SOLUTIONS TO THE FIELD EQUATIONS AND THE DECELERATION PARAMETER

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Abstract. We utilise a form for the Hubble parameter to generate a number of solutions to the Einstein field equations with variable cosmological constant and variable gravitational constant. The Hubble law utilised yields a constant value for the deceleration parameter. A variety of solutions is presented in the Robertson–Walker spacetimes. A generalisation of the cosmic scale factor is utilised in the anisotropic Bianchi I spacetime to illustrate that new solutions may also be found in spacetimes with less symmetry than Robertson–Walker. We also show that the constant deceleration parameter used is consistent with alternate theories of gravity by considering the scalar–tensor theory of Lau and Prokhovnik with a $k = 0$ Robertson–Walker background.

1. Introduction

The Einstein field equations are a coupled system of highly nonlinear differential equations and we seek physical solutions to the field equations for applications in cosmology and astrophysics. In order to solve the field equations we normally assume a form for the matter content or suppose that spacetime admits Killing vector symmetries (Kramer *et al.*, 1980). Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter which was proposed by Berman (1983). It is interesting to observe that this law yields a constant value for the deceleration parameter. Forms for the deceleration parameter which are variable have been investigated recently by Beesham (1993). The variation of Hubble's law assumed is not inconsistent with observation and has the advantage of providing simple functional forms of the scale factor. In the simplest case the Hubble law yields a constant value for the deceleration parameter. In earlier literature cosmological models with a constant deceleration parameter have been studied by Berman (1983), Berman and Gomide (1988) and others. The case of a perfect fluid Robertson–Walker spacetime with variable gravitational and cosmological constants has been pursued by Berman (1991). A treatment may also be performed in alternate theories of gravity; for example Berman and Gomide (1988) consider applications to the Price–Hoyle and Brans–Dicke theory.

Our intention in this paper is to extend the results obtained by Berman (1983), Berman and Gomide (1988) and Berman (1991) by obtaining solutions to the Einstein field equations, with variable gravitational and cosmological constants, in the Robertson–Walker spacetimes. The scale factor is explicitly determined by the law of variation for Hubble's parameter. Explicit forms for the gravitational constant, cosmological constant, scale factor, energy density and pressure are ob-

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tained for various cases. As a second application we consider a gravitational field with less symmetry than the Robertson–Walker spacetimes, namely the Bianchi I spacetimes. We show that the solutions to the classical Einstein field equations are consistent with a generalisation of the Hubble law used in the Robertson–Walker models. Furthermore we consider an example of a scalar-tensor theory, the theory of Lau and Prokhovnik (1986), and find that the theory is consistent with our specified form of the deceleration parameter. These examples suggest the possibility of seeking solutions to the field equations in spacetimes with less symmetry than Robertson–Walker and in alternate theories of gravity involving scalar fields.

2. Robertson–Walker Spacetimes

In standard coordinates $(x^a) = (t, r, \theta, \phi)$ the Robertson–Walker line element has the form

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1}$$

where $S(t)$ is the cosmic scale factor. Without loss of generality the constant k takes on only three values: 0, 1 or -1. The constant k is related to the spatial geometry of a 3-dimensional manifold generated by $t =$ constant. The Robertson–Walker spacetimes are the standard cosmological models and are consistent with observational results. For the case of variable cosmological constant $\Lambda(t)$ and gravitational constant $G(t)$ the Einstein field equations

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{2}$$

become

$$\frac{3}{S^2}(\dot{S}^2 + k) = 8\pi G\mu + \Lambda, \tag{3}$$

$$2\frac{\ddot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -8\pi G p + \Lambda \tag{4}$$

for the line element (1). From Equations (3) and (4) we obtain the generalised continuity equation

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) + \frac{\dot{G}}{G}\mu + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{5}$$

This reduces to the conventional continuity equation when Λ and G are constants. In an attempt to obtain solutions to the field equations we assume, as is often done, that the classical conservation law, ${T^{ab}}_{;b} = 0$, also holds. Then we have that equation (5) implies the two relationships

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) = 0, \tag{6}$$

$$8\pi\mu\dot{G} + \dot{\Lambda} = 0 \tag{7}$$

which facilitate the solution of the field equations. The result (6) is just the conventional continuity equation, and (7) simply relates G and Λ and does not explicitly contain the scale factor $S(t)$.

In this section we consider the generalised Einstein field equations (3)–(4) with variable gravitational constant $G(t)$ and variable cosmological constant $\Lambda(t)$ for the Robertson–Walker metric (1). We assume that the variation of the Hubble parameter is given by the equation

$$H = DS^{-m} \tag{8}$$

where D and m are constants. Then the deceleration parameter

$$q = -\frac{S(t)\ddot{S}(t)}{\dot{S}^2(t)}$$

and the definition

$$H = \frac{\dot{S}(t)}{S(t)}$$

imply that the deceleration parameter is constant:

$$q = m - 1.$$

Other forms of q have been investigated by Beesham (1993).

The form of the Hubble parameter (8) was first utilised by Berman (1983) and Berman and Gomide (1988) for the case of the classical Einstein field equations with $\dot{\Lambda} = 0$ and $\dot{G} = 0$. Berman (1991) presented a solution to the field equations (2) for the $k = 0$ Robertson–Walker spacetime:

$$S = (C + mDt)^{1/m}, \tag{9}$$

$$\Lambda = BS^{-2m}, \tag{10}$$

$$G = \beta S^{mB/(4\pi A)}, \tag{11}$$

$$\mu = \frac{A}{\beta} S^{-2m-mB/(4\pi A)}, \tag{12}$$

$$p = \frac{A}{3\beta}\left[m\left(2 + \frac{B}{4\pi A}\right) - 3\right] S^{-2m-mB/(4\pi A)} \tag{13}$$

where A, B, C, β are constants and are subject to the following condition

$$3D^2 = 8\pi A + B.$$

We note that the equation (16) given by Berman (1991), corresponding to our equation (13), has an incorrect coefficient on the right hand side. Equations (9)–(13) comprise the general solution to the generalised Einstein field equations (3)–(4) with variable cosmological and gravitational constants for the Hubble law (8). It is interesting to observe that in this solution we obtain the equation of state for an ideal gas given by

$$p = \alpha\mu$$

where we have set the constant α to be

$$\alpha = \frac{1}{3}\left[m\left(2 + \frac{B}{4\pi A}\right) - 3\right].$$

It is possible to avoid the horizon and monopole problem with the above variable $G(t)$ and $\Lambda(t)$ solutions as suggested by Berman (1991). Other models considered which also have the relationship

$$\Lambda \propto \frac{1}{t^2}$$

include Berman (1990), Berman and Som (1990), Berman *et al.* (1989) and Bertolami (1986a, b). This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe. A decreasing functional form permits Λ to be large in the early universe. A partial list of cosmological models in which the gravitational constant G is a decreasing function of time are contained in Grøn (1986), Hellings *et al.* (1983), Rowan-Robinson (1981), Shapiro *et al.* (1971) and Van Flandern (1981). The possibility of the G increasing with time, at least in some stages of the development of the universe, has been investigated by Abdel-Rahman (1990), Chow (1981), Levitt (1980) and Milne (1935).

Now we present a number of classes of new solutions for all cases of $k : 0, 1, -1$ for variable cosmological constant Λ and variable gravitational constant G for the Hubble law (8). These solutions cover both the cases of $m = 0$ and $m \neq 0$ for the scale factor S:

$$S = \begin{cases} [C + mDt]^{1/m} & \text{for } m \neq 0, \\ Ee^{Dt} & \text{for } m = 0 \end{cases}$$

where C, D, E are constants. Our new classes of solutions, extending those of Berman (1991), are found by assuming an ansatz that immediately leads to a solution of the Einstein field equation (3). In an attempt to solve the Einstein field equations (3)–(4) we adopt the ansatz

$$\frac{3D^2}{S^{2m}} - \Lambda = K, \tag{14}$$

$$8\pi G\mu - \frac{3k}{S^2} = K \tag{15}$$

where K is a constant. This ansatz has the advantage of providing further classes of solutions. With the equations (14)–(15) we observe that the Einstein field equation (3) is identically satisfied. From equation (14) we have that the cosmological constant takes the following form for all classes of solution:

$$\Lambda = \frac{3D^2}{S^{2m}} - K. \tag{16}$$

From equation (15) we can express the energy density μ in terms of the gravitational constant G and the scale factor S:

$$\mu = \frac{1}{8\pi G}\left[\frac{3k}{S^2} + K\right]. \tag{17}$$

Then to obtain the pressure p we utilise the continuity equation (6) with the above forms of the scale factor S and the energy density μ. On substituting (17) and the derivative of equation (16) with respect to the time coordinate t into equation (7) we obtain the differential equation

$$\frac{\dot{G}}{G} = 6mD^2 \frac{\dot{S}}{S^{2m+1}} \frac{1}{3k/S^2 + K}$$

relating G to S. Thus as the scale factor S is specified by our assumed form for the Hubble parameter, the gravitational constant G is known in principle. The ansatz (14)–(15) enables us to integrate all the Einstein field equations for a number of values of m, k and K. In the remainder of this section we present a variety of classes of solutions to the Einstein field equations for each of the cases considered. We list the form of the scale factor S, the variable cosmological constant Λ, the variable gravitational constant G, the energy density μ and the pressure p. There are other classes of solution possible for other values of m. However the integration process becomes extremely complicated and here we present only the simple cases that follow easily from the integration process.

(a) $m = 0$:

$$S = Ee^{Dt},$$

$$\Lambda = 3D^2 - K,$$

$$G = A,$$

$$\mu = \frac{1}{8\pi A}\left[\frac{3k}{S^2} + K\right],$$

$$p = -\frac{1}{8\pi A}\left[\frac{k}{S^2} + K\right].$$

In these de Sitter-type solutions Λ and G are strictly constants because of the restriction $m = 0$. The cosmological constant Λ vanishes when $K = 3D^2$ and is positive for $K < 3D^2$ The scale factor S is exponential in t, so that if $D > 0$ then the universe is exponentially expanding always. Such a model is not a physical description of our present universe but could be applicable in the early universe in the inflationary scenario. For $m = 0$ we get the deceleration parameter $q = -1$ for this class of solutions. The equation of state is given by

$$p = -\frac{1}{3}\mu - \frac{K}{12\pi A}$$

for $k \neq 0$ and for the $k = 0$ Robertson–Walker model we have

$$p = -\mu$$

and the pressures are negative.

(b) $m \neq 0, K = 0, k \neq 0$:

$$S = [C + mDt]^{1/m},$$

$$\Lambda = \frac{3D^2}{S^{2m}},$$

$$G = \alpha \exp\left\{\frac{mD^2}{k(1-m)}S^{2-2m}\right\},$$

$$\mu = \frac{3k}{8\pi\alpha}S^{-2}\exp\left\{\frac{mD^2}{k(m-1)}S^{2-2m}\right\},$$

$$p = -\frac{1}{8\pi\alpha}\left[\frac{4mD^3}{S^{3m}} + \frac{3k}{S^2}\right]\exp\left\{\frac{mD^2}{k(m-1)}S^{2-2m}\right\}.$$

This case shares the common feature that G may be increasing in time in certain regions of spacetime with the model proposed by Abdel-Rahman (1990). The relationship between the energy density and the pressure is given by

$$p = -\left[1 + \frac{4mD^2}{3kS^{3m-2}}\right]\mu.$$

With $m = \frac{2}{3}$ this relationship becomes

$$p = -\left(1 + \frac{8D^3}{9k}\right)\mu$$

which is the equation of state of an ideal gas. The positivity of the pressure is dependent on the values of D and k.

(c) $m \neq 0, K \neq 0, k = 0$:

$$S = [C + mDt]^{1/m},$$

$$\Lambda = \frac{3D^2}{S^{2m}} - K,$$

$$G = \alpha \exp\left\{\frac{3D^2}{KS^{2m}}\right\},$$

$$\mu = \frac{K}{8\pi\alpha} \exp\left\{-\frac{3D^2}{KS^{2m}}\right\},$$

$$p = -\frac{1}{8\pi\alpha}\left[\frac{2mD^2}{S^{2m}} + K\right] \exp\left\{-\frac{3D^2}{KS^{2m}}\right\}.$$

The relationship between μ and p is given by

$$p = \left[-\frac{2mD^2}{KS^{2m}} - 1\right]\mu.$$

In this case it is not possible to have an equation of state for an ideal gas as $m \neq 0$ by assumption. However if $m = \frac{1}{2}$ we obtain a simple relationship relating the energy density to the pressure

$$p = \left[-\frac{D^2}{K}S^{-1} - 1\right]\mu$$

from the above. This has the asymptotic behaviour that as t increases

$$p \approx -\mu$$

so that the pressure becomes negative.

(d) $m = 2, K \neq 0, k \neq 0$:

$$S = [C + 2Dt]^{1/2},$$

$$\Lambda = \frac{3D^2}{S^4} - K,$$

$$G = \alpha \left[\frac{(3kS^{-2} + K)^{K/k}}{\exp\{S^{-2}\}} \right]^{2D^2/k},$$

$$\mu = \frac{1}{8\pi\alpha} \left[3kS^{-2} + K \right] \left[\frac{\exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}} \right]^{2D^2/k},$$

$$p = -\frac{1}{8\pi\alpha} \left[3kS^{-2} + K \right] \left[\frac{\exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}} \right]^{2D^2/k} \times$$

$$\times \left[\frac{4D^2S^{-4}(2K - 3kS^{-2})}{k(3kS^{-2} + K)} + 1 \right] + \frac{kS^{-2}}{4\pi\alpha} \left[\frac{\exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}} \right]^{2D^2/k}.$$

Unlike the cases considered thus far we have a specific value for m. This gives a value $q = 1$ for the deceleration parameter. A wide range of behaviour is possible for the gravitational constant. The relationship between the energy density and the pressure is given by

$$p = -\frac{1}{k(3kS^{-2} + K)} \left[4D^2S^{-4} \left(2K - 3kS^{-2} \right) + k \left(kS^{-2} + K \right) \right] \mu$$

which differs substantially from the equation of state for an ideal gas.

(e) $m = -2, K \neq 0, k \neq 0$:

$$S = \frac{1}{\sqrt{C - 2Dt}},$$

$$\Lambda = \frac{3D^2}{S^{-4}} - K,$$

$$G = \alpha \frac{\exp\{(3k/K^2)S^2 - (1/2K)S^4\}}{[S^{-2}(3kS^{-2} + K)]^{9k^2/K^3}},$$

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right]\left[S^{-2}(3kS^{-2} + K)\right]^{9k^2/K^3} \exp\left\{\frac{1}{2K}S^4 - \frac{3k}{K^2}S^2\right\},$$

$$p = \frac{1}{4\pi\alpha}\left[3kS^{-2} + K\right]^{9k^2/K^3+1}\left[\frac{6k^2}{K^3} + \frac{3k - S^6}{3S^2(3kS^{-2} + K)} - \frac{1}{2}\right] \times$$

$$\times\, S^{-18k^2/K^3} \exp\left\{\frac{1}{2K}S^4 - \frac{3k}{K^2}S^2\right\}.$$

Here the deceleration parameter has the value $q = -3$ as $m = -2$. The relationship between the energy density and the pressure is given by

$$p = 2\left[\frac{6k^2}{K^3} + \frac{3k - S^6}{3S^2(3kS^{-2} + K)}\right]\mu.$$

As for case (d) we do not obtain the equation of state for an ideal gas.

(f) $m = \frac{1}{2}, K \neq 0, k \neq 0$:

$$S = \left[C + \frac{1}{2}Dt\right]^2,$$

$$\Lambda = \frac{3D^2}{S} - K,$$

$$G = \alpha \exp\left\{\sqrt{3/kK}D^2 \arctan(\sqrt{K/3k}\ S)\right\},$$

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right]\exp\left\{-\sqrt{3/kK}D^2 \arctan(\sqrt{K/3k}\ S)\right\},$$

$$p = -\frac{1}{8\pi\alpha}\left[\left(k - D^2S\right)S^{-2} + K\right]\exp\left\{-\sqrt{3/kK}D^2 \arctan(\sqrt{K/3k}S)\right\}.$$

For this class of solutions the deceleration parameter has the value $q = -\frac{1}{2}$ as $m = \frac{1}{2}$. The relationship between the energy density and the pressure is given by

$$p = \left[\frac{2k + D^2S}{3k + KS^2} - 1\right]\mu.$$

In this case it is not possible to have an equation of state for an ideal gas. As in case (c) as t increases we have the asymptotic relationship

$$p \approx -\mu$$

so that the pressures may be again negative.

(g) $m = \frac{2}{3}, K \neq 0, k \neq 0$:

$$S = \left[C + \frac{2}{3}Dt\right]^{3/2},$$

$$\Lambda = \frac{3D^2}{S^{4/3}} - K,$$

$$G = \alpha \left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2} \exp\left\{6 \arctan\left(\frac{2S^{2/3} - a}{\sqrt{3}\,a}\right)\right\}\right]^{D^2/Ka^2},$$

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right] \times$$

$$\times \left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} \exp\left\{-6 \arctan\left(\frac{2S^{2/3} - a}{\sqrt{3}\,a}\right)\right\}\right]^{D^2/Ka^2},$$

$$p = -\frac{1}{8\pi\alpha}\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} \exp\left\{-6 \arctan\left(\frac{2S^{2/3} - a}{\sqrt{3}\,a}\right)\right\}\right]^{D^2/Ka^2} \times$$

$$\times \left[kS^{-2} + K + \frac{2D^2}{3Ka}\left(3kS^{-2} + K\right)\left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2}\right]\right] \times$$

$$\times \left[\frac{S^{2/3} - a}{(S^{2/3}+a)^3} - \frac{4\sqrt{3}\left(S^{4/3} - aS^{2/3} + a^2\right)}{(S^{2/3}+a)^2\,(3a^2 + (2S^{2/3} - a)^2)}\right].$$

Clearly in this case we cannot obtain the equation of state for an ideal gas from the above equation.

In the above we have presented a number of new solutions to the Einstein field equations with variable cosmological constant and gravitational constant which satisfy the Hubble variation law given by Equation (8). It is remarkable that this simple law leads to a wide class of solutions. It is clear that other choices of m will lead to complex forms of the dynamical variables which makes studying the physical properties of these solutions in any detail difficult. It is interesting to observe that solutions are admitted in which the gravitational constant may be increasing with time (cf. Abdel-Rahman, 1990). The ansatz utilised to solve the Einstein field equations (3)–(4) is very simple. It might be worthwhile to investigate other possibilities that lead to solutions to the Einstein field equations with interesting behaviour for the gravitational constant and cosmological constant.

3. Bianchi I Spacetime

We now analyse the spatially homogeneous and anisotropic Bianchi I spacetime described by the line element

$$ds^2 = -dt^2 + A^2(t)\,dx^2 + B^2(t)\,dy^2 + C^2(t)\,dz^2. \tag{18}$$

This spacetime is a generalisation of the $k = 0$ Robertson–Walker spacetime and is often utilised in the study of anisotropic models. For a perfect fluid energy momentum tensor T_{ab} the Einstein field equations (2) with variable cosmological constant $\Lambda(t)$ and gravitational constant $G(t)$ can be written as the coupled system of differential equations

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \Lambda = 8\pi G\mu, \tag{19}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C} - \Lambda = -8\pi Gp, \tag{20}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{C}}{C} - \Lambda = -8\pi Gp, \tag{21}$$

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} - \Lambda = -8\pi Gp. \tag{22}$$

With $\Lambda = 0$ and G a constant we regain the classical Einstein field equations from the above equations. In the case $\mu = 0 = p$ and $\Lambda = 0$ we obtain the vacuum Kasner solution

$$A = [\alpha + \beta t]^{p_1},$$

$$B = [\alpha + \beta t]^{p_2},$$

$$C = [\alpha + \beta t]^{p_3}$$

where α and β are constants and

$$p_1 + p_2 + p_3 = 1,$$

$$p_1^2 + p_2^2 + p_3^2 = 1$$

must be satisfied for a consistent solution. The constants α and β are not essential to the solution and may be eliminated using the transformation $\tilde{t} \rightarrow \alpha + \beta t$. We note that the general Bianchi I solution for dust ($\mu \neq 0, p = 0$) is also known and is listed by Stephani (1990). The form of solution for dust is similar to the Kasner solution given above. Other special cases of solution are listed by Kramer *et al.* (1980).

We will show that the Hubble variation (8) used previously is consistent with the Bianchi I spacetime (18) for the vacuum field equations. It is possible in principle to perform a similar analysis for the Einstein field equations (19)–(22) with variable cosmological constant and gravitational constant. To perform an analogous discussion to the previous section we need to define the function

$$S = (ABC)^{1/3}$$

as an "average" of the anisotropy. Clearly this definition for the Bianchi I spacetime reduces to the scale factor of the flat $k = 0$ Robertson–Walker spacetime when we have $A = B = C$. Then the above definition gives the following form for Hubble's constant

$$H = \frac{\dot{S}}{S} = \frac{1}{3}(\ln ABC)^{\cdot} .$$

This form of the Hubble parameter was utilised by Misner *et al.* (1973) in studying adiabatic cooling of anisotropy in the early universe. For the vacuum Kasner solution the Hubble law is of the form

$$H = \frac{1}{3}\left[\frac{\beta}{\alpha + \beta t}\right] \tag{23}$$

from definition. Is this form of solution consistent with the Berman variation law? To answer this question we must compare this result with the Hubble law obtained from (8). Using the scale factor S defined above for the anisotropic Bianchi I spacetime we obtain the following form

$$H = DS^{-m} = \frac{D}{(\alpha + \beta t)^{m/3}}. \tag{24}$$

On comparing equations (23) and (24) we have

$$m = 3 \qquad D = \frac{1}{3}\beta.$$

Thus we have verified that the vacuum Kasner solution is consistent with the Hubble variation law

$$H = \frac{D}{C + mDt}$$

with $m = 3$. In fact the vacuum Kasner solution remains unchanged with this variation of the Hubble law as the only modification involves a rescaling of the arbitrary constant β.

The above solution for the Bianchi I spacetime is interesting as it suggests that the class of solutions presented for Robertson–Walker spacetimes may be extended to other spacetimes with less symmetry. It is possible that this approach may lead to new solutions of the Einstein field equations. We may extend the arguments given above in the Bianchi I spacetime to include the case of variable cosmological constant and gravitational constant. We illustrate this possibility with an elementary solution of the Einstein field equations (19)–(22). It is interesting to note that the vacuum Kasner solution consistent with the Berman law $H = DS^{-m}$ extends to the case of variable cosmological constant and gravitational constant. It is clear by simple inspection that this solution is admitted by the field equations (19)–(22) if the cosmological constant satisfies

$$\Lambda = 8\pi G p$$

and we have for the pressure

$$p = -\mu.$$

Thus the pressures are negative for a Kasner-type solution with variable cosmological constant and gravitational constant. We note that there is freedom in the solution as we can arbitrarily specify the behaviour of the cosmological constant or the gravitational constant. Even though this solution is very simple it illustrates that there are solutions to the Einstein field equations with variable cosmological constant Λ and gravitational constant G consistent with the Berman law (8). The simplest starting point for other solutions in the Bianchi I spacetime would be to choose the form of Λ and G so that the metric functions generate a behaviour which is similar to that of the Kasner solution.

4. Scalar–Tensor Theory of Lau and Prokhovnik

The law for the variation of Hubble's parameter (8) is also consistent with scalar–tensor theories of gravity that reduce to Einstein's general relativity. We illustrate this with the scalar–tensor theory of Lau and Prokhovnik (1986). This is a theory with variable cosmological constant and gravitational constant but, in addition, it has a scalar field ψ. The theory was structured so that it is consistent with the Dirac Large Numbers Hypothesis (Dirac, 1938, 1979). This theory was also investigated by Maharaj and Beesham (1988) who presented solutions to the field equations of Lau and Prokhovnik (1986) for the flat $k = 0$ Robertson–Walker spacetime. The generalised field equations in the scalar–tensor theory of Lau and Prokhovnik (1986) are given by

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} + \psi_{,a}\psi_{,b} \tag{25}$$

$$\dot{\psi}\Box\psi + \dot{\Lambda} + \frac{1}{2}\dot{g}^{00}\dot{\psi}^2 + g^{00}\dot{\psi}\ddot{\psi} + 8\pi\dot{G}L_m = 0 \tag{26}$$

where

$$\Box\psi = g^{ab}\psi_{;ab}.$$

Here L_m is the matter Lagrangian density including all non-gravitational fields. The quantity

$$\Lambda = \lambda(t) - \frac{1}{2}g^{00}\dot{\psi}^2$$

is a generalisation of the normal cosmological constant but is equivalent to the cosmological constant used before in Robertson–Walker spacetimes. The field equation (25) is a generalisation of the classical Einstein field equation to incorporate variable cosmological constant Λ, gravitational constant G and scalar fields ψ. The other field equation (26) governs the behaviour of the scalar field ψ. For details of the derivation of (25)–(26) see Lau and Prokhovnik (1986).

In this section we follow the notation of Lau and Prokhovnik (1986). This enables us easily to compare our results with those of Maharaj and Beesham (1988). In the case of the $k = 0$ Robertson–Walker spacetimes for compatibility with the Dirac Large Numbers Hypothesis (Dirac, 1938, 1979; Lau, 1985) we can show

$$S^2(t) = \beta_1(\alpha + \beta t)^{2/3}, \tag{27}$$

$$G(t) = \beta_2(\alpha + \beta t)^{-1}, \tag{28}$$

$$(L_m =)\mu(t) = \beta_3(\alpha + \beta t)^{-1}, \tag{29}$$

$$\psi = \frac{1}{\beta}\left(\frac{2}{3}\beta^2 - 8\pi\beta_2\beta_3\right)^{1/2}\ln(\alpha + \beta t) + A, \tag{30}$$

$$\Lambda = -\frac{1}{3}\beta^2(\alpha + \beta t)^{-2} \tag{31}$$

as established by Maharaj and Beesham (1988). The solution to the field equations in the theory of Lau and Prokhovnik (1986) is given by Equations (27)–(31). The solutions presented are analogous to those of Maharaj and Beesham (1988). It is interesting to observe that the cosmological constant has the behaviour

$$\Lambda \propto \frac{1}{t^2}.$$

This is the same form as the solutions presented by Berman (1991) for variable cosmological constant and gravitational constant without a scalar field ψ (also see Section 2). This form of the cosmological constant is consistent with observations of present day values for the cosmological constant which are small.

Using the definition for the Hubble parameter we have that

$$H = \frac{\dot{S}}{S} = \frac{\frac{1}{3}\beta}{\alpha + \beta t}.$$

However from Section 2 for $m \neq 0$ in the Robertson–Walker spacetimes we have that

$$H = \frac{D}{C + mDt}$$

which follows from the Berman hypothesis that the deceleration parameter is constant. Thus we have established that if

$$m = 3, \qquad C = 3\alpha, \qquad D = \beta,$$

then the dust solutions, for the $k = 0$ Robertson–Walker spacetime, in the theory of Lau and Prokhovnik (1986) are consistent with the Hubble variation law $H = DS^{-m}$. This example shows that the Hubble variation utilised in this paper may be useful in studying solutions of the field equations in scalar–tensor theories. It has the advantage of immediately specifying the scale factor. This is helpful in alternate theories of gravity as the normal variables are supplemented with the cosmological constant, gravitational constant and scalar fields. The Berman (1983) ansatz provides a mechanism to reduce the number of variables in an undetermined system of differential equations.

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