrm{d}S$ in Eq. (23) to be an exact differential one finds that one must have

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{1}{T}(\rho + P). \tag{26}$$

One can then integrate Eq. (23), thus obtaining

$$S = \frac{V}{T}(\rho + P), \tag{27}$$

where the integration constant has been taken to vanish. Making use of equations (5), (24), (26), and (27), one finds

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0; \tag{28}$$

so that there is entropy conservation in a fixed volume V_0.

If we put P from Eq. (9) into Eq. (26) and integrate the latter, we get

$$\rho(1 - \rho/\rho_p)^7 = \sigma T^4, \tag{29}$$

with the integration constant σ taken equal to the Stefan-Boltzmann constant, $\sigma = 6.24 \times 10^{-64}\ \mathrm{cm}^{-2}\,\mathrm{K}^{-4}$, so that for $\rho \ll \rho_p$ one gets the Stefan-Boltzmann law for black-body radiation. If in Eq. (29) we take ρ as given by Eq. (10), we get

$$T = \left(\frac{\rho_p}{\sigma}\right)^{1/4} \frac{aR^7}{(a^4 + R^4)^2}, \tag{30}$$

with $(\rho_p/\sigma)^{1/4} = 1.574 \times 10^{32}$ K. As $t \to -\infty$, $R \to 0$, and $T \to 0$. For $t = 0$, $R = R_I(\ll a)$, one gets $T = 2.65 \times 10^{-180}$ K. As R increases, T increases, attaining its maximum value, $T = 7.41 \times 10^{31}$ K, for $R^4 = 7a^4$. During the radiation-dominated period ($R \gg a$) Eq. (30) gives

$$T = (\rho_p/\sigma)^{1/4} a/R = C/R, \tag{31}$$

with $C = 1.574 \times 10^{29}$ cm K. If we assume that Eq. (31) holds for the cosmic microwave radiation, which has a temperature of 2.73 K, then one gets for the present value of R

$$R_N = 5.77 \times 10^{28}\ \mathrm{cm}. \tag{32}$$

From Eq. (27), if one expresses V, T, ρ, and P as functions of R according to equations (24), (30), (10), and (11), one finds

$$S = \frac{4}{3}(\rho_p/\sigma)^{-1/4}\rho_p a^3 V_0 = 3.24 \times 10^{25} V_0\ \mathrm{cm\,K}^{-1}, \tag{33}$$

thus verifying that S is indeed constant during the prematter and radiation periods as Eq. (28) asserts.

We have seen from Eq. (30) that the temperature T has a maximum for $R^4 = 7a^4$. It is interesting to note that the pressure P given by Eq. (11) also has its maximum value for $R^4 = 7a^4$ namely, $\rho_p/48$. For this value of R Eq. (10) gives $\rho = \rho_p/8$. For $R^4 > 7a^4$, T, ρ, and P all decrease with increasing R, and hence with increasing time t.

5. Radiation-Dust Periods

For $R = 20a = 2 \times 10^{-2}$ cm one gets from Eqs. (10) and (11) $\rho = 3.000075P$, so that the universe has essentially entered the radiation-dominated period characterized by Eq. (7). According to Eq. (22) this happens at a time $t = 1.50 \times 10^{-31}$ cm = 5.0×10^{-42} sec, and the temperature given by Eq. (30) is $T = 7.87 \times 10^{30}$ K. From this time on the universe is radiation dominated and is described by equations (17), (19) and (20) (with $t_0 = 0$), so that our model has essentially become the standard model (Weinberg, 1972). One can therefore discuss the development of our model and its transition to the matter-dominated state in terms of the astrophysical processes of the standard model. However, instead of that, we will assume a phenomenological equation of state relating P and ρ during the radiation-dominated period, the matter-dominated (or dust) period, and the transition period between them.

For the present, dust-like state in which Eq. (8) holds, Eq. (5) gives

$$\rho = B/R^3 \quad (B = \text{constant}). \tag{34}$$

Let us use the subscript N to denote the value which a quantity has now. Then Eq. (34) gives

$$B = \rho_N R_N^3. \tag{35}$$

With the Hubble constant defined by

$$H = \dot{R}/R, \tag{36}$$

Eq. (3) can be written as

$$H^2 = (8\pi/3)\rho. \tag{37}$$

Let us take $H_N = 5 \times 10^{-29}\ \text{cm}^{-1}$ (for our purpose we do not need a very accurate value). Then Eq. (37) gives

$$\rho_N = 2.98 \times 10^{-58}\ \text{cm}^{-2} \tag{38}$$

If we take R_N as in Eq. (32), then Eq. (35) yields

$$B = 5.72 \times 10^{28}\ \text{cm}. \tag{39}$$

Let us now take, as the equation of state, that given in (I),

$$P = \frac{1}{3}\rho^2/(\rho + \rho_T), \tag{40}$$

where ρ_T is a parameter, the transition density characterizing the change from the radiation state, described by Eqs. (7) and (17) for $\rho \gg \rho_T$, to the dust-like state, described by Eqs. (8) and (34) for $\rho \ll \rho_T$. Eq. (40) has been chosen because of its simplicity. If one uses Eq. (40) in Eq. (5), one gets

$$\rho^4 R^{12}/(\rho + \frac{3}{4}\rho_T) = C(= \text{constant}). \tag{41}$$

In the radiation period ($\rho \gg \rho_T$) this gives

$$\rho^3 R^{12} = C. \tag{42}$$

Comparing this with Eq. (17), we see that

$$C = a^{12}\rho_p^3. \tag{43}$$

In the dust period ($\rho \ll \rho_T$) Eq. (41) gives

$$\rho^4 R^{12} = \frac{3}{4} C \rho_T. \tag{44}$$

If we compare this with Eq. (34), we get

$$C = \frac{4}{3} B^4/\rho_T \tag{45}$$

It follows from Eqs. (43) and (45) that

$$\rho_T = \frac{4}{3} B^4/a^{12}\rho_p^3. \tag{46}$$

With B given by Eq. (39) one gets

$$\rho_T = 2.55 \times 10^{-46}\ \text{cm}^{-2}. \tag{47}$$

For this value of ρ one gets the corresponding value of R from Eqs. (41) and (43),

$$R_T = (7/4)^{1/12} a(\rho_P/\rho_T)^{1/4} = 6.52 \times 10^{24}\ \text{cm}. \tag{48}$$

Taking for R this value in Eq. (31), we get the corresponding value of T, which one might call the transition temperature,

$$T_T = 2.41 \times 10^4\ \text{K}. \tag{49}$$

This is in the range 10^3–10^5 K at which the universe is considered to have entered the matter-dominated era (Weinberg, 1972, p. 529).

Now let us consider the transition from radiation to dust on the basis of Eq. (40). By combining Eqs. (3), (5) and (40) one gets

$$\dot{\rho} + \left(\frac{8\pi}{3}\right)^{1/2} \frac{\rho^{3/2}(4\rho + 3\rho_T)}{\rho + \rho_T} = 0. \tag{50}$$

so that one can write

$$\left(\frac{8\pi}{3}\right)^{1/2}(t - t_2) = -\int_{\rho_2}^{\rho} \frac{\rho + \rho_T}{\rho^{3/2}(4\rho + 3\rho_T)}\, d\rho = 0, \tag{51}$$

Let us take ρ_2 and t_2 as corresponding to the radiation period, so that $\rho_2 \gg \rho_T$, and ρ_2 and t_2 satisfy Eq. (20) with $t_0 = 0$. One finds that Eq. (51) takes the form

$$(8\pi/3)^{1/2} t = \frac{2}{3}\rho^{-1/2} + (3\rho_T)^{-1/2}\left[\frac{1}{3}\tan^{-1}(4\rho/(3\rho_T)^{1/2} - \pi/6\right]. \tag{52}$$

For $\rho \gg \rho_T$, corresponding to the radiation period, Eq. (52) goes over into Eq. (20) with $t_0 = 0$. For $\rho \ll \rho_T$, corresponding to the dust period, we get

$$\rho = \frac{1}{6\pi t^2}. \tag{53}$$

Putting this into Eq. (34) gives

$$R^3 = 6\pi B t^2. \tag{54}$$

This could also have been obtained from Eqs. (3) and (4) under the assumption that $R = 0$ for $t = 0$. The situation is as if we had a universe filled with dust (P = 0) that started at $t = 0$ with a big bang. From Eqs. (54) and (36) we get

$$t_N = \frac{2}{3H_N}, \tag{55}$$

so that, with the value $H_N = 5 \times 10^{-29}$ cm^{-1}, we get $t_N = 1.33 \times 10^{28}$ cm = 1.4×10^{10} yr as the age of the universe.

6. A History of the Universe

In the previous sections we developed analytical descriptions of our model in various periods. In order to provide a clearer picture we present here a table of numerical values associated with various states in the history of the universe. The values have been calculated with the help of the equations given above. All numbers are in general-relativity units except the temperature T. To go over to conventional units, for the time t one takes 1 cm = 0.333×10^{-10} s, for the density ρ: 1 cm^{-2} = 1.349×10^{28} g cm^{-3}, and for the pressure P: 1 cm^{-2} = 1.214×10^{49} dyn cm^{-2}.

As the table shows, the universe originates at $t = -\infty$ from a geometric singularity, $R = 0$. However, since ρ and P are finite, the physical state is nonsingular. The density has its maximum value $\rho = \rho_p$, and the situation is characterized by the equation of state (1), so that there is tension rather than pressure. The prematter period continues until about $t = 3.63 \times 10^{-32}$ cm. During this period, with the exponential dependence of R on t as given by Eq. (14), the universe undergoes

TABLE I

t (cm)	R (cm)	ρ (cm^{-2})	P (cm^{-2})	T (K)	$\dot{R}$	$\ddot{R}$ (cm^{-1})
$-\infty$	0	3.83×10^{65}	-3.83×10^{65}	0	0	0
0	5.58×10^{-34}	3.83×10^{65}	-3.83×10^{65}	2.65×10^{-180}	1	1.79×10^{33}
3.37×10^{-32}	1×10^{-7}	3.83×10^{65}	-3.83×10^{65}	1.57×10^{4}	1.79×10^{26}	3.21×10^{59}
3.63×10^{-32}	1×10^{-5}	3.83×10^{65}	-3.83×10^{65}	1.57×10^{18}	1.79×10^{28}	3.21×10^{61}
3.85×10^{-32}	5×10^{-4}	3.60×10^{65}	-3.32×10^{65}	1.09×10^{30}	8.69×10^{29}	1.33×10^{63}
3.89×10^{-32}	1×10^{-3}	1.92×10^{65}	-6.38×10^{64}	3.93×10^{31}	1.27×10^{30}	0
3.92×10^{-32}	$3^{1/4} \times 10^{-3}$	9.58×10^{64}	0	6.73×10^{31}	1.18×10^{30}	-5.28×10^{62}
3.95×10^{-32}	$7^{1/4} \times 10^{-3}$	4.79×10^{64}	7.98×10^{63}	7.41×10^{31}	1.03×10^{30}	-4.89×10^{62}
6.67×10^{-32}	1×10^{-2}	3.83×10^{61}	1.28×10^{61}	1.57×10^{31}	1.79×10^{29}	-3.21×10^{60}
1.50×10^{-31}	2×10^{-2}	2.39×10^{60}	7.98×10^{59}	7.87×10^{30}	8.96×10^{28}	-4.01×10^{59}
2.79×10^{-8}	1×10^{10}	3.83×10^{13}	1.28×10^{13}	1.57×10^{19}	1.79×10^{17}	-3.21×10^{24}
2.79×10^{12}	1×10^{20}	3.83×10^{-27}	1.28×10^{-27}	1.57×10^{9}	1.79×10^{7}	-3.21×10^{-6}
2.79×10^{20}	1×10^{24}	3.83×10^{-43}	1.28×10^{-43}	1.57×10^{5}	1.79×10^{3}	-3.21×10^{-18}
1.15×10^{22}	6.52×10^{24}	2.55×10^{-46}	4.25×10^{-47}	2.41×10^{4}	3.01×10^{2}	-1.04×10^{-20}
9.63×10^{23}	1×10^{26}	5.72×10^{-50}	4.28×10^{-54}	1.57×10^{3}	69.2	-2.40×10^{-23}
3.05×10^{25}	1×10^{27}	5.72×10^{-53}	4.28×10^{-60}	1.57×10^{2}	21.9	-2.40×10^{-25}
1.33×10^{28}	5.77×10^{28}	2.98×10^{-58}		2.73	2.88	-7.20×10^{-29}

inflationary expansion. This is exemplified by the fact that during the short time interval from $t = 0$ to $t = 3.63 \times 10^{-32}$ cm the scale parameter R increases by a factor of 1.79×10^{28}. During this interval the temperature grows from 2.65×10^{-180} K to 1.57×10^{18} K. On the other hand, the density ρ remains practically constant.

The prematter period is followed by a period of transition to the radiation-dominated state (from about $t = 3.63 \times 10^{-32}$ cm to about $t = 1.50 \times 10^{-31}$ cm). Some interesting events take place during this period. At $t = 3.89 \times 10^{-32}$ cm, when $R = a = 10^{-3}$ cm, one has $\ddot{R} = 0$ and the expansion rate has its maximum $\dot{R} = 1.27 \times 10^{30}$. Therefore gravitational attraction begins to brake the expansion, and deceleration sets in. The negative pressure decreases to zero, becomes positive and grows rapidly. The temperature and the pressure both reach their maximum values at $t = 3.95 \times 10^{-32}$ cm, when $R = 7^{1/4}a$. At this stage $T = 7.41 \times 10^{31}$ K, and $P = 7.98 \times 10^{63}$ cm^{-2}.

It should be noted parenthetically that during this transition period radical changes take place in the properties of matter. As the density decreases, the completely homogeneous prematter converts into ordinary matter consisting of radiation and high-energy elementary particles. One can expect that during this process small local perturbations develop. These perturbations subsequently serve as condensation centers for the formation of galaxies.

At $t = 1.50 \times 10^{-31}$ cm, when $R = 20a$, one has $P = \frac{1}{3}\rho$, so that the model has entered the radiation-dominated era. At this stage the universe has a temperature $T = 7.87 \times 10^{30}$ K, and a density $\rho = 2.39 \times 10^{60}$ cm^{-2}. From now on the universe behaves as if started from the big bang. During the radiation and dust periods our model behaves like the standard model (Weinberg, 1972). For convenience we have introduced equation of state (40) to describe the transition from radiation to dust. This process is characterized by a transition density $\rho_T = 2.55 \times 10^{-46}$ cm^{-2}, corresponding to a time $t = 1.15 \times 10^{22}$ cm and a temperature $T = 2.41 \times 10^{4}$ K. In the matter-dominated period (starting at about $t = 9.63 \times 10^{23}$ cm) there is no longer any interaction between the radiation and the matter. The temperatures T in the table now refer to the background radiation. In the last line the present data are given. The value of the density ρ_N is based on the assumption $H_N = 5 \times 10^{-29}$ cm^{-1}.

7. Discussion

The model considered here and in the earlier work (I) are characterized by the fact that in the state of maximum contraction the universe is filled with prematter having the equation of state (1). As the expansion takes place, the prematter goes over into ordinary matter. In both models it was assumed that $a = 1 \times 10^{-3}$ cm and $H_N = 5 \times 10^{-29}$ cm^{-1}. There is some arbitrariness in the choice of these values. However, the purpose of the work is to present a general picture of the possible behavior of the universe, and changing these parameters by reasonable amounts will not change the qualitative behavior.

In the earlier work (I) a closed model of the universe (with $k = 1$) was considered. Such a model oscillates in time so that it describes a universe existing forever, without a beginning. In the present work the spatially-flat model (with $k = 0$) describes a universe that had its beginning an infinitely long time ago ($t = -\infty$). There remains the possibility of a model of a universe that began its existence at a finite time ago ($t = 0$). This case will be investigated in a future work.

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