A METHOD OF COMPUTING PERIODS OF CYCLIC PHENOMENA

I. JURKEVICH

Flower and Cook Observatory, University of Pennsylvania, Philadelphia, Pa, and Space Sciences Laboratory, General Electric Company, Valley Forge, Pa, U.S.A.

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Abstract. Use of an analysis of Expected Mean Square Deviations to search for periodicities in an observational data sample is described. The statistic for testing the null hypothesis of non-periodicity is derived from a partitioning of the total sum of squared deviations from the mean.

Unlike most existing methods, the present one does not require equally spaced observations. No assumptions are made concerning the statistical nature of spacing intervals.

The method is illustrated by numerical examples.

1. Introduction

The need for precisely determining periods of rhythmic phenomena is well known and numerous methods have been devised for this purpose (e.g., Stumpff, 1937; Tsessevich, 1947; Kozik, 1964; Kwee and van Woerden, 1956; Lafler and Kinman, 1965; Blackman and Tukey, 1959; Deeming, 1970; and others). With the exception of the methods due to Lafler and Deeming, available methods are not suitable for investigating sporadically observed phenomena or phenomena observed at irregularly spaced intervals. Yet, many observational procedures in astronomy preclude, by their very nature, achievement of equally spaced observations. Existing techniques of time series analysis, so powerful for equally spaced observations, could be applied to irregularly spaced data by replacing actual observations by new data generated from the original set by interpolation. Effects of such a procedure are generally not known and, therefore, this approach is at best an artificial expedient.

Furthermore, the availability of high speed computers makes the use of equidistant ordinates of lesser importance. Consequently, for unequally spaced data, the development of methods directly applicable to the problem is desirable. It is fully realized that, in doing so, we sacrifice mathematical elegance and computational economy of standard methods and may, in some cases, lose a significant fraction of the body of mathematical proofs justifying certain computational results.

The problem at hand is basically a special case of a broader class of problems dealing with the spectral analysis of irregularly observed time series. Essentially, there exists no literature on this subject. In fact, the entire mathematical literature of this field consists of approximately three papers; one by Parzen (1963), another by Bloomfield (1970), and a recent one by Shaw (1971).

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The present method, developed for a special case of randomly spaced observations in 1964 (Jurkevich, 1964), differs from some described subsequently (Lafler and Kinman, 1965; Deeming, 1970), primarily in the statistic that is employed to establish periodicity. In practice, the statistic based on the Expected Mean Square Deviation is believed to be statistically more natural and possibly more sensitive in detecting periodicities than those used by Lafler and Deeming. Although ideas underlying these three methods are straightforward and have been used in part by unsophisticated presentations of rhythmic data, e.g., as a graph of means and standard errors of data from successive cycles overlying one another, a thorough discussion of a properly generalized approach is not available.

The remainder of the paper is concerned with the description of the proposed method.

2. Periodicity Search in Terms of Expected Mean Squares

A. GENERAL REMARKS

The purpose of the proposed analysis is to determine whether a time series observed at irregularly spaced intervals contains periodicities. As in all existing methods, the essential step is to construct the curve exhibiting the observed phenomenon as a function of phase for an assumed period, and then to decide whether the result yields an acceptable representation of observations. The quantity to be used in examining the fit, henceforth referred to as a statistic, must be selected so as to exhibit the properties of the periodic variation of observations as sharply as possible in the presence of observational errors. Once the statistic has been established, the period is obtained by examining the fit for a range of trial periods until the best representation is obtained.

B. SELECTION OF THE STATISTIC

Consider a data sample of size N and denote the individual observation by x_i, the overall mean by $\bar{x}$, the overall sum of squared deviations by V^2, and the expected value of the overall variance by S^2. These quantities are given by

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i ,$$

$$V^2 = \sum_{i=1}^{N} x_i^2 - N\bar{x}^2 ,$$

$$S^2 = \frac{V^2}{N-1} .$$

If the data sample is now divided into m groups, the corresponding statistical parameters of the lth group are given by

$$\bar{x}_l = \frac{1}{m_l} \sum_{i=1}^{m_l} x_i ,$$

$$V_l^2 = \sum_{i=1}^{m_l} x_i^2 - m_l \bar{x}_l^2 ,$$

$$S_l^2 = \frac{V_l^2}{m_l - 1} ,$$

where m_l is the number of observations in the lth group. The sum of the squared deviations in m groups is then given by

$$V_m^2 = \sum_{i=1}^{m} V_l^2 . \tag{1}$$

By virtue of the well-known theorem on the addition of variances, we generally have

$$V_m^2 < V^2 .$$

A consequence of this theorem is the fact that, when several samples of the same argument are combined into a single sample, the expected variance of the latter exceeds the mean of group variances if the group means are different, or is equal to the mean of group variances if the group means are identical. In any specific case, the amount by which V^2 exceeds V_m^2 is computed from

$$V_{BG}^2 = \sum_{i=1}^{m} m_l (\bar{x}_l - \bar{x})^2 . \tag{2}$$

A little reflection shows that, for a given data sample which is to be analyzed for periodicities, the quantity V^2 is not a function of the trial period. However, parameters V_l^2 and $\bar{x}_l$ are very sensitive functions of the latter. For instance, if the trial period is simply incorrect, phase-reduced observations, when represented by a curve, scatter irregularly over the diagram. In this case, group means and variances are likely to be nearly the same and the quantity V_{BG}^2 is quite small relative to V_m^2. As one approaches the true period, the group means become quite different; V_{BG}^2 becomes considerably larger than V_m^2 which, in turn, must decrease relative to V^2. In fact, when the trial period is exactly equal to the true period, V_m^2 reaches its minimum and V_{BG}^2 its maximum. The reduction of V_m^2 relative to V^2 can conceivably be caused by statistical fluctuations. A decision on whether this is so or not can be made either by examining the fit of the phase diagram or, since observational errors can safely be assumed to be normally distributed, by the use of the statistical F-test. In fact, the possibility of using this test provided the primary motivating factor for selecting Expected Mean Squares as the test statistic.

It must be mentioned, however, that within the present formulation, the F-test must be used with caution. All cells into which the period under investigation is divided are assumed to contain observations. For a reasonably large data sample and a relatively modest number of groups (e.g., $N=200$, $m=10$), this condition was found to be satisfied. If, in addition, the observational noise is 'white', the use of the F-test is meaningful. In general, m, the number of groups containing observations, is itself a random variable and, consequently, for a given choice of period P, the degrees of freedom of the F ratios are random variables which follow essentially a Poisson distribution. Another complication is essentially non-normal distribution of means within groups which arises because the times, in the case of random spacing of observations, are randomly distributed within a selected group interval. Hence, the distribution of means within groups would have a variance component above that due to noise. All this could drastically affect the use of the F-test and possibly render invalid the testing of the null hypothesis of non-periodicity. Such difficulties, connected with the validity of statistical theory under given circumstances, do not arise if the decision about periodicity is based on the quality of the phase diagram. The quantity V_m^2 exhibits a sharp minimum with relatively broad wings in the vicinity of the true period. This minimum is taken as an indication that, for the trial period in question, the best fit to the phase diagram has been obtained.

In the following paragraphs we shall consider several properties of the selected statistic.

C. LIMITING VALUES OF DETECTABLE PERIODS

It is well known that no method of periodicity analysis can yield periods either much shorter than the precision to which observation times are given or much longer than the total interval over which observations are scattered.

Although not surprising, the proposed method clearly exhibits these two facts.

The phase, φ, of a given observation is computed from

$$\frac{t - t_0}{P} \pmod 1 \quad \text{such that} \quad 0 \leqslant \varphi \leqslant 1$$

with negative arguments being allowed. In the above, t represents the time of an observation and t_0 denotes time origin. It is clear that the phase will vanish whenever $(t-t_0)/P$ is exactly equal to its own characteristic, that is, whenever the mantissa of the resulting number is zero. This will occur for any period such that $P \leqslant 10^{-k}$ where k is the number of decimal digits in $(t-t_0)$. Since for any such P the resulting phase is zero, every observation will be assigned to the same group. Thus, all group variances vanish except one, and its variance is equal to the overall variance. Clearly, the quantity V_m^2 ceases to be meaningful for such periods.

A similar situation obtains for periods $P > \mathrm{Max}(t-t_0)$ where t is in the range of the data. Here, again, negative arguments need to be considered. Note that observations are assigned to groups according to index I_G computed by

$$I_G = \mathrm{IFIX}(m\varphi) = \mathrm{IFIX}\left[m \frac{t - t_0}{P} \pmod 1\right], \quad \text{such that} \quad 0 \leqslant \varphi \leqslant 1,$$

where m continues to denote the number of groups containing data and the operator IFIX indicates that only the characteristic of the argument is taken.

Clearly, I_G vanishes for any period exceeding the largest difference $t-t_0$ which can be produced in the data sample under consideration. Under these conditions, all observations will be assigned to a single group and, as in the case of small periods, no reduction in the sum of the expected group squared deviations is possible.

D. NATURE OF THE FUNCTION V_m^2

It was already mentioned that the quantity V^2 is not a function of the trial period. The value of the quantity V_m^2 depends, however, on the specific data content of the individual groups. A change in this content can be effected only when at least one of the observations changes its group association in response to changes in the trial period.

It is not too difficult to show that, for an observation having argument t, the group index I_G remains constant within the range given by

$$P_A = \frac{m(t-t_0)}{\mathrm{IFIX}\left\{\dfrac{m(t-t_0)}{P}\right\}} > P > \frac{m(t-t_0)}{\mathrm{IFIX}\left\{\dfrac{m(t-t_0)}{P}\right\}+1} = P_B\,.$$

Within a given data sample, one expects to find one or more observations which yield the maximum value of P_A, P_{AMAX}, and, similarly, observations which produce the minimum value of P_B, P_{BMIN}. Within the range $P_{BMIN}<P<P_{AMAX}$ not a single observation changes its group assignment and, therefore, the quantity V_m^2 remains constant. Consequently, V_m^2 exhibits stepwise variation with P.

E. SELECTION OF THE ORIGIN t_0

For fixed values of m, t, and P, the range over which V_m^2 remains constant becomes smaller with increasing t_0. This is seen from the fact that for large values of t_0, $1 \ll \mathrm{IFIX}[m(t-t_0)/P]$ and, numerically,

$$\mathrm{IFIX}\left[\frac{m(t-t_0)}{P}\right] \approx \frac{m(t-t_0)}{P}.$$

Consequently, for large values of t_0 we have

$$\Delta P = P_A - P_B \approx \frac{P^2}{m(t-t_0)}.$$

This implies that the more remote an observation is from t_0, the greater is the likelihood that it will change its group association with a slight change in the trial period. If t_0 were placed outside the data range or even at one of the ends of the data range, fluctuations in V_m^2 would be considerably larger, even though more uniformly distributed, than for t_0 placed approximately at the midpoint of the range. In the former case, the detail to which the stepwise variation of V_m^2 must be investigated in order to locate the minimum would be computationally excessive. Therefore, it was found convenient to place the time origin at the weighted midpoint of the data range despite the fact that fluctuations in V_m^2 due to group jumps exhibit peaks near the ends of the data range. For a moderate data range, this behavior causes only a minor inconvenience because it occurs at the ends where the present method loses its power in any event since it runs into the problem of limiting periods. Nevertheless, when the data range is very large and consequently a wide range of periods can be investigated, fluctuations in question may narrow down the effective range of detectable periods.

F. PRECISION OF THE METHOD

The range P over which the function V_m^2 maintains a constant minimum is not to be interpreted as the true precision with which the period has been established. However, the true precision obviously cannot be greater than the pseudo precision provided by the present method. The pseudo precision specified herein can be made different in a number of ways. For instance, if the use of the F-test is justified, one could investigate how the significance level of the acceptable hypothesis changes with not only the trial period, but also with t_0. Thus, after choosing the trial period, one may further choose t_0 to obtain various values of F with various degrees of freedom producing the largest or smallest significance level for that trial period. Such considerations would result in different spans of constant V_m^2 with respect to the trial period; wherein allowing t_0 to vary results in a greater number of possible jump points, but the choosing of a maximum or minimum may subsequently eliminate some of them.

In view of the above comments, we choose to estimate the precision of the computed period by examining the changes in the phase diagram in response to disturbances in the period which corresponds to the minimum value of V_m^2.

It is also possible that many tests on noise series on many kinds of time sets may establish empirical relations for precision of this method.

G. MULTIPLE PERIODS

The ability of the present method to separate two or more cyclic patterns which may be present in the data has not been studied in detail. At the present time, it is not known what effect the presence of another period would have on the F-test of the null-hypothesis of the first period. Related to this consideration is the known fact that correlations among errors in a least squares analysis using a linear model of time series results in undersized estimates of variability for the parameters estimated. Whether something similar occurs in the present method has not been studied.

However, limited numerical tests indicate that, in cases when the two periods are not too closely subharmonic, fluctuations in V_m^2 for one period include irregular fluctuations due to the presence of the other period and separation is effected without difficulty. Computations using synthetic data generated from superposition of the first and second harmonics of a sinusoidal signal of equal amplitude encountered no difficulties.

These tests, incomplete as they are, indicate that for a large number of time series which may occur in astronomy, the method is capable of separating multiple periods.

H. SELECTION OF m, THE NUMBER OF GROUPS

No a priori statement can be made concerning a most efficient grouping of observations for a time series with unknown periods.

However, it appears that $m=3$ is the practical minimum if t_0 is fixed because with $m=2$ the center of both of the intervals may occur close to the region where the data crosses the overall mean in case the true periodicity is present. Furthermore, for a simple signal, such as a sinusoid, $m=4$ certainly guarantees significant differences between group means no matter where the origin t_0 is located.

As far as the upper limit of m is concerned, all one can say is that it must not be taken so large as to lead to empty groups thereby causing difficulties described in Section 2.B.

3. Examples

The application of the method outlined in Section 2 is demonstrated by three examples. The first treats synthetic data which closely resemble the light variation of the eclipsing binary VW Cephei. The simulated light curve is assumed invariant with time. The second example is concerned with the periodicity analysis of the light variation of the Seyfert galaxy 3C120. In this example, at least two periodicities are present. Finally, the third example compares the results of the present method with those of Deeming's method for the spectroscopic binary HD 217 792 $=\pi$PsA.

A. SYNTHETIC VW CEPHEI DATA

This example was set up to insure a clean test case with precisely known spacing of observations and known errors in both the ordinates and the argument of the light curve. For this purpose, a single cycle of the system light curve, observed by K. K. Kwee and shown in Figure 1, was replaced by the curve marked by crosses. The ordinates of this simulated curve are derived from a four-harmonic Fourier series approximation of the original data. The fundamental period used to generate

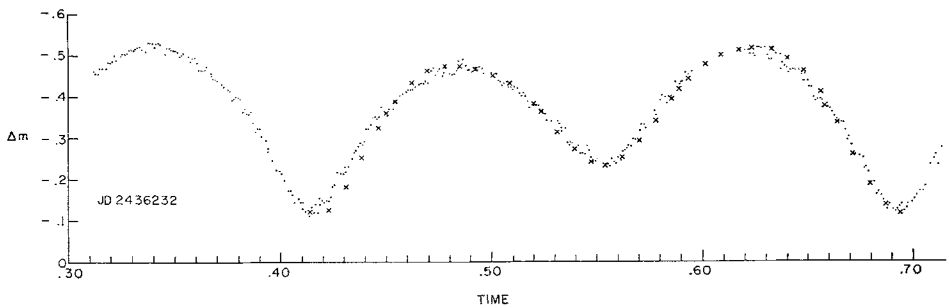

Fig. 1. Observations used to generate data for example (A). Points are observations of VW Cephei. Crosses represent an approximation for one cycle of the light curve.

these data was set exactly equal to 0.2783 days and the reference time t_0 was taken equal to 0.4120 days. Subsequently, using a table of random numbers, a random sequence of arguments was produced for which the exact value of the corresponding ordinate was computed. Finally, the latter was degraded by a normally distributed noise. Simulated observations, thus produced, are shown in Figure 2. The scale of this figure is just sufficient to indicate statistical fluctuations in the synthetic data.

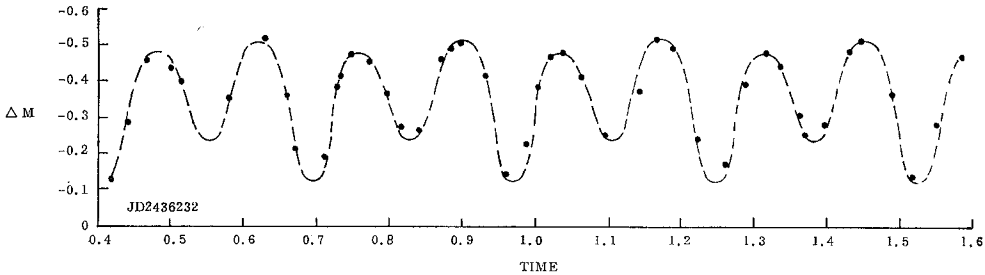

Fig. 2. Synthetic data employed in computations of example (A).

Results of the application of the present method to these data are shown in Figures 3, 4 and 5. Figure 3 shows that, for $m=2$, the minimal number of groups which can be used in the analysis, the method fails to indicate the presence of a period in the range of tested values. However, for $m=4$, there is an unmistakable indication of a minimum in the computed value of V_m^2. With the subsequent increase in m, the quantity V_m^2 displays deeper and deeper minima in the neighborhood of 0.28 days.

Figure 4 displays the behavior of V_m^2 with the increasing time span which, in the present case, also implies the increasing number of observations. In this particular

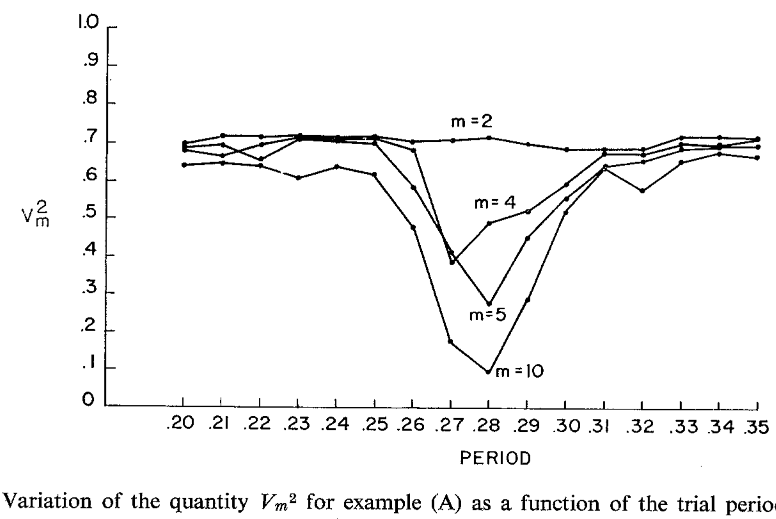

Fig. 3. Variation of the quantity $V_m{}^2$ for example (A) as a function of the trial period and the number of groups, m.

case, the first cycle contains only 12 observations, and V_m^2 shows no evidence of a period. In fact, the function exhibits large fluctuations which are not apparent in Figure 4 because of the scale used. However, for two cycles and 26 observations, these sharp fluctuations are smoothed out and V_m^2 shows a well-defined, even though shallow, minimum.

Finally, Figure 5 shows the detailed stepwise changes of V_m^2 in the vicinity of its absolute minimum. The latter is maintained over an interval 0.2780 to 0.2781. Note that these values are within about 0.1% of the value used to generate the data. It

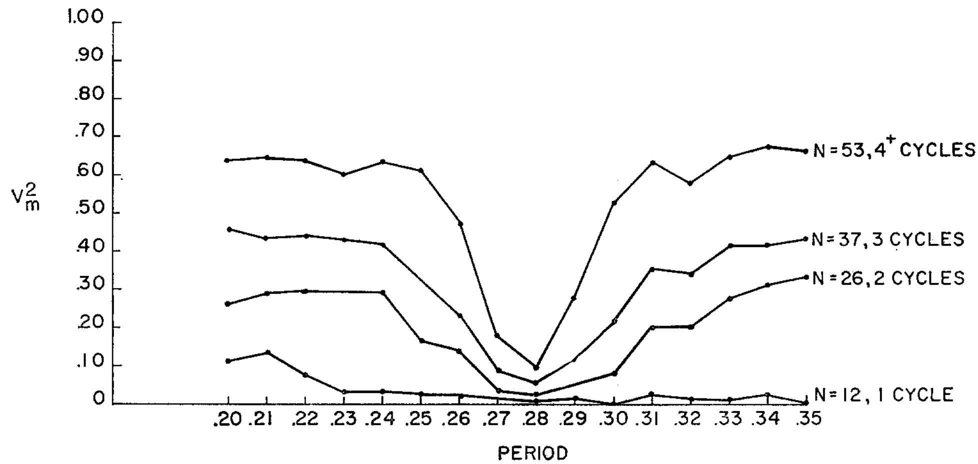

Fig. 4. Variation of the quantity $V_m{}^2$ for example (A) with the number of observations N, and the time span of data.

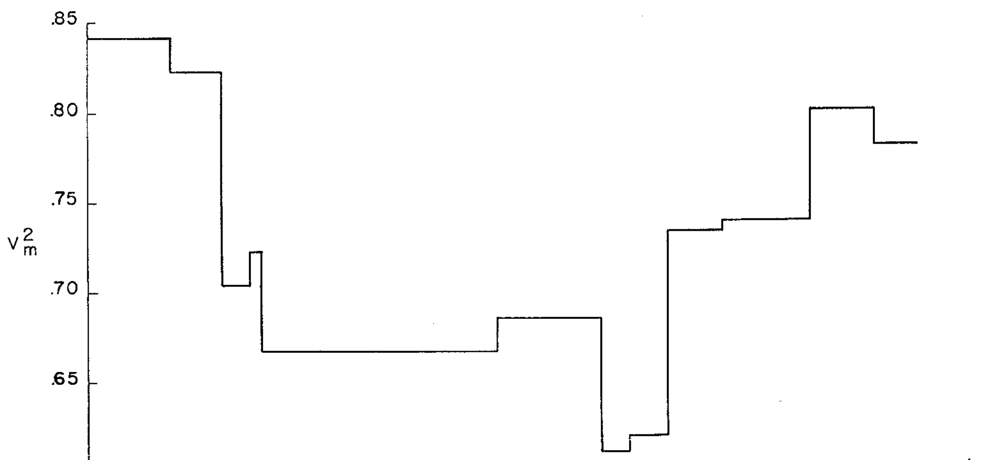

Fig. 5. Detailed variation of $V_m{}^2$ for example (A) in the neighborhood of the fundamental period. ($m = 10$, the time span covers four cycles.)

is again appropriate to point out that the above interval is not to be confused with the precision of period determination. Within the present scheme of things, this can only be done by examining the phase diagram.

Also, it is not surprising that, at least in the present case, a continuing increase in the data time span and in the number of observations displaces the minimum value of V_m^2 closer to the period that was used to generate the data. Although not displayed in figures, V_m^2 has a second minimum in the neighborhood of the second harmonic. As expected, the latter minimum has broader wings and is shallower in depth than the minimum corresponding to the fundamental period. Higher order harmonics cannot be isolated with as few as 50 observations scattered over approximately four cycles.

This simple example provides sufficient evidence that implementation of the method for real observations is quite practical.

B. SEYFERT GALAXY 3C120

The second example uses data of Usher *et al.* (1969, 1970) for the Seyfert galaxy 3C120. These observations are typical of the fragmentary information existing for many variable objects which was gathered irregularly over extended periods of time. The data sample used in computations spans the interval from 1905 to 1970 and contains approximately 300 observations. Points in Figure 6 show the best documented portion of the sample. This figure indicates that the object may contain two reasonably well-defined periods of vastly different length. This fact, surmised previously by Usher and co-workers, is of considerable interest for our purpose since it may test the ability of the method to separate such periods.

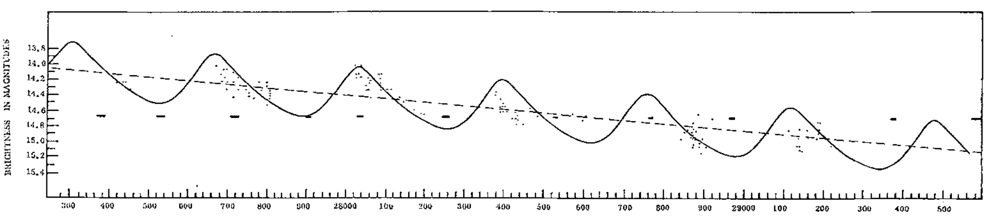

Fig. 6. Representation of photometric data of the Seyfert galaxy 3C120 between 1934 and 1939 by a synthetic light curve generated from periods of 353 days and 22.3 years. Abscissa is in JD 2400000 plus the indicated epoch; ordinates are in photographic magnitudes.

For details of the periodicity analysis of 3C120, the reader is referred to the paper by Jurkevich *et al.* (1971). Here, it is sufficient to show the variation of V_m^2, given in Figures 7 and 8. The first of these presents a clear indication of the minima near 353 and 703 days. Within the uncertainty of their location, these periods constitute the two detectable harmonics of the short period term.

Usher *et al.* suggested that the long period term may have a period of 30 years. However, the more reliable observations extend over no more than 38 years so that

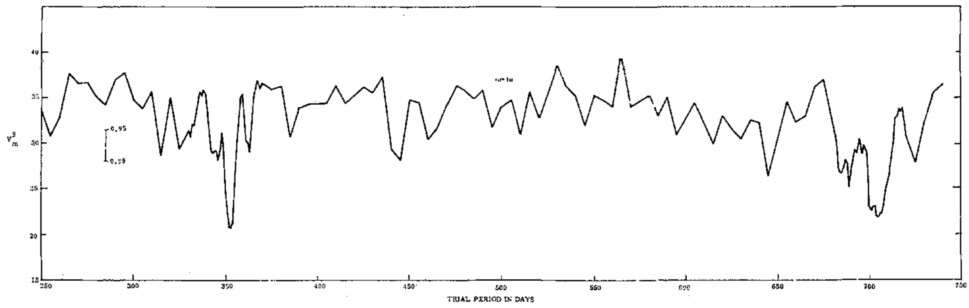

Fig. 7. The run of statistic V_m^2 for the photometric data of 3C120 in the short period region.

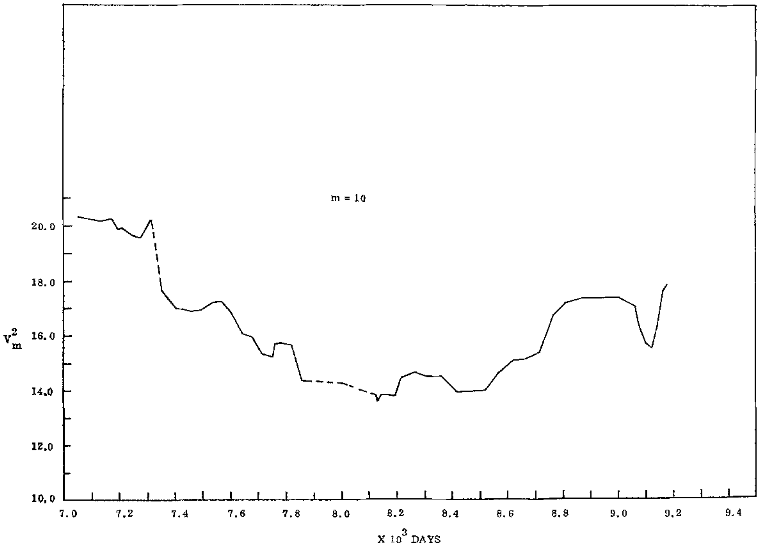

Fig. 8. Variation of statistic V_m^2 for the photometric data of 3C120 in the long period region.

search for the long period seems hardly justified. If, nevertheless, the search is carried out, an indication of a period of 22.4 years is obtained. The variation of V_m^2 in the vicinity of this period is shown in Figure 8. As expected, this minimum has very broad wings.

Reduction of the data to a single-cycle modulo 353 days, with no regard for the existence of the long period, produces a diagram which has all the features of the short period variation indicated by the solid line in Figure 6. The fact that the long period term is ignored in this reduction causes an exaggerated scatter of observations about the short period trend line. The computed periods yield a very satisfactory representation of light variations of 3C120 over the entire data sample (Jurkevich *et al.*, 1971). It remains to be seen whether the synthesized light curve is suitable for extrapolation outside the data range.

On the basis of this example, the method certainly appears capable of yielding useful information from highly sporadic observations containing two reasonably different periods.

C. SPECTROSCOPIC BINARY HD 217792

The data for the final example were taken from the paper of Bopp *et al.* (1970). The sample contains 52 observations, extends over 19376 days, and exhibits a very high scatter in radial velocities. These observations were subjected to periodicity analysis both by the present method and by the method of Deeming. Description of the latter method is contained in the appendix of the reference cited above.

The detailed examination of the behavior of Deeming's statistic r shows that the latter reaches the minimum in the interval from 178.3516 to 178.3539. It is interesting to note that this interval does not include the value of 178.3177 days given in the paper of Bopp *et al.* For the present purpose, however, this difference is not too significant. The variation of the quantities r and V_m^2 is shown in Figure 9. The 95 to 99.9% significance levels for V_m^2 are also shown. The quantity V_m^2 attains its minimum in the interval 178.118 to 178.141 days. Note that at the 99% significance level, the reduction of V_m^2 relative to V^2 is maintained for periods from 177.38 to 178.65 days. Within this range, the hypothesis of non-periodicity is rejected.

If, now, the midpoints of the intervals over which V^2 and r maintain constancy of their minima are taken as the estimates of the periods, the two values agree to about 0.1%. Examination of the respective phase curves indicates that the two periods represent the data equally well.

It is clear that the two methods produce closely comparable results even with data as inadequate as the present ones.

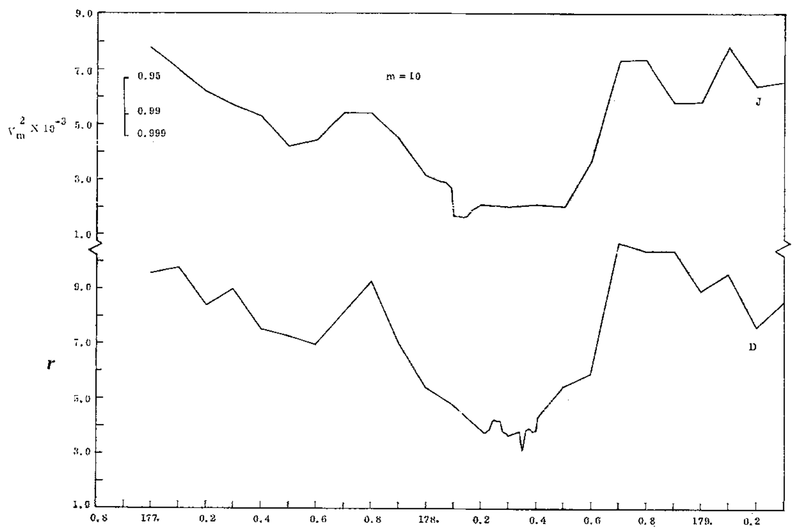

Fig. 9. Variation of statistics $V_m{}^2$, and r of Deeming, for the spectroscopic binary HD 217792.

4. Concluding Remarks

Computationally, the Expected Mean Squares Analysis presented in this paper is very attractive due to its simplicity. It incorporates the simplest arithmetical operations needed to compute the basic statistical parameters. Since no complex computations whatever are involved, the computation is easily mechanized. Among the known methods applicable directly to irregularly spaced observations, only the method of Deeming has comparable features.

Among drawbacks of the present method, perhaps the major one stems from the stepwise variation of the test statistic with trial period. Chance fluctuations in V_m^2 result in numerous dips throughout the range of tested periods. Whenever such fluctuations occur near the true minimum, identification of the latter becomes troublesome. In order to locate the smallest of the minima, it is necessary to explore in detail the behavior of the test statistic in the vicinity of the suspected minimum. To do this automatically would require a complex search algorithm. It was found that the most practical way to avoid complicated search routines is to conduct computations at a remote computer terminal so that results can be examined immediately and computations modified as needed. The author carried out his computations using a General Electric 605 Time Sharing System.

It must be noted that the above remarks apply fully to Deeming's method. It is likely that they will remain valid for any conceivable method designed to treat unequally spaced discrete data.

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