REMARKS ON THE VISCOSITY CONCEPT IN THE EARLY UNIVERSE

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Abstract. Some aspects of viscous cosmological models, mainly of Bianchi type-I, are studied, in particular with the purpose of trying to obtain a natural explanation of why the entropy per baryon in the universe, $\sigma \sim 10^9$, is so large. Using the FRW metric it is first shown, in agreement with previous workers, that the expressions for the bulk viscosity as derived from kinetic theory in the plasma era is incapable of explaining the large value of σ. However it is possible to imagine the viscosity to be an "impulse" viscosity operative in one or several phase transitions in the early universe. This is the main idea elaborated on in the present paper. It is shown that in the $k = 0$ FRW space, an impulse bulk viscosity $\zeta_{\rm infl} \sim 10^{60}$ g cm^{-1} s^{-1}) acting at the phase transition at the end of the inflationary epoch corresponds to the correct entropy. If the space is anisotropic, it is natural to exploit the analogy with classical fluid dynamics to introduce the turbulent viscosity concept. This is finally discussed, in relation to an anisotropy introduced in the universe via the Kasner metric.

1. Introduction and Summary

Perfect fluid models have for a long time been used in cosmological theory. The introduction of viscosity concepts came later, and there has in fact been an open question as to whether the viscosity concept is needed for an explanation of the observed quantities in the universe. Misner (1968) was probably the first to introduce viscosity in cosmological theory in connection with his study of how initial anisotropies in the early universe become relaxed. Cosmological models with viscosity terms have later now and then been discussed in the literature, from quite different points of view. Grøn (1990) has recently given an extensive review of the subject, and the reader is referred to it for detailed information and more references to the literature.

From a physical point of view it would in our opinion be almost surprising if the viscosity concept were *not* of importance in cosmology. An essential ingredient of the Riemannian model of the universe is after all to borrow the energy-momentum tensor from *nonviscous* fluid dynamics and insert it into the Einstein equations. From fluid dynamics we know that in many situations the nonviscous theory is inadequate, in particular if anisotropies, or turbulence effects, are involved. Use of the molecular shear and bulk viscosities means in effect an expansion of the theory to first order in the deviation from thermal equilibrium. Moreover, once turbulence occurs in the velocity field, it is the Reynolds stresses that essentially govern the

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behaviour of the fluid. In view of these known facts from ordinary fluid dynamics we are somewhat reluctant in accepting cosmological theory as an exceptional case wherein the viscosity concepts are of no use. One main reason why the viscous cosmological theory has so far not gained large popularity is undoubtedly that the viscosity-produced entropy – at least in the lepton or plasma eras – seems to be small. The nondimensional entropy σ per baryon is observed to be very large, $\sigma \sim 10^9$, and previous investigations have shown that a straightforward use of the kinematically derived bulk viscosity in the isotropic and homogeneous universe is unable to explain this large magnitude (see, for instance, Weinberg 1971, 1972; Johri and Sudarshan 1988).

The fluidlike behaviour of the universe during its early epochs is however in all probability very complicated. We shall in the following focus attention on two facets of the problem that have so far been little studied:

1) Phase transitions in the very early universe may lead to sudden changes whose description may conveniently be given in terms of phenomenological viscosity coefficients. An eclatant example of this sort is the transition from the de Sitter universe back to the Riemann universe at the end of the inflationary era, at $t \sim 10^{-33}$ s. We shall call an effective viscosity operative during such a phase transition (*bulk* viscosity in case of an isotropic universe) an "impulsive" viscosity. It is conceivable that the impulse-viscosity picture is applicable at later times in the universe's history also, at least at times prior to the lepton era beginning at $t \sim 10^{-5}$ s.

2) It may turn out to be necessary to introduce *turbulence* when the early universe is pictured as a fluid. What we wish to focus attention on, is the possibility one has to introduce the turbulent viscosity concept in cases when the fluid possesses shear. We present a simple picture in which there occurs an "impulse" turbulent shear viscosity in a universe whose anisotropy is governed by the Kasner metric. The impulse viscosity is for definiteness taken to be operative at the beginning of the lepton era.

The organization of the paper is as follows. In the next section we summarize useful information about the general form of the energy-momentum tensor in the presence of expansion, shear, vorticity, and thermal conduction, and give the covariant expression for the rate of entropy production. The formalism is thereafter applied to the Friedmann–Robertson–Walker (FRW) metric; the Friedmann equations, and the Saha equation, are discussed. In Section 3 we consider the entropy production during the plasma era, basing the analysis on the kinematically derived expressions for the viscosity coefficients by Caderni and Fabbri (1977), van Leeuwen *et al.* (1973, 1975) and others. The numerical results derived by solving the Friedmann equations are qualitatively in agreement with those obtained earlier by other workers: the amount of produced entropy predicted by this kind of formalism in the plasma era is far too small to account for the large observed value of σ. In Section 4 we then turn to the "impulse" viscosity idea, requiring that a bulk viscosity ζ_{infl} is operative at the end of the inflationary period. We find that if $\zeta_{\mathrm{infl}} \simeq 10^{60}\ \mathrm{g\,cm^{-1}\,s^{-1}}$, the produced entropy in the phase transition is of the right magnitude. Finally, in Section 5 we reconsider the impulse viscosity idea but now from a different point of view, namely as a shear turbulent viscosity in a universe which is made anisotropic by adoption of the Kasner metric. We base this analysis upon analogy with the immense usefulness of the turbulence viscosity concept in classical fluid mechanics, in cases when the fluid possesses shear. It is well known that in fluid mechanics the mere use of molecular viscosity coefficients would in many cases be quite inadequate; it becomes necessary to model the turbulent Reynolds stresses in some way, and here the use of turbulent viscosities quite generally stands out as a very simple and effective tool. In case of the Kasner universe, we find $\eta_{\mathrm{turb}} \simeq 5 \times 10^{31}\ \mathrm{g\,cm^{-1}\,s^{-1}}$ operative at the beginning of the lepton era corresponds to the correct magnitude of σ.

We use cgs units, although the speed of light is taken to be unity in the theory. Full cgs units are reinstated in expressions requiring actual evaluation.

Finally, as already mentioned, the viscosity concept in cosmology has been looked upon from quite different points of view. Without going into any detail, let us just mention three different lines of thought that have been discussed in the recent literature.

First, is has been argued that there are strong similarities between isotropic cosmological theories involving bulk viscosity coefficients and theories involving creation of matter. As Lima and Germano (1992) have pointed out, if one requires equality between the dynamic pressures predicted by the two theories, then one can express the particle creation rate in terms of known parameters and moreover obtain the same kind of cosmological solution as in the viscous model of Murphy (1973). [The paper of Murphy sent shock waves through the cosmological community since it claimed the initial singularity to be avoidable by introducing the bulk viscosity concept in the formalism. Although Belinskii and Khalatnikov (1975, 1977) later criticised Murphy's model for corresponding to very peculiar parameter choices, the model has nevertheless continued to attract interest during the years.] Theories involving creation of matter have been discussed from a thermodynamic point of view by Prigogine and coworkers (Prigogine, 1989; Prigogine *et al.*, 1989), and by Calvao *et al.* (1992).

Secondly, it is natural to link the cosmological viscosity to the dark matter problem, the explanation of which is one of the great challenges at present. About 90 percent of the matter in the universe is non-luminous. It is currently thought that massive dark matter particles (such as WIMPs) may well account for the missing mass. [See, for instance, the recent review article by Pretzl (1993).] If the mixture of ordinary (luminous) and dark matter can be considered as an imperfect fluid, then the situation is immediately open to a natural use of the cosmological viscosity concept. Pavon and Zimdahl (1993) have discussed the magnitude of the bulk dissipative stress which can be ascribed to the presence of dark matter. It is also worth noticing that recent numerical calculations seem to select a mixture of hot and cold dark matter (Davis *et al.*, 1992; Taylor and Rowan-Robinson, 1992).

As the third and final point we mention the possibility of taking magnetic fields into account in viscous cosmology. In a series of papers, Coley and Tupper (1983, 1984) and Benton and Tupper (1986) have discussed cosmological models of this sort and shown that it is possible to satisfy the Einstein equations, the dominant energy condition (Hawking and Ellis, 1973), and the positivity conditions for the transport coefficients. Moreover, the authors show that their solutions satisfy the relativistic thermodynamic relations in the form originally given by Eckart (1940). In general, it seems that the viscous magnetic models deserve to be studied, since there is at present a weak cosmic magnetic field which may be the remains of a strong magnetic field in the plasma era.

2. Basic Formalism

2.1. Definition Equations. Energy–Momentum Tensor

We use the convention in which the Minkowski metric is (-+++). Greek indices are summed from 0 to 3, Latin indices are summed from 1 to 3. Let $U^\mu = (U^0, U^i)$ be the four-velocity of the cosmic fluid. In comoving coordinates, $U^0 = 1$, $U^i = 0$.

Let $g_{\mu\nu}$ be a general metric. Using the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \tag{1}$$

we define the rotation tensor as

$$\omega_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta U_{[\alpha;\beta]} = \frac{1}{2}(U_{\mu;\alpha} h_\nu{}^\alpha - U_{\nu;\alpha} h_\mu{}^\alpha) \tag{2}$$

and the expansion tensor as

$$\theta_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta U_{(\alpha;\beta)} = \frac{1}{2}(U_{\mu;\alpha} h_\nu{}^\alpha + U_{\nu;\alpha} h_\mu{}^\alpha). \tag{3}$$

The scalar expansion is $\theta = \theta_\mu{}^\mu = U^\mu_{;\mu}$. The shear tensor, as defined by

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\theta, \tag{4}$$

is traceless, i.e. $\sigma_\mu{}^\mu = 0$. The following decomposition of the covariant derivative of the fluid velocity is often useful:

$$U_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} h_{\mu\nu}\theta - A_\mu U_\nu. \tag{5}$$

Here $A_\mu \equiv \dot{U}_\mu = U_\nu U_{\mu;\nu}$ is the four-acceleration of the fluid.

Let us write down the expression for the energy-momentum tensor $T_{\mu\nu}$ of the viscous fluid, taking into account also the conduction of heat. Let η be the shear viscosity, ζ the bulk viscosity, and κ the thermal conductivity, all quantities taken in accordance with their nonrelativistic definitions. Then, if

$$Q^{\mu} = -\kappa h^{\mu\nu}(T_{,\nu} + TA_{\nu}) \tag{6}$$

denotes the spacelike heat flux density four-vector, we have

$$T_{\mu\nu} = \rho U_{\mu}U_{\nu} + (p - \zeta\theta)h_{\mu\nu} - 2\eta\sigma_{\mu\nu} + Q_{\mu}U_{\nu} + Q_{\nu}U_{\mu}, \tag{7}$$

where ρ is the mass-energy density and p the isotropic pressure, both taken in the local rest inertial frame. The last term in (6), containing TA_{ν}, is of relativistic origin. If one ignores this term, one is left with $Q^{\mu} = -\kappa h^{\mu\nu}T_{,\nu}$. This expression is defined such that in a local rest inertial frame (designated by a "hat") $Q_{\hat{0}} = 0$, whereas $Q_{\hat{i}} = -\kappa T_{,\hat{i}}$ is the heat energy per unit time crossing a unit surface orthogonal to the unit vector $\mathbf{e}_{\hat{i}}$.

Consider finally the production of entropy. It is here physically instructive to start from the nonrelativistic theory. Let u_i be the nonrelativistic velocity components, and let σ be the nondimensional entropy per particle (baryon). The ordinary entropy per unit volume is thus $S = nk_B\sigma$, where n is the baryon number density and k_B is Boltzmann's constant. Then (Landau and Lifshitz, 1959)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{2\eta}{T}(\theta_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\mathbf{u})^2 + \frac{\zeta}{T}(\nabla\cdot\mathbf{u})^2 + \frac{\kappa}{T^2}(\nabla T)^2, \tag{8}$$

where $\theta_{ik} = u_{(i,k)}$. The transition to relativistic theory may be made by means of the effective substitutions

$$\theta_{ik} \to \theta_{\mu\nu}, \quad \delta_{ik} \to h_{\mu\nu}, \quad \nabla\cdot\mathbf{u} \to \theta, \quad -\kappa T_{,k} \to Q_{\mu}, \tag{9}$$

from which we obtain

$$S^{\mu}_{;\mu} = \frac{2\eta}{T}\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{\zeta}{T}\theta^2 + \frac{1}{\kappa T^2}Q_{\mu}Q^{\mu}\,. \tag{10}$$

Here, S^{μ} is the entropy current four-vector

$$S^{\mu} = nk_B\sigma U^{\mu} + \frac{1}{T}Q^{\mu}. \tag{11}$$

The same result follows from a more careful analysis taking into account the relativistic thermodynamic equations (Weinberg, 1971; Taub, 1978).

2.2. On the Friedmann Cosmology

Let us summarize some of the characteristic properties of this theory: we start from the FRW line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{12}$$

where $R(t)$ is the scale factor and $k = 1, 0, -1$ the curvature parameter. The coordinates are numerated such that $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$. In comoving coordinates $U_{\mu;\nu} = \Gamma^0_{\mu\nu}$, so that in view of standard relations for the Christoffel symbols we have

$$U_{0;0} = 0, \qquad U_{1;1} = \frac{R\dot{R}}{1 - kr^2}, \qquad U_{2;2} = R\dot{R}r^2, \qquad U_{3;3} = R\dot{R}r^2 \sin^2\theta \tag{13}$$

with $\dot{R} = \mathrm{d}R/\mathrm{d}t$. The other covariant derivatives are zero. Equations (13) can alternatively be written as $U^{\mu}_{;\nu} = h_\nu{}^\mu \dot{R}/R$. Using the same equations we see that the rotation and shear tensors both vanish,

$$\omega_{\mu\nu} = \sigma_{\mu\nu} = 0, \tag{14}$$

whereas the scalar expansion is

$$\theta = 3\frac{\dot{R}}{R} \equiv 3H, \tag{15}$$

where H is the Hubble parameter. The fluid's four-acceleration is zero, $A_\mu = 0$. The expression (7) for the energy-momentum tensor yields

$$T_{00} = \rho, \qquad T_{0k} = 0, \qquad T_{ik} = (p - \zeta\theta)g_{ik}. \tag{16}$$

There is thus no conduction of heat in the FRW space, which is a homogeneous space. The entire effect of the bulk viscosity is to reduce the pressure by an amount $\zeta\theta$. Note that, since $g_{00} = -1$, the temperature T which per definition is pertaining to a local rest inertial frame, is in the present case identical to the temperature $T\sqrt{-g_{00}}$; which according to Tolman (1934) is a constant throughout a body if it is in thermal equilibrium.

When applied to the FRW space, Equations (10) and (11) yield

$$S^{\mu}_{;\mu} = \frac{\zeta}{T}\theta^2, \qquad S^0 = nk_B\sigma, \qquad S^i = 0. \tag{17}$$

Consider now the Einstein equations. If the cosmological constant $\Lambda = 0$, we have

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{18}$$

Inserting the energy-momentum tensor (16) we obtain the Friedmann equations for the viscous fluid,

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = 8\pi G p, \tag{19}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G(p - \zeta\theta). \tag{20}$$

Next, the differential conservation equation for energy, $T^{0\nu}_{;\nu} = 0$, yields

$$\dot{\rho} + (\rho + p)\theta = \zeta\theta^2. \tag{21}$$

The conservation equation for baryon particle number is

$$(nU^\mu)_{;\mu} = 0, \tag{22}$$

which means that nR^3 = constant in the comoving frame. Finally from (17) we obtain, when substituting S^μ from (11) and observing that $Q^0 = 0$ in the comoving frame,

$$\dot{\sigma} = \frac{\zeta}{nk_BT}\theta^2 . \tag{23}$$

This gives the rate of change of the nondimensional entropy per particle, in the Friedmann universe.

It seems to be worthwhile to summarize briefly how the value of σ itself is calculated, at the time of recombination. The starting point is the Saha equation for the equilibrium fractional ionization x of hydrogen (Lightman *et al.*, 1975; Zel'dovich and Novikov, 1971):

$$\frac{x^2}{1-x} = \frac{(2\pi m_e k_B T)^{3/2}}{n(2\pi\hbar)^3} \exp\left(-\frac{1}{2}\alpha^2\frac{m_e}{k_BT}\right), \tag{24}$$

where m_e is the electron mass and $\alpha = 1/137$. Recombination occurs when the left hand side of (24) is about unity. This gives us one relation between the temperature T and the number density of baryons n. A second relation between T and n is obtained by requiring the energy density of radiation aT^4 to be equal to the energy density of matter nmc^2 at recombination. Here $a = \pi^2 k_B^4/(15\hbar^3 c^3) = 7.56 \times 10^{-15}\ \mathrm{erg\,cm^{-3}\,K^{-4}}$ is the radiation constant (we ignore the presence of neutrinos). Solving the two equations, we find that the recombination temperature is $T \simeq 4000$ K, the particle number density is $n \simeq 10^3\ \mathrm{cm}^{-3}$, and the entropy per particle is

$$\sigma = \frac{4aT^3}{3k_Bn} \simeq 4 \times 10^9. \tag{25}$$

This large entropy is believed to have lasted up to the present.

The natural question now is: can the entropy (25) be explained on the basis of reasonable assumptions about the bulk viscosity ζ in (23)? In the two following sections we shall consider this problem from two quite different points of view: first, a standard kind of approach consisting in solving the Friedmann equations and using a kinematically derived expression for the bulk viscosity; thereafter, an approach involving the impulse viscosity idea. In both cases, the FRW metric is assumed. The equation of state for the cosmic fluid will be adopted in the conventional form

$$p = (\gamma - 1)\rho, \quad 0 \leq \gamma \leq 2. \tag{26}$$

Here, $\gamma = 1$ corresponds to a pressure-free fluid appropriate to the universe today; $\gamma = 4/3$ corresponds to a radiation dominated universe, and $\gamma = 2$ yields the relativistic Zel'dovich fluid in which the velocity of sound equals the velocity of light. Finally, $\gamma = 0$ corresponds to the peculiar tensile-stress vacuum "fluid" of the de Sitter universe.

The curvature parameter k is not believed to be of main importance in the present problem, and we accordingly put $k = 0$ in the following.

3. Entropy Production in the Plasma Era

It is natural to focus attention first on the plasma era (also called the radiation era). It is defined to start at the time when electron pairs e^+ and e^- annihilate at a temperature of $T = 2m_e c^2/3k_B \simeq 4 \times 10^9$ K. The universe is then about $t = 24$ s old; its total density is $\rho \simeq 2.1 \times 10^3\ \mathrm{g\,cm^{-3}}$, and its baryon density is $n_b \simeq 9.3 \times 10^{21}\ \mathrm{cm^{-3}}$ (Harrison, 1973; Börner, 1988). The annihilation leaves radiation as the dominant constituent of the universe. In addition to a surviving small fraction of electrons, there are also left non-interacting neutrinos, protons, and slowly decaying neutrons. A little later, at $t \simeq 1000$ s ($T \simeq 4 \times 10^8$ K, redshift $z \simeq 10^8$), the nuclear reactions have terminated, and one is left essentially with a fully ionized plasma consisting of the ordinary species of electrons (density n_e) and protons (density n_p), being neutral in the unperturbed state ($n_e = n_p \simeq 10^{19}\ \mathrm{cm^{-3}}$ at $t = 1000$ s). The plasma era lasts for a long time, until recombination of hydrogen occurs at about 4000 K ($z \simeq 1400$) at $t \simeq 4 \times 10^5$ yr.

Let us investigate the following idea: from $t \simeq 1000$ s onwards, when the universe is characterized by ionized H and He in equilibrium with radiation, one may calculate the magnitude of the bulk viscosity ζ, as a function of the temperature, on the basis of conventional kinetic theory. If the Friedmann equations are solved to give the time variation of $n(t)$ and $T(t)$, one can then in principle use (23) to calculate the entropy production during the plasma era. The physical reason for the dissipation in this era is that matter and electromagnetic radiation are not in thermal equilibrium with each other. The mean free path of photons exceeds that of electrons by a factor of 10^8.

Caderni and Fabbri (1977), in their approach to this problem, made use of prior results obtained by van Leeuwen *et al.* (1973, 1975; cf. also de Groot *et al.*, 1980) in relativistic kinetic theory. Here general expressions were given for the transport coefficients in multicomponent fluids. We shall write down the simplifying polynomial approximations that Caderni and Fabbri worked out for η and ζ in the plasma era (cgs units):

$$\eta = \frac{5m_e^6c^8\zeta(3)}{9\pi^3\hbar^3e^4n_e}x^{-4}(1 - 0.305x^{-1} + 8.218x^{-2} - 35.80x^{-3} + 68.90x^{-4} - 61.59x^{-5} + 20.75x^{-6}), \tag{27}$$

$$\zeta = \frac{\pi c^2\hbar^3 n_e}{2^8e^4\zeta(3)}x^3(1 + 2.273x^{-1} - 21.80x^{-2} + 62.54x^{-3} - 92.74x^{-4} + 70.04x^{-5} - 21.08x^{-6}). \tag{28}$$

Here $x = m_ec^2/k_BT$, n_e is replaceable by n_p, and $\zeta(3) = 1.202$ is the Riemann zeta function with 3 as argument. We note that $x = 1$ for $T = 5.9 \times 10^9$ K. In our present case, where the highest temperature in the period is $T \simeq 4 \times 10^8$ K, we see that the corresponding value of x is 14.8. At later times x takes larger values, and as regards order-of-magnitude estimates one can restrict oneself to the prefactors in (27) and (28). It is instructive to calculate η and ζ at the limits of the considered era. In its beginning when $T \simeq 4 \times 10^8$ K, $n_e \simeq 10^{19}$ cm^{-3},

$$\eta_{1000\text{s}} \simeq 2.8 \times 10^{14}\ \text{g cm}^{-1}\,\text{s}^{-1}, \qquad \zeta_{1000\text{s}} \simeq 7.0 \times 10^{-3}\ \text{g cm}^{-1}\,\text{s}^{-1}, \tag{29}$$

whereas at its termination, when $T \simeq 4000$ K, $n_e \simeq 4 \times 10^3$ cm^{-3} (Harrison, 1973), $x = 1.48 \times 10^6$,

$$\eta_{\text{recomb}} \simeq 6.8 \times 10^{9}\ \text{g cm}^{-1}\,\text{s}^{-1}, \qquad \zeta_{\text{recomb}} \simeq 2.6 \times 10^{-3}\ \text{g cm}^{-1}\,\text{s}^{-1}. \tag{30}$$

The shear viscosity is thus vastly greater than the bulk viscosity, even though the first one is diminishing by more than four orders of magnitude throughout the plasma era whereas the latter one stays almost constant.

We now need to calculate how θ and T vary with t in the plasma era. Using the Friedmann equations (19) and (20) for a flat universe ($k = 0$), and the equation of state (26), we obtain the following differential equation for the scalar expansion:

$$\dot{\theta}(t) + \frac{1}{2}\gamma\theta^2(t) - 12\pi G\zeta(t)\theta(t) = 0. \tag{31}$$

We have here allowed for a general time dependence of the bulk viscosity $\zeta(t)$. From now on let us use the subscript "in" to distinguish the initial instant of the plasma period that we consider. Thus, to summarize,

$$t_{\text{in}} = 1000\ \text{s}, \qquad T_{\text{in}} = 4 \times 10^8\ \text{K}, \qquad n_{\text{in}} = 10^{19}\ \text{cm}^{-3}. \tag{32}$$

Moreover, we conform to usual practice in letting subscript zero refer to the present time. The solution of (31) can now be written as (cgs units)

$$\theta(t) = \frac{\theta_{\rm in}\, e^{F(t)}}{1 + \dfrac{\gamma\theta_{\rm in}}{2}\displaystyle\int_{t_{\rm in}}^{t} e^{F(t')}\, dt'}\,, \tag{33}$$

where

$$F(t) = \frac{12\pi G}{c^2}\int_{t_{\rm in}}^{t} \zeta(t')\, dt'. \tag{34}$$

This implies the following scale factor:

$$\frac{R(t)}{R_0} = \frac{R_{\rm in}}{R_0}\left[1 + \frac{\gamma\theta_{\rm in}}{2}\int_{t_{\rm in}}^{t} e^{F(t')}\, dt'\right]^{2/(3\gamma)}; \tag{35}$$

cf. also Johri and Sudarshan (1988) and Padmanabhan and Chitre (1987). The initial value $\theta_{\rm in}$ will be assumed to be given by usual theory for the nonviscous radiation dominated Friedmann universe: since $H = 1/(2t)$ according to that theory, one has

$$\theta_{\rm in} = \frac{3}{2t_{\rm in}} = 1.5 \times 10^{-3}\ {\rm s}^{-1}. \tag{36}$$

From Table 15.4 in Weinberg's book (1972) we moreover adopt the value

$$\frac{R_{\rm in}}{R_0} = 5 \times 10^{-9}. \tag{37}$$

When giving a quantitative estimate for the entropy production in the plasma era, we can safely put $\gamma = 4/3$. This is so because in the major part of this era the universe was radiation dominated. The density n of protons is found from the conservation equation (22) for particle number which implies

$$n(t) = \frac{n_{\rm in}R_{\rm in}^3}{R^3(t)} = \frac{n_0 R_0^3}{R^3(t)}. \tag{38}$$

Because of the radiation dominance we shall take the relation between the density ρ and the temperature T to be

$$\rho(t) = aT^4(t). \tag{39}$$

[It ought to be noted that the quantity ρR^4, which is a constant in a nonviscous radiation dominated universe, is not constant here: manipulating the Friedmann equations (19) and (20) we obtain ($k = 0$):

$$\frac{d}{dR}(\rho R^3) + 3pR^2 = 3\zeta\theta R^2, \tag{40}$$

which, in view of the relationship $p = 1/3\rho$, leads to

$$\frac{\mathrm{d}}{\mathrm{d}R}(\rho R^4) = 3\zeta\theta R^3. \tag{41}$$

The bulk viscosity thus makes ρR^4 time dependent.] Finally we note the relationship

$$\rho(t) = \frac{\theta^2(t)}{24\pi G}, \tag{42}$$

which is actually a rewriting of the Friedmann equation (19) when $k = 0$. [Equation (42) corresponds to $\Omega = 1$.] Imagine now, in principle, that the expression (28) for ζ is inserted into Equation (34). We have then five equations at our disposal for the calculation of the five unknowns θ, R, n, T, and ρ as functions of time in the plasma era, viz. Equations (33), (35), (38), (39) and (42). Fortunately, Equations (29) and (30) tell us that to obtain order of magnitude estimates it is sufficient to put ζ = constant. This simplifies the situation enormously. For definiteness we assume in the following

$$\zeta = \zeta_{\text{in}} = 7 \times 10^{-3}\ \mathrm{g\,cm^{-1}\,s^{-1}}. \tag{43}$$

Introducing for brevity the symbol

$$t_c = \left(\frac{12\pi G}{c^2}\zeta_{\text{in}}\right)^{-1}, \tag{44}$$

we then obtain, for $t_{\text{in}} \leq t < 4 \times 10^5$ yr,

$$\theta(t) = \frac{\theta_{\text{in}}\,\mathrm{e}^{(t-t_{\text{in}})/t_c}}{1 + \frac{1}{2}\gamma\theta_{\text{in}}t_c[\,\mathrm{e}^{(t-t_{\text{in}})/t_c} - 1]}, \tag{45}$$

$$\frac{R(t)}{R_0} = \frac{R_{\text{in}}}{R_0}\left[1 + \frac{1}{2}\gamma\theta_{\text{in}}t_c(\,\mathrm{e}^{(t-t_{\text{in}})/t_c} - 1)\right]^{2/(3\gamma)}. \tag{46}$$

We now insert the calculated expressions for θ, n, and T into expression (23) for the rate of entropy production. Putting $\gamma = 4/3$ the result becomes very simple:

$$\dot{\sigma}(t) = \left(\frac{24\pi aG}{c^2}\right)^{1/4}\frac{\zeta_{\text{in}}\theta_{\text{in}}^{3/2}}{k_B n_{\text{in}}}\,\mathrm{e}^{(3/2)(t-t_{\text{in}})/t_c}. \tag{47}$$

According to this model, the rate of entropy production increases exponentially during the plasma era. Integrating over time from t_{in} to $t_f = 4 \times 10^5$ yr $= 1.26 \times 10^{13}$ s, we obtain the total entropy production in the plasma era

$$\Delta\sigma = (24a)^{1/4}\left(\frac{c^2}{\pi G}\right)^{3/4}\frac{\theta_{\rm in}^{3/2}}{18k_B n_{\rm in}}\left[e^{(3/2)(t_f - t_{\rm in})/t_c} - 1\right]. \tag{48}$$

Since the exponent is very small, we can finally write this as

$$\Delta\sigma = (24a)^{1/4}\left(\frac{c^2}{\pi G}\right)^{3/4}\frac{\theta_{\rm in}^{3/2}}{12k_B n_{\rm in}}\frac{t_f}{t_c}. \tag{49}$$

Now t_c is in our case a very large quantity, $t_c = 5.1 \times 10^{28}$ s, and we end up with

$$\Delta\sigma \simeq 3 \times 10^{-7}, \tag{50}$$

which is extremely small. We conclude that the bulk viscosity derivable from kinetic theory is completely unable to explain the large entropy per baryon observed in the universe.

4. "Impulsive" Viscosity at Inflationary Times

Let us consider quite a different way of explaining the large entropy in the universe, namely to take σ to be a result of the violent GUT phase transitions that one believes took place in the very early universe, at inflationary times. We recall the usual picture for the very early universe (see, for instance, Grøn 1986; Soleng 1987): From big bang at $t = 0$ until Planck time at about 10^{-43} s, there was the quantum era about which virtually nothing is known. Classical spacetime acquires a meaning onwards from the instant when the action integral for the geometry becomes of the order of Planck's constant. This occurs at $t \simeq 10^{-39}$ s, thus larger that the Planck time but smaller than the time for a typical GUT inflation which starts at $t_1 \simeq 10^{-39}$ s and lasts until $t_2 \simeq 1.4 \times 10^{-33}$ s. The inflationary period is characterized by a de Sitter geometry with line element

$$ds^2 = -dt^2 + R_1^2 \exp\left(2\sqrt{\frac{\Lambda}{3}}\,t\right)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{51}$$

according to which there is an exponential scale factor $R(t) = R_1 \exp(\sqrt{\Lambda/3}\,t)$. We have here required continuity of R at the starting point $t = t_1$, when compared with the pre-inflationary $k = 0$ Riemann metric, Equation (12). During inflation, the vacuum acts as "fluid" under extreme tensile stress; its equation of state is $p = -\rho$, in accordance with the Einstein equations which are

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{52}$$

when the vacuum energy outweighs the matter energy completely. In this brief period from t_1 to t_2, an enormous supercooling takes place in the universe; the temperature drops "smoothly" from about 10^{27} K to about 10^{22} K, whereafter there occurs a sudden (first order) phase transition back to a Friedmann universe of an ordinary radiation dominated type at $t = t_2$.

Let us investigate the following idea: Ignore any other viscosity in the universe than the "impulsive" bulk viscosity

$$\zeta(t) = \zeta_{\text{infl}} t_2 \delta(t - t_2), \tag{53}$$

imagined to be operative at the end of the inflationary period, and estimate the amount of entropy σ produced per baryon by integrating Equation (23) over time from $t = t_2-$ to $t = t_2+$:

$$\sigma = \frac{\zeta_{\text{infl}} t_2}{k_B} \left[\frac{\theta^2}{nT} \right]_{t_2+} . \tag{54}$$

We have here made what seems to be a reasonable choice, namely to take the appropriate value of θ^2/nT to be inserted in (23) to be the Riemannian value just *after* the termination of the phase transition. The natural question is: What is the magnitude of the "effective" viscosity ζ_{infl} that will lead to the correct entropy, $\sigma \simeq 4 \times 10^9$?

The value of $R(t)$ after the inflationary period is believed to be the same as if no inflation were present. In our context it is natural to relate the physical quantities at $t = t_2+$ to those at $t = t_{\text{in}} = 1000$ s, considered earlier. For $t_2 < t < t_{\text{in}}$ the universe is assumed nonviscous and radiation dominated, so that $R(t) \propto t^{1/2}$. The conservation of particle number implies $n_2 R_2^3 = n_{\text{in}} R_{\text{in}}^3$, whereas the conservation of entropy in this time interval implies $R_2 T_2 = R_{\text{in}} T_{\text{in}}$. Thus σ can be written

$$\sigma = \frac{9 t_2 \zeta_{\text{infl}}}{4 k_B n_{\text{in}} T_{\text{in}} t_{\text{in}}^2} . \tag{55}$$

Requiring this to be equal to 4×10^9 we then obtain, by using the data of Equation (32),

$$\zeta_{\text{infl}} \simeq 10^{60} \text{ g cm}^{-1} \text{ s}^{-1}. \tag{56}$$

An impulse viscosity of this magnitude at the end of the inflationary era is thus reconcilable with the observed entropy in the universe.

5. Anisotropy and Turbulent Viscosity

As a general remark, from a fluid dynamical viewpoint the introduction of viscosity coefficients in cosmological theory is usually done in an almost surprisingly simple manner. There are two reasons why we consider this to be so. First, the coefficients η and ζ are the "molecular" viscosities, known to be important for laminar flows in fluid dynamics but usually incapable of describing processes such as energy dissipation once *turbulence* sets in. In such a case, one has to take also the Reynolds stresses (usually written as $-\rho\overline{u_i' u_k'}$) into account. A realistic description of a complex turbulent flow will often imply a highly nontrivial Reynolds number-dependent parameterization in the governing equations. Secondly, even if the cosmological theory is formulated on the laminar level, the limitation to a completely *isotropic* space, as following from the FRW metric, is in our view an extreme idealization: from Equation (10) it is apparent that in the presence of shear *both* viscosity coefficients, η as well as ζ, contribute to the entropy production. Moreover, at least in the situation discussed above, Equation (29), the magnitude of η is enormously larger than that of ζ. In view of these circumstances it appears very natural to modify the standard cosmological theory in such a way that an initial anisotropy is allowed for in the early universe, and thereafter relate this anisotropy to the production of entropy via Equation (10).

As anisotropic space we shall choose the Kasner space which, similarly to the $k = 0$ FRW space, is of Bianchi type-I. The Kasner line element is

$$ds^2 = -dt^2 + t^{2p_1}\,dx^2 + t^{2p_2}\,dy^2 + t^{2p_3}\,dz^2, \tag{57}$$

where p_1, p_2, p_3 are constant parameters satisfying the relations

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{58}$$

This space, in spite of its anisotropy, is homogeneous. The Kasner solution is a quite interesting model for the very early universe. The metric (57) is an exact solution of the vacuum Einstein equations $G_{\mu\nu} = 0$, which are believed to be the appropriate governing equations in the very early universe. At later times, when the matter terms begin to become important in the energy-momentum tensor, there occurs a gradual transition to the $k = 0$ FRW universe (Misner *et al.*, 1973, p. 802). This transition is believed to take place within, or near to, the lepton era, i.e., 10^{-5} s to 24 s, or equivalently 10^{12} K to 4×10^9 K.

The metric (57) implies, with the obvious numeration $(x, y, z) \to (x^1, x^2, x^3)$,

$$\omega_{\mu\nu} = 0, \qquad \theta = \frac{1}{t}, \tag{59}$$

$$\begin{aligned} \sigma_{11} &= \left(p_1 - \frac{1}{3}\right) t^{2p_1-1}, \qquad & \sigma_{22} &= \left(p_2 - \frac{1}{3}\right) t^{2p_2-1}, \\ \sigma_{33} &= \left(p_3 - \frac{1}{3}\right) t^{2p_3-1}, \qquad & &\text{other } \sigma_{\mu\nu} = 0. \end{aligned} \tag{60}$$

These equations, together with (58), imply $\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = 1/(3t^2)$. Thus, the contribution to the entropy production from the shear viscosity in the Kasner universe is independent of which set of values is actually given to the parameters p_1, p_2, p_3.

We now take into account the possible presence of turbulence in the universe in the following indirect way. The key point is the turbulent viscosity concept

η_{turb}, borrowed from classical fluid dynamics. We may notice from this classical theory that η_{turb} is usually introduced via the generalized Boussinesq hypothesis for incompressible flow:

$$-\rho\overline{u_i'u_k'} = \eta_{\text{turb}}(u_{i,k} + u_{k,i}) - \frac{2}{3}\rho K\delta_{ik}. \tag{61}$$

Here u_i is the averaged velocity, u_i' its fluctuating part, and $K = \frac{1}{2}\overline{u_i'u_k'}$ is the turbulent kinetic energy per unit mass. (Equation (61) is actually the basis for the frequently used $K - \varepsilon$ method in computational fluid dynamics, ε denoting the energy dissipation.) If we assume that it is the shearing stresses that are the dominant stresses here, then we see that for $i \neq k$, Equation (61) becomes formally the same as in the case of laminar incompressible flow. The molecular viscosity is relatively unimportant in the presence of turbulence, and so the averaged Navier–Stokes equations become formally the same as in the laminar case, only with η_{turb} in place of η.

It ought to be emphasized that we expect n_{turb} to be a large quantity; the inclusion of turbulence will normally imply that the viscosity increases by several orders of magnitude. An example, again taken from classical fluid dynamics, is so typical in this respect that it is worthwhile to include it here. Consider the classic oscillating water-tunnel experiments of Jonsson (1963) and Jonsson and Carlsen (1976), where water was forced to oscillate horizontally above a rippled bed. In such a case there is established an overlap layer, several centimetres thick, in which the turbulent viscosity at height z above theoretical bed level can be written as $\eta_{\text{turb}} = 0.40\rho u_* z$. Here 0.40 is the von Karman constant and u_* is the friction velocity. A theoretical analysis of the Jonsson-Carlsen experiment (Brevik, 1981) shows that η_{turb} can be as large as about 25 g cm^{-1} s^{-1}, thus 2500 times as large as the molecular viscosity $\eta = 0.01$ g cm^{-1} s^{-1} for water.

After this detour into classical fluid dynamics we return to cosmology. In accordance with the remarks above we suggest that the turbulent generalization of the entropy production Equation (10) is

$$S^{\mu}_{;\mu} = \frac{2}{T}\eta_{\text{turb}}\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{62}$$

The bulk viscosity is now neglected; we base this assumption upon analogy with the dominant role played by the turbulent shear viscosity in classical fluid dynamics. The last term to the right in (10) does not give any contribution: the acceleration $A_\mu = 0$ in the Kasner space, and Q^μ, which according to (6) becomes proportional to the spatial gradients of T, vanishes since the Kasner space is homogeneous. In view of the relationship $\sqrt{-g} = t$ the conservation Equation (22) for particle number implies $(nt)_{,0} = 0$, which in turn implies $S^{\mu}_{;\mu} = k_B n\dot{\sigma}$. Thus Equation (62) can finally be written

$$\dot{\sigma} = \frac{4}{3nk_BTt^2}\eta_{\text{turb}}. \tag{63}$$

In principle, $\eta_{\rm turb}$ may depend on t. We note again that the entropy production is independent of which values are given to p_1, p_2 and p_3.

The actual value of $\eta_{\rm turb}$ in (63), in analogy with the situation of classical fluid dynamics, has to be calculated on the basis of comparison with observations. Very little seems to be known about the behaviour of viscosity in the Kasner universe. In order to get an estimate about the magnitude of $\eta_{\rm turb}$ we shall follow an approach similar to that of the previous section, namely to interpret $\eta_{\rm turb}$ as an "impulsive" viscosity acting at a phase transition in the early universe. For definiteness we shall locate this sudden transition to take place at the border between the lepton era and the previous epoch (at later times, the correctness of the standard Riemannian model is almost universally trusted). The change in entropy $\Delta\sigma$ in the phase transition can according to (63) be written as

$$\Delta\sigma = \frac{4\eta_{\rm turb}}{3k_B}\left[\frac{1}{nTt}\right], \tag{64}$$

where [ ] signifies the beginning of the lepton era. Using the data of Harrison (1973) we have at this instant $n = 6 \times 10^{29}$ cm^{-3}, $T = 10^{12}$ K, $t = 2 \times 10^{-4}$ s. If we moreover require $\Delta\sigma$ in (63) to be equal to the real entropy content $\sigma \simeq 4 \times 10^9$, we obtain for the effective turbulent viscosity

$$\eta_{\rm turb} \simeq 5 \times 10^{31}\ {\rm g\,cm^{-1}\,s^{-1}}. \tag{65}$$

This viscosity is seen to be much smaller that the previous inflationary-related viscosity (56), as one might expect.

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