AN EXACT ANALYTICAL SOLUTION IN RADIATION GAS DYNAMICS

(Letter to the Editor)

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Abstract. The Brinkley–Kirkwood theory, as modified by Bhatnagar and Kushwaha for the inclusion of radiation pressure, is applied to obtain an exact analytical solution for radiation pressure, shock velocity, etc., when a strong explosion takes place in a cold undisturbed gas obeying an exponential density distribution. Cases involving spherical symmetry, axial symmetry or spheroidal symmetry are also considered.

1. Introduction

In this paper we review the problem of the effects of a strong explosion in different types of exponential medium when radiation pressure is considered to be important. This study is of particular interest in the theory of the formation of novae and supernovae. Many attempts have already been made to solve different types of problems relative to shock wave and stellar structure.

In the present letter we have applied the Brinkley–Kirkwood theory (1947) modified by Bhatnagar and Kushwaha (1961) to study the effects of a strong explosion in an undisturbed media where the density law obeys a relation of the type

$$\rho_0 = \rho_c \, e^{-rf(\theta)} .$$

Similar cases have been studied in detail by Laumbach and Probstein (1969, 1970a, b), Oppenheim *et al.* (1975), DebRay and Bhowmick (1974), Bhowmick (1975), Sakashita (1971) and others. Radiation effects are not taken into account by any of these authors, although the effect of radiation flux was considered by Laumbach and Probstein (1970a).

We have considered the effect of radiation in the form of radiation pressure, disregarding the radiation flux. The effective ratio of specific heats when radiation pressure is important is taken into account to study the problem (cf. Pai, 1966).

We have obtained an exact analytical solution for radiation pressure, gas pressure and, hence, shock velocities when radiation is important and when it is not important. The ratio of these two shock velocities is obtained.

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2. Theory

Following Bhatnagar and Kushwaha (1961), we have

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\nu \frac{x^\alpha p^3}{D(x)} \frac{1}{\rho_0 U^2} \frac{G}{2(k+1)-G} - \frac{\alpha p}{2x} \frac{4\frac{\rho_0}{\rho} + 2\left(1 - \frac{\rho_0}{\rho}\right) G}{2(k+1)-G}, \tag{1}$$

where

$$G = 1 - \left(\frac{\rho_0 U}{\rho_c}\right)^2 ; \; k = \rho_0 U \frac{\mathrm{d}u}{\mathrm{d}p} ; \; c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{2}$$

and

$$\frac{\mathrm{d}D}{\mathrm{d}x} = -x^\alpha \rho_0 h(p), \tag{3}$$

in which D is the energy of explosion of the shock wave ρ_0 the density in the undisturbed medium, ρ the density just behind the shock front, p the combined radiation pressure and material pressure at the shock, $\alpha = 0$, 1, 2 for plane, cylindrical and spherical cases, respectively, U the shock velocity, and ν is a similarity parameter.

We now assume the shock to be infinitely strong. Following Sachdev (1972) and Pai (1966) we obtain the single differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}R} + \frac{2\gamma_e}{5\gamma_e - 1}\left[-\frac{\mathrm{d}}{\mathrm{d}R}(\log \rho_0)\right] + \frac{1}{R} \frac{2(2\gamma_e^2 - \gamma_e + 1)}{(\gamma_e + 1)\gamma_e} p = -\frac{8\pi\nu\gamma_e}{E(5\gamma_e - 1)} R^2 p^2, \tag{4}$$

$$\gamma_e = \frac{4(\gamma - 1)Rp_2 + \gamma}{3(\gamma - 1)Rp_2 + 1} = \text{effective ratio of specific heats in radiation gas dynamics}, \tag{5}$$

$$R_{p_2} = \frac{\text{radiation pressure}}{\text{gas pressure}} \text{ at the shock front}, \tag{6}$$

$$\gamma = \text{ratio of specific heats}, \tag{6}$$

p = radiation pressure p_r + material pressure p_m,

$\alpha = 2$, and D is replaced by $E/4\pi$

$$\frac{\rho}{\rho_0} = \frac{\gamma_e + 1}{\gamma_e - 1} ; \; \frac{p}{\rho_0} = \frac{2}{\gamma_e + 1} U^2 ; \; c^2 = \gamma_e \frac{p_r + p_m}{\rho} \tag{7}$$

and

$$h(p) \to 0 \quad \text{(Schatzman assumption)}.$$

We now consider models for which $f(\theta) = \Theta$ and, therefore,

$$\rho_0 = \rho_e \, \mathrm{e}^{-r_0\Theta} ;$$

where

$$\Theta = \pm 1 , \tag{8a}$$

$$\Theta = \pm \frac{\cos\theta}{\Delta} , \tag{8b}$$

$$\Theta = \pm \left[1 + \frac{e^2}{1-e^2} \cos^2\theta \right]^{1/2} ; \tag{8c}$$

and θ is the angle measured from the vertical axis of symmetry. Now, putting $\gamma_e = R =$ the shock radius, after using the transformation

$$\xi = R\Theta \tag{9}$$

we find that

$$\frac{\mathrm{d}p}{\mathrm{d}\xi} + \frac{2\gamma_e}{5\gamma_e - 1}\left[1 + \frac{1}{\xi}\frac{2(2\gamma_e^2 - \gamma_e + 1)}{(\gamma_e + 1)\gamma_e}\right] p = \frac{-8\pi\nu\gamma_e}{E(5\gamma_e^{-1})}\frac{\xi^3}{\Theta^3} . \tag{10}$$

After further transformation, given by

$$P = \frac{4\pi\nu}{E\Theta^3} p , \quad -\rho = \frac{1}{p} ;$$
$$B = \frac{2\gamma_3}{5\gamma_e - 1} , \quad A = \frac{2(2\gamma_e^2 - \gamma_e + 1)}{(\gamma_e + 1)\gamma_e} ; \tag{11}$$

Equation (10) assumes the simple form

$$\frac{\mathrm{d}\rho}{\mathrm{d}\xi} - B\left[1 + \frac{A}{\xi}\right]\rho = -B\xi^2 . \tag{12}$$

3. Solution

An exact analytical solution of Equation (12) can be obtained in the form

$$P = \frac{-1}{\xi^2 + \frac{2-AB}{B}\xi + \frac{(2-AB)(1-AB)}{B^2(1+1/B\xi)} AB} \tag{13}$$

or

$$p = \frac{-E\Theta^3/4\pi\nu}{\xi^2 + \frac{2-AB}{B}\xi + \frac{(2-AB)(1-AB)}{B^2(1+1/B\xi)} AB} . \tag{14}$$

This is an exact analytical expression for radiation pressure and gas pressure, valid for Equation 8(a), 8(b) and 8(c).

If we put $p_r = 0$, then $\gamma_e = \gamma$ from Equation (5). If, for monoatomic gases, we put $\gamma = \frac{5}{3}$, then there is no radiation pressure and the expression for material pressure is given by

$$p_m = \frac{-E\Theta^3/4\pi\nu}{\xi^2 + \frac{11}{5}\xi}. \tag{15}$$

However, for $P_m \to 0$, we have

$$\gamma = \tfrac{5}{3} \text{ and } \gamma_e = \tfrac{4}{3} \text{ when } R_p \to \infty .$$

Therefore, an exact analytical expression for radiation pressure is given by

$$p_r = \frac{-E\Theta^3/4\pi\nu}{\xi^2 + \frac{61}{28}\xi + \frac{183}{1568}(1/(1+17/8\xi)^{116/119})}. \tag{16}$$

Expressing the pressure in terms of shock velocity, from (15) and (16) we obtain

$$U_1^2 = \frac{-E\Theta^3}{4\pi\nu}\,\frac{4}{3\rho_c}\,\frac{e^{\xi}}{\xi^2 + \frac{11}{5}\xi}. \tag{17}$$

Similarly,

$$U_2^2 = \frac{-E\Theta^3}{4\pi\nu}\,\frac{7}{6\rho_c}\,\frac{e^{\xi}}{\xi^2 + \frac{61}{28}\xi + \frac{183}{1568}(1/(1+17/8\xi)^{116/119})}, \tag{18}$$

Dividing (17) by (18), we have

$$\frac{U_1^2}{U_2^2} = \frac{8}{7}\,\frac{1 + \frac{61}{28}\,\frac{1}{\xi} + \frac{1}{\xi^2}\,\frac{183}{1568}/(1 + \frac{17}{8}\,\xi)^{116/119}}{1 + \frac{11}{5}\,\frac{1}{\xi}}, \tag{19}$$

where U_1 is the shock velocity when radiation pressure is not important, and U_2 is the shock velocity when radiation pressure is of importance.

4. Discussion

The results in (15), (16), (17) and (18) are valid when the density is increasing exponentially. Of course, (19) is valid always. If we take $\Theta = -1$, $\rho_0 = \rho_c\, e^R$, all these results are valid.

When $\Theta = (R \cos \theta)/\Delta$, where Δ is some scale height, the results represented by Equations (15)–(18) are valid for θ between $\pi/2$ and π – i.e., when the explosion has taken place at certain heights and explosion waves move downwards.

In the case of spheroidal symmetry, we must take

$$\Theta = -\left[1 + \frac{e^2}{1-e^2}\cos^2\theta\right]^{1/2}$$

to obtain real values of the expressions (15)–(18). Of course, the ratio of shock velocities given by Equation (19) continues to remain valid in all cases. If we remember that $\xi = R\Theta$, where Θ is defined in Equations (8), $\xi \to 0$ means that either $R \to 0$ or $\Theta \to 0$ whereas $\xi \to \infty$ means that only $R \to \infty$ *as* Θ is bounded above.

When radiation is important, the shock velocity tends to infinity almost in the same way as the shock velocity when radiation is not of importance (i.e., when $\xi \to 0$.

The ratio of two shock velocities tend to a constant value as $\xi \to \infty$, as seen from Equation (19) which gives

$$\frac{U_1^2}{U_2^2} \to \tfrac{8}{7} \text{ as } \xi \to \infty .$$

As evident from the structure of the solution and its discussion, the behavioural pattern of the pressure and shock velocity structure will be similar regardless of the effects of radiation.

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