# DYNAMICAL EVOLUTION OF TRIPLE SYSTEMS

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(Received 10 February, 1986)

**Abstract.** This article reviews numerical experiments on the three-body problem carried out at the Leningrad University Astronomical Observatory during the past 20 years. Systematic studies of triple systems with negative total energy have yielded the following main results. Most ($\approx 95\%$) of the systems decay; the decay always occurs after a close triple approach of the components. In a system with unequal masses, the escaping body usually has the smallest mass. A small fraction ($\approx 5\%$) of quasi-stable systems is formed if the angular momentum is non-zero. The qualitative evolution in three-dimensional cases is the same as for planar systems. Small changes in initial conditions sometimes lead to substantial differences in the final outcome. The decay of triple systems is a stochastic process similar to radioactive decay. The estimated mean lifetime is $\approx 100$ crossing times for equal-mass components and decreases for increasing mass dispersion.

A classification of the close triple approaches which lead to immediate escape is given for equal-mass systems as well as for selected sets of unequal components. Detailed studies of close triple approaches by computer simulations reveal that the early evolution is determined by the initial ratio of the interaction forces. The review concludes by discussing applications of the results to observational problems of stellar and extragalactic systems.

## 1. Introduction

The gravitational $N$-body problem was applied to celestial mechanics soon after the discovery of Newton's universal law. However, in the course of three centuries the analytical studies have not provided an effective solution to the main problems. Today, numerical methods may be used for obtaining detailed solutions. Computer simulations of he $N$-body problem were first performed by von Hoerner (1960) who studied the evolution of small systems ($N$ = 4–25). Further advances of the direct method as well as increased computer power have yielded significant results for systems with $N$ = 500 components (Aarseth, 1974; Wielen, 1974).

It should be noted that the study of large $N$-body systems requires considerable amounts of computer time (roughly proportional to $N^3$), hence, the number of different initial conditions which can be examined is relatively limited. On the other hand, small $N$ systems offer good prospects of systematic investigations, particularly in the case of $N = 3$. The shortening of computer process-time permits statistical methods to be used for studying the behaviour of triple systems. This is achieved by selecting a representative sample of initial conditions which then reveal general features of the evolution.

A wide range of triple systems occur among stars and galaxies, and their kinematics and dynamics are of considerable cosmogonic interest. Moreover, studies of dynamical evolution of $N$-body systems with $N$ = 10–500 have shown that three-body interactions play a crucial role in the central regions of open and globular clusters, as well as galaxy

*Astrophysics and Space Science* **124** (1986) 217–241.

clusters. Consequently, the study of the three-body problem is relevant to the subjects of celestial mechanics and stellar dynamics alike.

## 2. Method and Initial Conditions

Computer simulations of the three-body problem were introduced at the Leningrad University Observatory in 1964 under the leadership of Professor T. A. Agekian. The methods used for this investigation were developed by Agekian and Anosova (1967, 1968) and consist of the following steps:

(1) choice of system units;
(2) generation of initial conditions;
(3) numerical integration of the Lagrangian equations of motion;
(4) Sundman's regularization of close two-body approaches.

Note that very close triple approaches (collisions) are practically absent even in the case of three equal-mass bodies with zero total angular momentum.

A standard set of system units and initial conditions for triple systems with negative total energy has been adopted for the statistical studies (Agekian and Anosova, 1967). Scaled-system units are used, in which the distance $d = 1$ is the mean harmonic separation between the bodies in virial equilibrium, and the velocity $v = 1$ is the corresponding r.m.s. velocity. The unit of time, $\tau = 1$, is then the mean crossing time of a particle through the system; it is defined by

$$\tau = \frac{G\sqrt{\sum_{i=1}^{3} M_i \sum_{i \neq j} M_i M_j}}{(-2E)^{3/2}} \; ; \tag{1}$$

and the mean size is given by

$$d = \frac{G \sum_{i \neq j} M_i M_j}{2E} \, , \tag{2}$$

where $G$ is the gravitational constant; $M_i$ and $M_j$, the masses of the bodies; and $E$, the total energy of the triple system. This system of units provides a uniform description which enables quantitative results to be compared for different initial conditions.

A convenient method for generating initial configurations is achieved in the following way (Agekian and Anosova, 1967, 1968). The components $A$ and $B$ of the triple system shown in Figure 1 are placed at the points $(-0.5, 0)$ and $(0.5, 0)$ in the Cartesian coordinate system $\xi$, $\eta$. A circle of unit radius centred on $(-0.5, 0)$ is then drawn. The initial positions of the third component $C$ are distributed with uniform probability within the positive quadrant ($\xi > 0$, $\eta > 0$). In this way, all possible configurations of the triple system with equal masses are sampled. If the components have different masses, it is also necessary to take permutations of the positions $A$, $B$, and $C$, thereby obtaining six

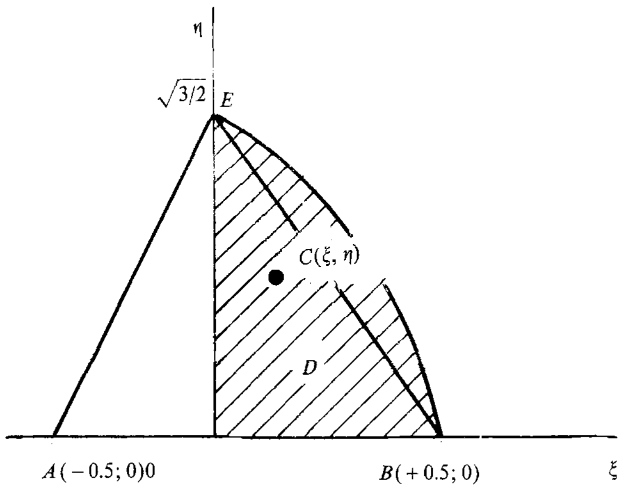


Fig. 1. All possible initial configuration regions $D$ of the triple systems. The components of the triple systems are placed at the points $A$ $(-0.5, 0)$, $B$ $(+0.5, 0)$, $C$ $(\xi, \eta)$ in the Cartesian coordinate system.

initial states each time. The initial velocities of the bodies are selected with virial coefficients (kinetic energy over potential energy) $k_0$ in the range (0, 0.5).

The dynamical evolution of the triple systems is studied by numerical integration of the equations of motion using a standard method. Since the calculation time is roughly proportional to $N^3$, it is desirable to choose an optimal integration method that gives both sufficient accuracy and a minimum computer time. We have adopted a fourth-order Runge–Kutta method for this purpose (see Hohl and Watt, 1976). Integration steps are chosen according to a specified tolerance and the constancy of the ten integrals of motion is checked after each step. The energy constant $E$ is the most sensitive to integration errors. Typical relative energy errors for the simulations at the moment of escape are $\Delta E/E \approx 10^{-4}$. The other integrals are usually preserved to two or more orders higher accuracy. In principle the accuracy can be improved at the expense of increased computer time. However, the present solution accuracy may be considered satisfactory for statistical investigations. Furthermore, not a single case among the $\approx 3 \times 10^4$ experiments gave rise to difficulties with the Sundman regularization method.

## 3. Main Results of the Simulations

### 3.1. Orbital Motions

Early investigations of triple systems with negative energy were concerned with the general behaviour of the motions (Agekian and Anosova, 1967, 1968). Some of the relevant processes may be mentioned briefly. Close triple approaches play an important role. As a result, one of the components may be ejected to great distances or even to infinity, leaving behind a temporary or permanent binary system. Once formed, a binary can disintegrate or exchange components by further interactions, whereas a simpler interplay of bodies occur when close approaches and ejections are absent. During the dynamical evolution of the triples, these states change continuously until a sufficiently close triple approach produces an escaping body, with the two remaining components forming a final binary system.

Figure 2 shows typical trajectories of motions in a triple system. Here the component masses are equal and the initial velocities are zero, resulting in planar motion with zero-angular momentum. Numbers along the trajectories denote the time in units of the mean crossing time $\tau$. For a stellar system of solar mass components and mean size $d = 0.01$ pc, one time unit corresponds to about $10^4$ yr, whereas for a typical system of

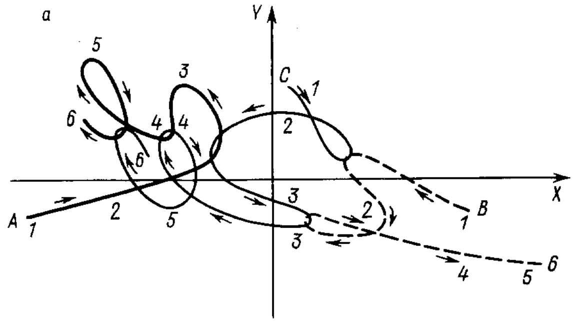


Fig. 2a.

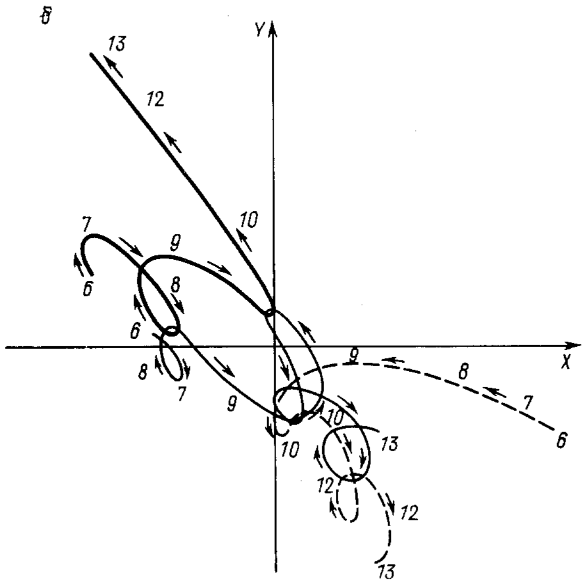


Fig. 2b.

Fig. 2(a–d). Typical trajectories of motions in a triple system. The points $A, B, C$ denote the initial positions (Figure 2(a)). Numbers denote the time in units of the crossing time $\tau$. The solid line shows the component $A$, the thin line shows component $B$ and the dashed line component $C$. Each figure follows sequentially from the previous one and arrows along the trajectories indicate the direction of the motion.

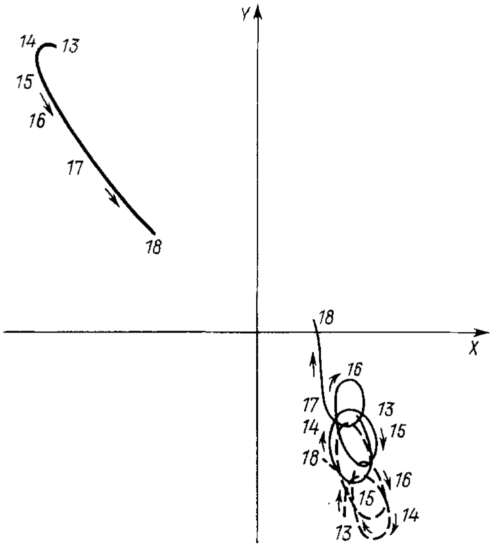


Fig. 2c.

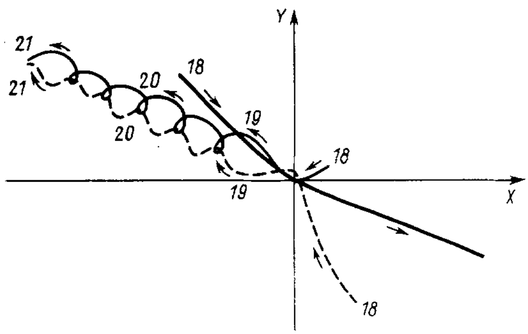


Fig. 2d.

galaxies it would correspond to about $10^9$ yr. The basic states of bound triple systems are illustrated in the figure.

On the basis of numerical studies of typical motions in the general three-body problem, Szebehely (1971) and Agekian and Martynova (1973) have introduced the following classification scheme: (0) close triple approach; (1) simple interplay; (2) ejection with return; (3) escape; (4) stable revolution; (5) Lagrangian equilibrium configurations; and (6) collisions and periodic orbits. The states 4, 5, and 6 are of a special type and can be studied by analytical methods. Stability criteria for the states 4 have been proposed by Golubev (1967), Szebehely and Zare (1977), Harrington (1972), and Black (1982). The three first authors obtained a criterion based on analytical methods, whereas the two others used computer simulations. The characteristic states 0, 1, and 2 follow each other in a definite order, whereas the states 3 are final. Moreover, the states 6 appear to be particular cases of motions in systems without hierarchical structure.

In general, a study of triple systems can only be effective by using methods of computer simulations. Criteria for distinguishing between the states 0, 1, and 2 combined with 3 for non-rotating planar triple systems have been suggested by Agekian and Martynova (1973). In order to separate the states 2 and 3, it is necessary to use criteria for return or escape; alternative expressions have been proposed by Birkhoff (1927), Tevzadze (1962), Standish (1971), Griffith and North (1974), Yoshida (1972), and Marchal (1974). A unified form of these criteria has been suggested by Szebehely (1973) as

$$\rho > a\,, \qquad \dot{\rho} > 0\,, \qquad \dot{\rho}^2 > b\,, \tag{3}$$

where $\rho$ is the distance from the remote body to the centre of mass of the two other particles. Expressions for $a$ and $b$ in the different criteria are quoted in Table I for the body of mass $M_3$. Each of these criteria provide a sufficient but not necessary condition for escape. Orlov (see Figure 3 of Anosova and Orlov, 1985) has shown that the criteria with the smallest proportion of undetected escapers are given by the four last authors. An even stronger escape criterion has recently been proposed by Kuznetsova and Orlov (1983) and by Marchal *et al.* (1984). However, for large distances ($\rho > 4d$) there is no essential differences between the suggested criteria (except for the simpler one due to

TABLE I

Escape criterion for triple systems

| Author | a | b |
|---|---|---|
| Birkhoff (1927) | $\dfrac{2G\mathfrak{M}^2}{3\,\lvert E\rvert}$ | $\dfrac{8G\mathfrak{M}}{\rho_0}$ |
| Tevzadze (1962) | $\dfrac{2G\mathfrak{M}}{3\,\lvert E\rvert}$ | $\dfrac{2G\mathfrak{M}}{\mu}\quad \dfrac{M_1}{\rho_0-\dfrac{M_2}{\mu}a}+\dfrac{M_2}{\rho_0-\dfrac{M_1}{\mu}a}$ |
| Standish (1971) | $\dfrac{G(M_1M_2+M_1M_3+M_2M_3)}{\lvert E\rvert}$ | $2G\mathfrak{M}\left[\dfrac{1}{\rho_0}+\dfrac{M_1M_2}{\mu^2}\,\dfrac{a^2}{\rho_0^2(\rho_0-a)}\right]$ |
| Griffith and North (1974) | $\dfrac{G(M_1M_2+M_1M_3+M_2M_3)}{\lvert E\rvert}$ | $2G\mathfrak{M}\left[\dfrac{1}{\rho_0}+\dfrac{M_1M_2a^2}{\mu^3\rho_0^2}\left(\dfrac{M_1}{\rho_0-\dfrac{M_1}{\mu}a}+\dfrac{M_2}{\rho_0-\dfrac{M_2}{\mu}a}\right)\right]$ |
| Yoshida and Marchal (1972, 1974) | $\dfrac{GM_1M_2^2}{\mu\,\lvert E\rvert}$ | $\dfrac{2G\mathfrak{M}}{\mu}\left(\dfrac{M_1}{\rho_0-\dfrac{M_2}{\mu}a}+\dfrac{M_2}{\rho_0+\dfrac{M_1}{\mu}a}\right)$ |

Comment:
$\mu = M_1 + M_2$; $\mathfrak{M} = M_1 + M_2 + M_3$

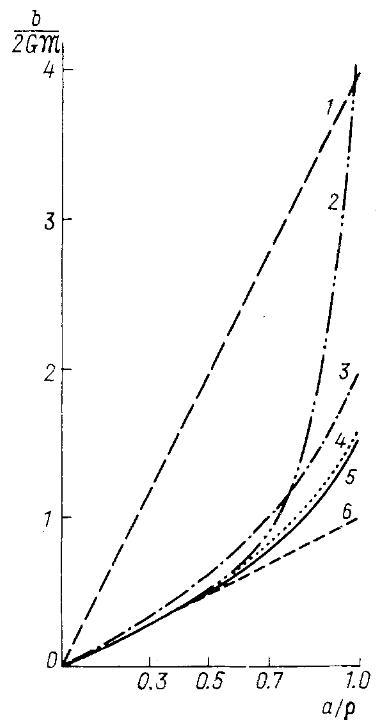


Fig. 3. Comparison of the escape criteria: (1) Birkhoff (1927); (2) Standish (1971); (3) Tevzadze (1962); (4) Griffith and North (1974); (5) Yoshida (1972); Marchal (1974); (6) Kuznetsova and Orlov (1983); Marchal *et al.* (1984). The values of the units are from Equation (3) and Table I.

Birkhoff). Thus, Tevzadze's criterion with $\rho > 4d$ has been used as an escape condition in the present work.

### 3.2. Qualitative results

The statistical material obtained at the Leningrad Observatory amounts to about $3 \times 10^4$ triple systems with negative total energy (see Anosova and Orlov, 1985; and references given therein). The basic qualitative results can be summarized in the following way:

(1) In the majority of cases, the dynamical evolution was completed by escape.

(2) For approximately 20% of the systems, the escape criterion was not satisfied but one of the components was ejected to a large distance from the two others. During such prolonged excursions, the direct integration requires considerable computer time. For such systems we have, therefore, introduced the notion of *conditional* escape. Thus, separations $\rho > \rho^*$ are used to terminate an integration, and we have adopted $\rho^* = 20d$ or $30d$ as a practical limit.

(3) The mean escape time $\langle T \rangle$ estimated for the triple systems (determined from actual or conditional escape) depend somewhat on the choice of the critical value $\rho^*$ in

TABLE II

Dependence of the estimated life time $\overline{T}$ on the distance $\rho^*$ of a remote component for the triples

| $\rho^*$ | 4 | 10 | 20 | 40 | 100 | 400 | 1000 | > 1000 |
|---|---|---|---|---|---|---|---|---|
| $\overline{T}(\rho^*)$ | 36.5 $\pm$ 1.6 | 78.7 $\pm$ 2.8 | 101.9 $\pm$ 3.4 | 126.1 $\pm$ 4.5 | 151.4 $\pm$ 5.2 | 236 $\pm$ 21 | 337 $\pm$ 49 | 780 $\pm$ 260 |

the conditional escape criterion. It can be seen from Table II that $\langle T \rangle$ increases for larger values of $\rho^*$.

(4) For isolated triple systems where $\rho \to \infty$, $\langle T \rangle$ also becomes arbitrarily large. Therefore, the mean escape time is not well defined, and one can only discuss the probability $P(t)$ that escape occurs after a time $t$, or the probability $P(\rho)$ that a remote component reaches a distance $\rho$ from the centre of mass of the two other bodies (see Table III).

TABLE III

Dependence of the escape probability $P$ on the distance $\rho$ and on the evolution time $T$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | 10 | 12 | 16 | 20 | 40 | 100 | 200 | 1000 | |
| $P(\rho)$ | 0.649 | 0.737 | 0.812 | 0.855 | 0.931 | 0.971 | 0.983 | 0.997 | |
| $T$ | 10 | 20 | 40 | 60 | 80 | 100 | 200 | 300 | 500 |
| $P(T)$ | 0.110 | 0.188 | 0.318 | 0.419 | 0.502 | 0.533 | 0.788 | 0.889 | 0.975 |

(5) Escape (actual or conditional) always arises from a close triple approach of the bodies, and the moment of escape has been assumed to coincide with the minimum triangular perimeter.

(6) Statistically, escape is more probable when the triple approach is closer.

(7) As a rule, the final binaries have orbits with large eccentricities, in agreement with theoretical considerations for an equilibrium distribution of binaries in an irregular field (Ambartsumian, 1937; Heggie, 1975). This also agrees with data of a statistical study of escape from triple systems (Monaghan, 1976).

(8) The escape phenomenon of triple systems is a random process analogous to radioactive disintegration. Thus, the distribution of escape times has an approximate exponential form and statistically the systems do not approach nearer to escape during the course of their evolution. Moreover, the distribution $P(t)$ in Table III demonstrates that there is a secular process opposite to ageing (Agekian *et al.*, 1983); thus, at a given time the most distant ejection may have a period exceeding the age of the system. The estimate of the half-life for true escape gives $T_{1/2} \approx 80\tau$.

(9) An increasing mass dispersion reduces the lifetime and in most cases, the body with smallest mass is ejected.

(10) The lifetime tends to increase with increasing angular momentum, especially by the appearance of a small proportion ($\approx 5\%$) of hierarchical triple configurations which are stable over long times. Triple approaches are absent in such systems, and the motions can be described by the superposition of two perturbed Keplerian orbits during intervals $\Delta t \approx 1000\tau$. Application of stability criteria (Szebehely and Zare, 1977; Harrington, 1972) to these systems confirms this result.

(11) The qualitative picture is essentially preserved during the transition from planar motions to the three-dimensional case.

(12) New qualitative results of three-dimensional calculations relate to the orientation of the final trajectories: (i) as a rule, the orbits of the binary and an escaper are not co-planar; (ii) in slowly rotating systems, the angular momentum of the escaper tends to be of opposite sign to that of the binary, whereas the spins are aligned for rapid rotation; and (iii) the velocity vector of an escaper is usually nearly perpendicular to the total angular momentum of the triple system.

### 3.3. Close Triple Approaches

Szebehely (1979) has shown analytically and by computer simulations that one of the components can attain a large velocity as the result of a close triple approach. In this way, a velocity $V$ of order $10^2$ km s$^{-1}$ can be reached in a system with solar mass components and initial dimension 0.01 pc; alternatively, $V \approx 10^3$ km s$^{-1}$ in a compact massive galaxy triplet.

A classification of close triple approaches which lead to escape has been suggested by Anosova and Zavalov (1981) for equal-mass components, and by Anosova and Orlov (1983a) for the case of unequal masses. It has been shown that a close triple approach (state 0 above) is necessary for escape to occur. The most effective triple approaches are those in which a temporary binary is first formed, whereupon the ejected body returns for a favourable interaction with the two binary components. This is a fly-by interaction and is denoted class I; see Figure 4(a) for an example. During the triple approach, the trajectory of the escaping body is often nearly rectilinear and the three-body configuration takes the form of an isosceles triangle in the equal-mass case. In systems with unequal masses, such approaches can sometimes produce escape of the most massive body. In those triple approaches which are separated by a sequence of exchanges, escape from an equal-mass system seldom takes place. Such temporary exchanges of components are denoted class II; an example is illustrated in Figure 4(b). In systems with unequal components, these approaches can occasionally result in the escape of bodies of intermediate and maximum mass.

Synchronization of motions also has an effect on triple approaches of class I (fly-by). Thus, if a single component passes through the centre of mass of the system just as the two other bodies are approaching, its velocity is decreased and escape does not occur, whereas escape is promoted if the binary components are expanding at the moment of closest approach.

Subregions $D_x$ (with $x = 1, 2, 3, \ldots$) inside the region $D$ of all possible initial configurations (cf. Figure 5) which lead to escape after the first triple approach have recently

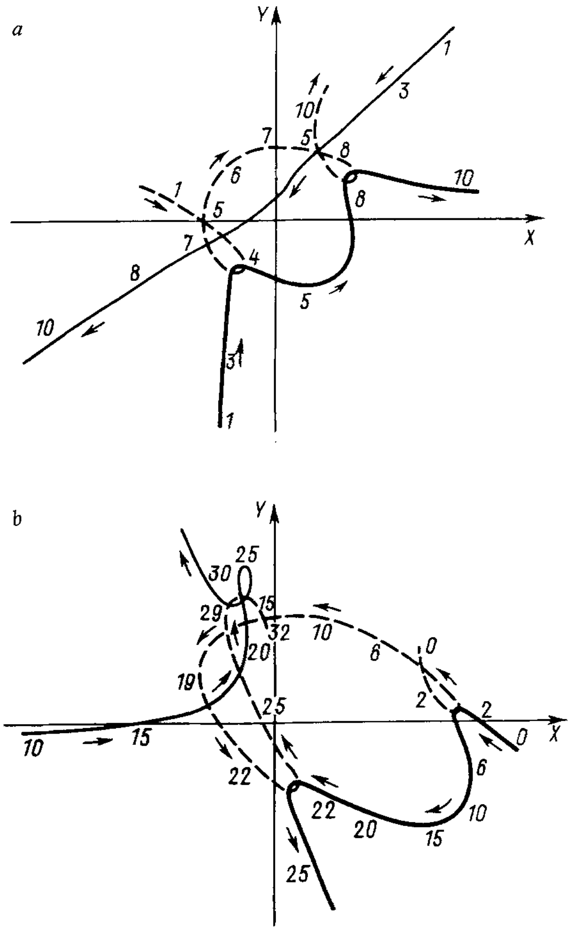


Fig. 4. (a) Fly-by interaction – close triple approach of class I. (b) Temporary exchange of components – close triple approach of class II. The symbols are analogous to those in Figure 2.

been discovered (Anosova and Zavalov, 1986). These regions which are shown in Figure 5 can be characterized as follows:

(1) The 'axis' of the regions $D_x$ are the contours

$$x = r_{AC}/r_{BC} = U_{BC}/U_{AC} = \text{const.}\,, \tag{4}$$

for which the ratio of interaction forces (and the ratio of relative potential energies $U_{AC}$ and $U_{BC}$) from the two first bodies $A$ and $B$ to the third body $C$ is equal.

(2) On the contours of the regions $D_x$ the values that are approximately equal to an integer appear to be connected with a resonance phenomenon.

(3) The boundaries of the regions $D_x$ correspond to condition escape; i.e., distant ejections resulting from the first triple approach.

(4) In the regions $D_x$ with increasing value of $x$, the number $n^*$ of double approaches

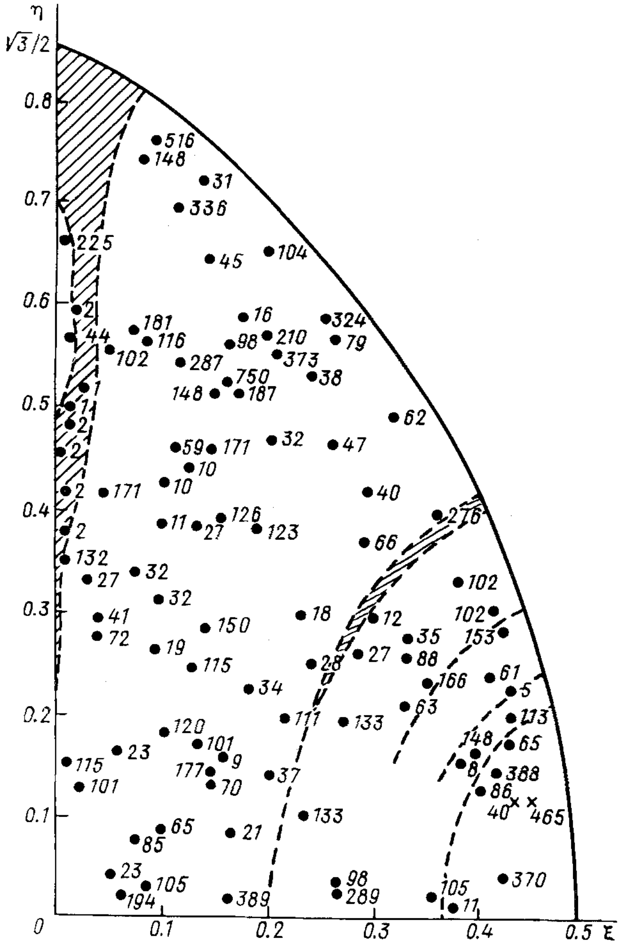

Fig. 5. The escape times $T(\xi, \eta)$ for case I*a* (see notes to the tables).

of the bodies $B$ and $C$ in a temporary binary before the triple approach also increases; for $x \approx 1$, $n^* = 0$, and for $x \approx 2$, $n^* = 1$, etc. Thus, the time of escape increases for larger values of $x$.

(5) The velocities of the escapers also increases with $x$; this is connected with a hardening of the final binaries with increasing $x$.

(6) The degree of closeness of triple approaches in all regions $D_x$ is roughly equal; this may be connected with non-synchronization between the minimum perimeter $p$ and the binary separation $r_{ij}$ in the different regions.

(7) The regions $D_x$ have a complicated structure; they are almost symmetrical relative to the contours but different bodies escape from each side; always a component $A$ escapes from the right side, a component $B$ from the left, and a component $C$ escapes from a narrow strip along the contours.

(8) On escape of components $A$ or $B$, the centre of mass of the other two bodies is

only approached once, whereas component $C$ escapes after two approaches (one weak, one strong) to the temporary binary $AB$.

(9) The orbits of the bodies during a triple approach is determined by the initial configuration. Thus, for hierarchical systems near an isosceles triangle with a base $r_{AC}$ or $r_{BC}$, the only passage of component $A$ or $B$ takes place through the centre of mass of the other two bodies. On the other hand, for configurations near an isosceles triangle with a base $r_{AB}$ greater than $r_{AC}$ (or $r_{BC}$), two passages take place, one slow and the other a rapid fly-by.

(10) The basic result of this study is that a determining parameter for the course of dynamical evolution in these triple systems is given by the value of the ratio $x = r_{AC}/r_{BC} = U_{BC}/U_{AC}$.

### 4.3. Quantitative Results

The main statistical results of the three-body study are summarized in Table IV. The following quantities are displayed: the number of experiments $N$; the average value and r.m.s. deviation of the escape time $T$ for true or conditional escape and of the eccentricity $e$; and the corresponding references of the results. The different types of initial conditions are indicated in the last column. A time limit of $t_k = 150\tau$ for the simulations was chosen by Szebehely (1972), whereas Standish (1972) adopted $t_k = 10^4$

TABLE IV

The mean quantities $\overline{T}$ and $\bar{e}$ in the three-body problem

| $N$ | $\overline{T} \pm \sigma_T$ | $\bar{e} \pm \sigma_e$ | Authors | Problem |
|---|---|---|---|---|
| 200 | $95.4 \pm 6.9$ | – | Agekian and Anosova (1968) | I*a* |
| 100 | $87.1 \pm 8.6$ | $0.742 \pm 0.024$ | Standish (1972) | |
| 1500 | $112.3 \pm 2.8$ | $0.707 \pm 0.009$ | Anosova (1977) | |
| 1000 | $115.7 \pm 4.0$ | $0.722 \pm 0.008$ | Agekian *et al.* (1983) | |
| 300 | $27.8 \pm 2.1$ | – | Anosova (1969) | I*b* |
| 92 | 48 | 0.76 | Szebehely (1972) | |
| 100 | $39.3 \pm 7.8$ | $0.901 \pm 0.015$ | Standish (1972) | |
| 1100 | $64.5 \pm 2.8$ | $0.820 \pm 0.008$ | Anosova and Polozhentsev (1978) | |
| 5500 | $59.4 \pm 1.1$ | $0.802 \pm 0.003$ | Anosova and Orlov (1983) | |
| 100 | $117.1 \pm 13.0$ | – | Anosova (1969a) | II*a* |
| 100 | $73.8 \pm 7.8$ | $0.851 \pm 0.016$ | Standish (1972) | |
| 5000 | $91.5 \pm 1.5$ | $0.833 \pm 0.003$ | Anosova *et al.* (1984) | |
| 400 | $66.4 \pm 6.1$ | $0.680 \pm 0.013$ | Standish (1972) | II*b* |
| 4500 | $89.0 \pm 3.0$ | $0.711 \pm 0.007$ | Anosova and Orlov (1985) | III*b* |

*Notes*:
I*a*: plane problem; equal masses; $k_0 = 0$, $L = 0$.
I*b*: plane problem; unequal masses; $k_0 = 0$, $L = 0$.
II*a*: plane problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
II*b*: plane problem; unequal masses; $k_0 \neq 0$, $L \neq 0$.
III*b*: three-dimensional problem; chance choice of masses and $k_0$.

interaction steps and sometimes $t_k = 1000\tau$. Anosova *et al.* (1984) assumed instead a distance of conditional escape $\rho^* = 20d$ and in other experiments discussed here, $\rho^* = 30d$.

Table IV shows that the lower value of the average escape time for equal-mass components is $\langle T \rangle \approx 100\tau$ for $\rho^* = 20d$. However, from Table V it can be seen that

TABLE V

The distribution of the escape time of the triple systems

| $\Delta T(\tau)$ | I*a* | II*a* | III*a* | III*b* |
|---|---|---|---|---|
| 0–10 | 0.168 | 0.134 | 0.077 | 0.137 |
| 10–20 | 0.081 | 0.087 | 0.061 | 0.115 |
| 20–30 | 0.078 | 0.077 | 0.053 | 0.085 |
| 30–40 | 0.066 | 0.066 | 0.049 | 0.074 |
| 40–50 | 0.057 | 0.056 | 0.045 | 0.065 |
| 50–100 | 0.212 | 0.217 | 0.194 | 0.202 |
| 100–200 | 0.216 | 0.198 | 0.215 | 0.197 |
| 200–300 | 0.085 | 0.068 | 0.086 | 0.076 |
| 300–400 | 0.028 | 0.026 | 0.050 | 0.032 |
| $>400$ | 0.009 | 0.071 | 0.170 | 0.017 |

*Notes*:
I*a*: plane problem; equal masses; $k_0 = 0$, $L = 0$.
II*a*: plane problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*a*: three-dimensional problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*b*: three-dimensional problem; chance choice of masses and $k_0$.

TABLE VI

The distribution of the eccentricities of the final binaries

| $\Delta e$ | Computer simulations | | | | Theory | |
|---|---|---|---|---|---|---|
| | I*a* | II*a* | III*a* | III*b* | I + II Monaghan | III Ambartsumian |
| 0 – 0.1 | 0.015 | 0.002 | 0.008 | 0.006 | 0.005 | 0.010 |
| 0.1 – 0.2 | 0.042 | 0.008 | 0.012 | 0.017 | 0.015 | 0.030 |
| 0.2 – 0.3 | 0.061 | 0.017 | 0.037 | 0.048 | 0.026 | 0.050 |
| 0.3 – 0.4 | 0.064 | 0.025 | 0.054 | 0.045 | 0.037 | 0.070 |
| 0.4 – 0.5 | 0.072 | 0.034 | 0.075 | 0.074 | 0.051 | 0.090 |
| 0.5 – 0.6 | 0.098 | 0.052 | 0.098 | 0.090 | 0.066 | 0.110 |
| 0.6 – 0.7 | 0.133 | 0.066 | 0.124 | 0.140 | 0.086 | 0.130 |
| 0.8 – 0.9 | 0.337 | 0.156 | 0.202 | 0.199 | 0.160 | 0.170 |
| 0.9 – 1.0 | | 0.539 | 0.229 | 0.246 | 0.440 | 0.190 |

*Notes*:
I*a*: plane problem; equal masses; $k_0 = 0$, $L = 0$.
II*a*: plane problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*a*: three-dimensional problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*b*: three-dimensional problem; chance choice of masses and $k_0$.

the distribution has a maximum in the interval $(0, 20\tau)$ which contains 20–30% of all events. This maximum is particularly large for triple systems with unequal mass components. The distribution of final eccentricities is represented in Table VI. As much as a third to a half of all eccentricities fall in the interval (0.9, 1.0) for different initial conditions. Figure 6 shows the dependence of the escape time on the mass dispersion $\sigma$, where the latter is expressed in terms of the maximum and minimum component mass and given (cf. Anosova and Orlov, 1983) by

$$\sigma = 1 - M_{\min}/M_{\max} \,. \tag{5}$$

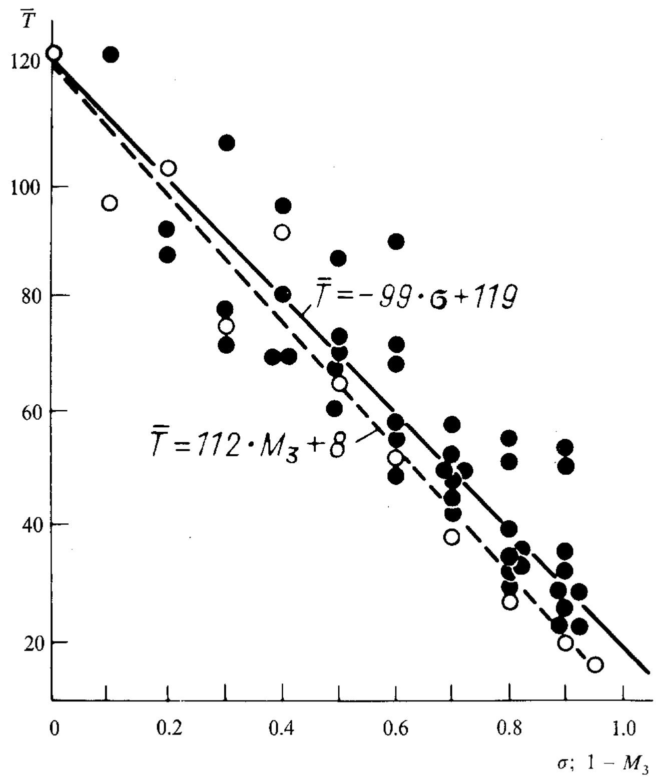


Fig. 6. The time dependence of escape for different mass dispersions $\sigma$. $\bar{T}(\sigma)$ is the solid line with filled circles and $T(M_3)$ is the dashed line with open circles ($M_3$ is the mass of component $C$). Time is in units of the mean crossing time $\tau$.

This study has shown that the evolution is mainly affected by the largest mass ratio, whereas the body of intermediate mass is less important. Figure 6 also shows that the escape time depends almost linearly on the coefficient of mass dispersion such that escape occurs more rapidly for larger values of $\sigma$.

Data obtained by different authors on the escape of bodies with maximum, intermediate, and minimum mass is displayed in Table VII. It can be seen that the

TABLE VII

The fraction of escaping bodies with maximum ($N_1/N$), intermediate ($N_2/N$), and minimum ($N_3/N$) mass in triple systems

| $N$ | $N_1/N$ | $N_2/N$ | $N_3/N$ | Authors |
|---|---|---|---|---|
| 300 | 0.00 | 0.05 | 0.95 | Anosova (1969) |
| 92 | 0.011 | 0.120 | 0.869 | Szebehely (1972) |
| 480 | 0.060 | 0.138 | 0.802 | Standish (1972) |
| 1100 | 0.000 | 0.245 | 0.755 | Anosova and Polozhentsev (1978) |
| 5500 | 0.040 | 0.164 | 0.796 | Anosova and Orlov (1983) |

lightest body escapes in approximately 80% of the cases, compared to about 16 and 4% for the intermediate and maximum mass. Results of three-body studies at the Leningrad Observatory since 1964 are summarized in Table VIII (see Anosova and Orlov, 1985). The first and second column contains the number of experiments $N$ and the maximum time $t_k$ for rotating triple systems without escape. In these systems, the component $C$ remains in the lower right-hand side of region $D$ (see Figures 1 and 8). It has been shown (Anosova *et al.*, 1984) that these systems satisfy the stability criteria of Golubev (1967), Szebehely and Zare (1977) and Harrington (1972). The third column gives the average and r.m.s. value of the reduced escape time $\langle T^* \rangle$ which includes unfinished cases, where it has been assumed that $T = t_k$. In the following five columns is given the corresponding values of the evolution parameters for escape (actual or conditional), as

TABLE VIII

Summary of results of three-body simulations at Leningrad Observatory

| $N$ | $t_k$ | $\overline{T^*}$ | $\overline{T}$ | $\overline{DE}$ | $\overline{p}$ | $\overline{a}$ | $\overline{e}$ | $N_{c.e.}/N$ | $\rho$ | Problem |
|---|---|---|---|---|---|---|---|---|---|---|
| 2500 | 1000 | | 113.7 | 0.751 | 0.600 | 0.237 | 0.713 | 0.158 | 30 | I*a* |
| | | | ±2.2 | ±0.038 | ±0.006 | ±0.003 | ±0.006 | | | |
| 5500 | 1000 | – | 59.4 | 0.752 | 0.876 | 0.435 | 0.794 | 0.131 | 30 | I*b* |
| | | | ±1.1 | ±0.027 | ±0.009 | ±0.006 | ±0.003 | | | |
| 5000 | 1000 | 140.0 | 91.5 | 0.398 | 1.167 | 0.276 | 0.833 | 0.210 | 20 | II |
| | | ±3.2 | ±1.5 | ±0.015 | ±0.007 | ±0.001 | ±0.003 | | | |
| 3000 | 500 | 172.9 | 116.1 | 0.314 | 1.040 | 0.284 | 0.710 | 0.253 | 20 | III*a* |
| | | ±2.8 | ±1.9 | ±0.013 | ±0.008 | ±0.002 | ±0.004 | | | |
| 1000 | 500 | 115.3 | 89.0 | 0.298 | 1.247 | 0.414 | 0.711 | 0.230 | 20 | III*b* |
| | | ±4.3 | ±3.0 | ±0.020 | ±0.019 | ±0.006 | ±0.007 | | | |

*Notes*:
I*a*: plane problem; equal masses; $k_0 = 0$, $L = 0$.
I*b*: plane problem; unequal masses; $k_0 = 0$, $L = 0$.
II*a*: plane problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*a*: three-dimensional problem; equal masses; $k_0 \neq 0$, $L \neq 0$.
III*b*: three-dimensional problem; chance choice of masses and $k_0$.

well as their dispersions; the escape time $T$ in units of the crossing time; the relative excess energy of the escaper ($DE$) evaluated with respect to the escape criterion; the perimeter $p$ of the configuration triangle at the closest triple approach; and the semi-major axis $a$ and eccentricity $e$ of the final binary. Finally, column 9 shows the proportion of conditional escape and column 10 contains the assumed value of $\rho^*$.

Correlation coefficients between the evolution parameters of $10^4$ triple systems have been obtained, and the main ones are shown in Table IX. In particular, there is a strong correlation between the values of $p$ and $DE$, $a$, and $DE$. Thus, $p$ and $a$ are also well correlated in the sense that close triple approaches produce energetic escapers and small final binaries. Results of rotating systems are presented in Table X. Here $k_0$ denotes the initial virial ratio of kinetic and potential energy, whereas the last two columns contain the proportion of final prograde and retrograde orbits of the escapers. It can be seen that prograde orbits dominate in cases of fast rotation, whereas retrograde escape is more prevalent for slow rotation ($k_0 = 0.1$).

TABLE IX

The correlation coefficients between the evolution parameters

| Problem | $(T, n)$ | $(DE, p)$ | $(DE, a)$ | $(p, a)$ | $(p, e)$ |
|---|---|---|---|---|---|
| I$a$ | +0.875 | −0.464 | −0.693 | +0.679 | +0.329 |
| II$a$ | +0.722 | −0.444 | −0.721 | +0.610 | +0.216 |
| III$a$ | +0.865 | −0.462 | −0.803 | +0.601 | +0.229 |
| III$b$ | +0.871 | −0.446 | −0.476 | +0.694 | +0.221 |

*Notes*:

$\sigma_{max} = 0.32$ – r.m.s. deviation of the correlation coefficients.

I$a$: plane problem; equal masses; $k_0 = 0$, $L = 0$.

II$a$: plane problem; equal masses; $k_0 \neq 0$, $L \neq 0$.

III$a$: three-dimensional problem; equal masses; $k_0 \neq 0$, $L \neq 0$.

III$b$: three-dimensional problem; chance choice of masses and $k_0$.

TABLE X

Fraction of the triple systems with prograde ($N_p/N$) and retrograde ($N_{rg}/N$) motions

| $k_0$ | $N_p/N$ | $N_{rg}/N$ |
|---|---|---|
| 0.1 | 0.406 | 0.594 |
| 0.3 | 0.600 | 0.400 |
| 0.5 | 0.644 | 0.356 |

## 5.3. Escape from Initial Configurations

Various studies have demonstrated that small changes of initial conditions can lead to widely different outcomes (Anosova, 1969a; Agekian and Anosova, 1974, 1977; Standish, 1976). It has been shown (Agekian and Anosova, 1974, 1977) that the region

$D$ of all possible configurations breaks up into a number of small islands $s$, inside which the escape time $T$ is a continuous function of the initial coordinates. At the boundaries of $s$ the function $T(\xi, \eta)$ has a discontinuity such that $T(\xi^*, \eta^*) \to \infty$. During a crossing of the boundary, the number of triple approaches $n(\xi, \eta)$ may change by an arbitrary value. Moreover, the islands $s$ may consist of subregions $s^*$, inside which the function $n(\xi, \eta) = \text{const}$. When crossing the boundary of $s^*$, the function $T(\xi, \eta)$ changes continuously and $n(\xi, \eta)$ changes by 1. These results are summarized in Figures 5, 7, and 8. The initial positions of component $C$ are indicated and the numbers shown correspond to the time of escape. The regions which include $n(\xi, \eta) = 1$ are shaded in Figures 5 and 7. These figures show that the average escape time increases for an increasing degree of hierarchical structure, as measured by the initial configuration parameter $\eta^* = \rho/r_{\min}$ where $r_{\min}$ is the distance between the closest components.

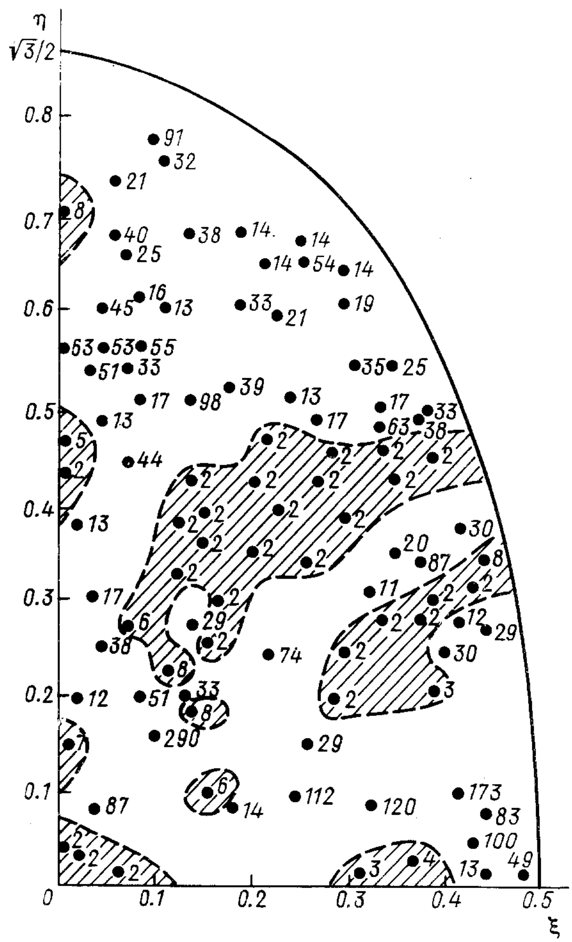

Fig. 7. The escape times $T(\xi, \eta)$ for case I$b$ (see notes to the tables).

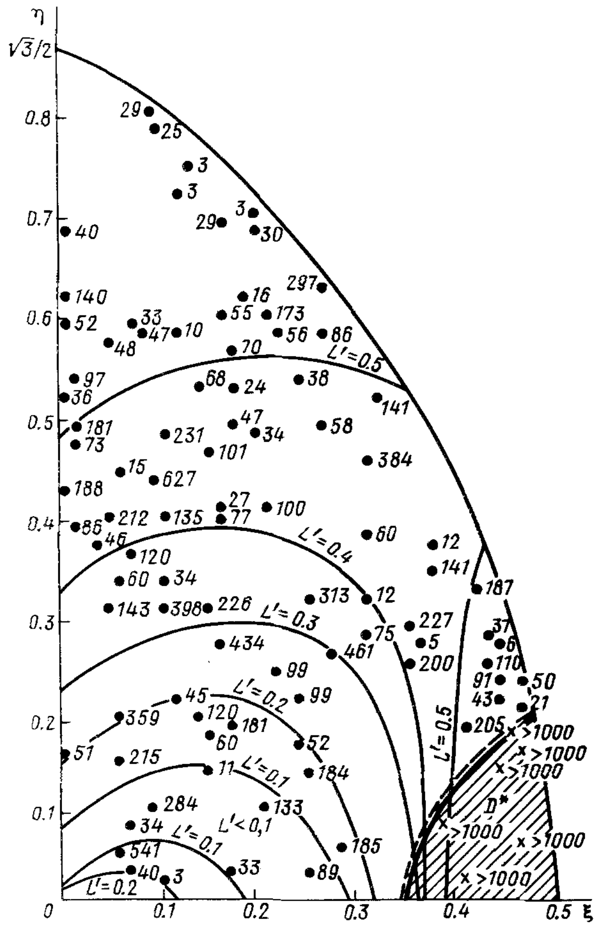

Fig. 8. The escape times $T(\xi, \eta)$ for case II$a$ (see notes to the tables). Solid lines are isolines of the angular momentum $L$ = const. The points on Figures 5, 7, and 8 denote initial positions of the component $C$ and corresponding numbers give the time of escape.

For systems with different masses and the same initial configuration, the escape time depends on the mass $M_d$ of a distant component. At first, for $M_d = M_{\max}$ the mean escape time is smaller than for $M_d = M_{\min}$. The former is due to the fact that the triple approach occurs earlier, whereas in the second case a massive binary forms which stabilizes the system as a whole by wide triple approaches. It has also been shown (Anosova and Orlov, 1984, Figure 7) that islands with $n(\xi, \eta) = 1$ increase in area for increased mass dispersion.

Two types of periodic orbits have been discovered in rotating systems with equal-mass components and no hierarchical structure (Anosova *et al.*, 1984): (i) in systems of a 'chain' type one body successively approaches the other two in turn without the latter approaching each other (three such chains in 5000 systems were noted in planar cases); (ii) in one 'toroidal' system successive approaches take place between all the

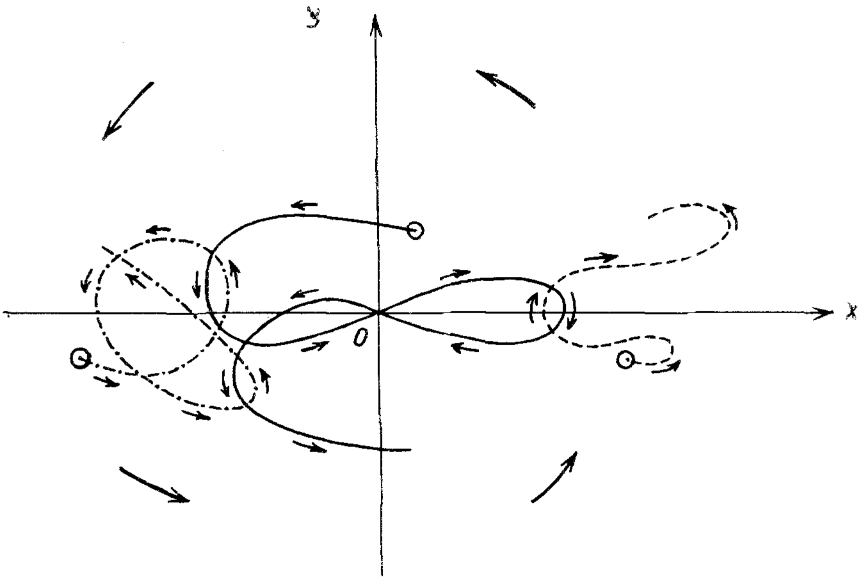


Fig. 9a.

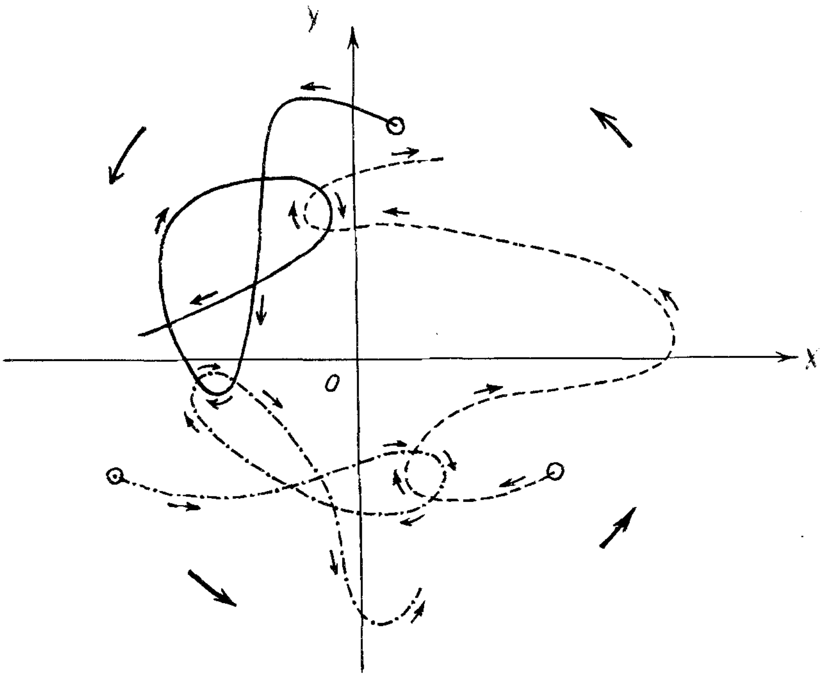


Fig. 9b.

Fig. 9. Periodic orbits in rotating systems with equal-mass components and no hierarchical structure. (a) Systems of the 'chain' type. (b) Systems of the 'toroidal' type. Open circles denote the initial positions of the components $A$, $B$, $C$.

bodies (only one case from 5000 systems was noted). Both types of periodic motions are illustrated schematically in Figure 9; the symbols are analogous to Figure 2. Note that the usual stability criteria which have been proposed for hierarchical systems refer to the minimum distance between the *same* pair of bodies during the whole evolution.

In 1981 a film was produced at the Leningrad Observatory which shows the complete evolution of a planar triple system with negative energy.

## 4. Applications to Observed Triple Systems

Since triple systems are common in the galactic field as well as in star clusters, the results of the three-body problem are of considerable importance. Questions relating to the age of clusters containing triple stars and a qualitative picture of their evolution are of special interest.

The computer simulations point to two types of behaviour for systems with negative energy. First, dynamically stable hierarchical systems exist in which the outer body executes an approximate Keplerian orbit with respect to the inner binary which retains its identity throughout. In these systems, the dynamical evolution may be effectively studied by analytical methods. In the second case, the motions of the bodies are of complicated form and the minimum two-body separation is associated with different pairs of bodies. Systems of the latter type are characteristically unstable and their investigation requires methods of computer simulations.

Triple stars and galaxies are referred to as having hierarchical (seldom as $\varepsilon$-Lyr type) or non-hierarchical structure (often as Trapezium type). Thus, the dynamically stable and unstable triple systems are usually distinguished observationally by the ratio of the maximum and minimum angular separation. However, the apparent configuration is not sufficient to decide on the relevant category: (i) projection effects may hide the true configuration; and (ii) apparent hierarchical forms may occur in both types, as numerical experiments have shown.

Data of astrometric, photometric, and spectroscopic observations of the components must be used to study the evolution of actual triple stars. The maximum attainable accuracy is required to obtain reliable information about parallaxes, relative positions, and velocities (from proper motions and radial velocities), as well as individual masses in order to exclude optical triples (see Anosova and Orlov, 1985; Anosova, 1984). One must also take into consideration the effects of observational errors. The above studies have shown that in order to obtain reliable results, a dynamical investigation can only be conducted for triple stars within 100 pc of the Sun. For larger distances, the present errors are such that even the correct sign of the total energy cannot be guaranteed with confidence. This study has also revealed that the catalogue stars do not contain sufficiently accurate information for a dynamical study, even for such well-known stars as $\alpha$ Cen and $\alpha$ Gem (Castor). However, an investigation of the character of the motions in triple stars and galaxies can still be undertaken, since the configurations are statistically related to these motions.

Statistical studies of the configuration distributions for simulated and observed triples have been conducted by Anosova and Orlov (1983b, 1985). Projection effects have been estimated theoretically (Ambartsumian, 1951; Agekian, 1954) and by computer simulations (Anosova, 1968; Anosova and Orlov, 1983b). These studies have shown that the projection effects do not influence the distributions significantly. Distributions of the true configurations of triple systems and their projection on the coordinate planes are displayed in Figure 10(a). The configurations have been scaled by the parameter $1/\eta^* = r_{\min}/\rho$, where $r_{\min}$ is the distance between two neighbouring components. Figure 10(b) shows the corresponding effect for a distribution $f(k)$ of virial coefficients $k = T_k/|U|$. These results do not depend on the choice of the mass dispersion coefficient $\sigma$ and of the initial values $k_0$.

Distributions of observed triple stars are based on data from the *Index Catalogue of Double and Multiple Stars* (Anosova and Orlov, 1983b), where probable optical systems are excluded (Anosova, 1969c). Distributions of galaxy triples have been analysed using available data (cf. Karachentseva *et al.*, 1979; Karachentsev and Karachentseva, 1981; Karachentseva and Karachentsev, 1982). Figure 11 gives results of distributions obtained by computer simulations (solid lines) and observations (dotted lines). Also shown (dashed lines) are configuration distributions chosen at random. It can be seen that the

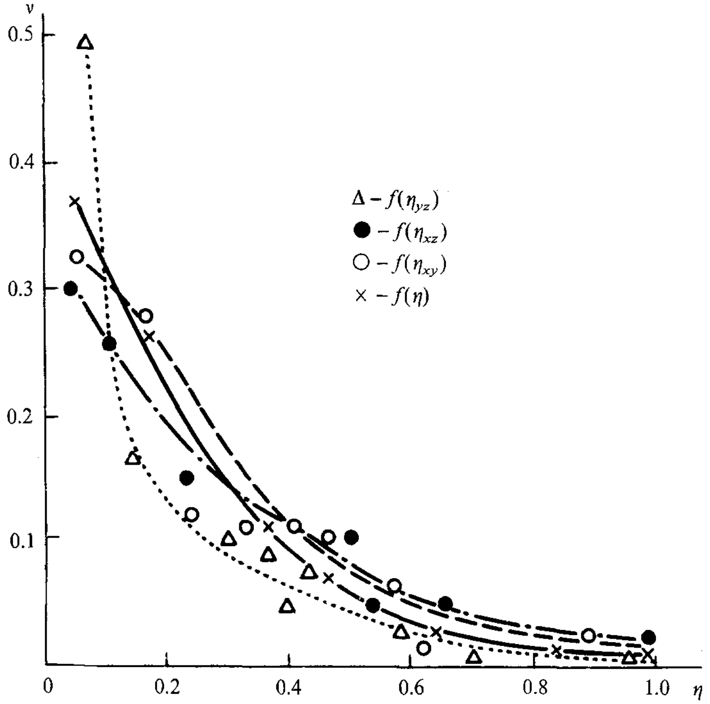


Fig. 10a.

Fig. 10. Distributions of triple systems. (a) True configurations and the projections on the coordinate planes. (b) Virial coefficients and the projections on the coordinate planes.

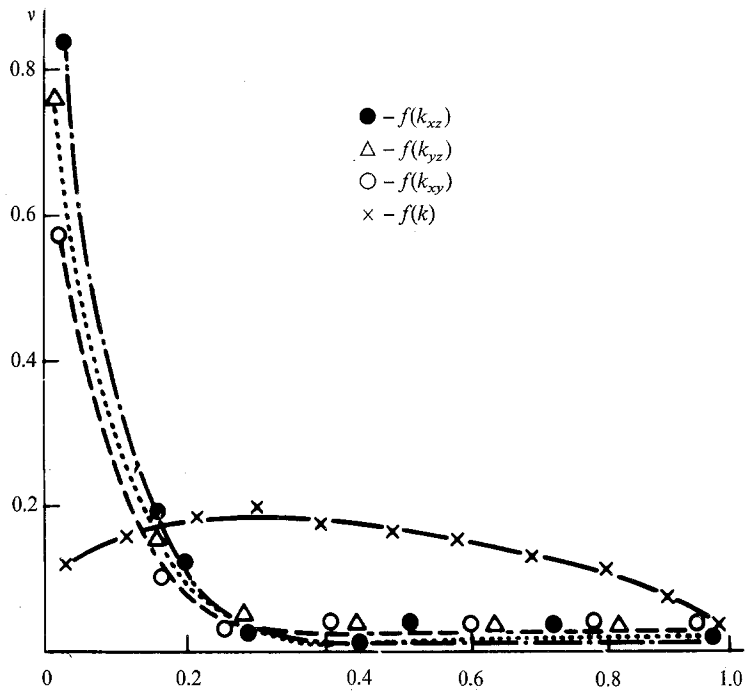


Fig. 10b.

results of numerical experiments are in good agreement with the observed data of triple stars but not with the randomly chosen configurations. In the case of galaxies, the small statistical sample prevents a definite conclusion. Moreover, these results are affected by the approximation of treating galaxies as point masses in the simulations and also by limitedness due to the selection criteria (Karachentseva *et al.*, 1979). Since the dynamical evolution of the great majority of the simulated systems terminate in escape, the similarity with the stellar configuration distributions can be considered as an argument for the instability of the observed triples.

The mean escape time of astronomical triple systems can now be estimated. Table IV gives $\langle T^* \rangle = 100\tau$ as a lower value of the average escape time of bound systems. Thus, for solar mass components and a typical dimension $d = 0.01$ pc, $\langle T \rangle = (1.6 \pm 1.5) \times 10^6$ yr, and for triple galaxies with $M_i = 10^{10} M_\odot$ and $d = 50$ kpc, $\langle T \rangle = (1.8 \pm 1.7) \times 10^{11}$ yr.

Actual triple systems are subject to regular and irregular forces due to other stars or galaxies. A distant component in bound orbit with respect to the binary itself may, therefore, become unbound by external effects, justifying the introduction of conditional escape. Critical values of $\rho^*$ corresponding to the tidal radius are given in Table XI for five external fields; the solar neighbourhood, open clusters, central parts of globular clusters, the Local Group, and clusters of galaxies. The additional parameters given in the table are: $\nu$, the mean particle density; $R$, the characteristic radius of the clusters;

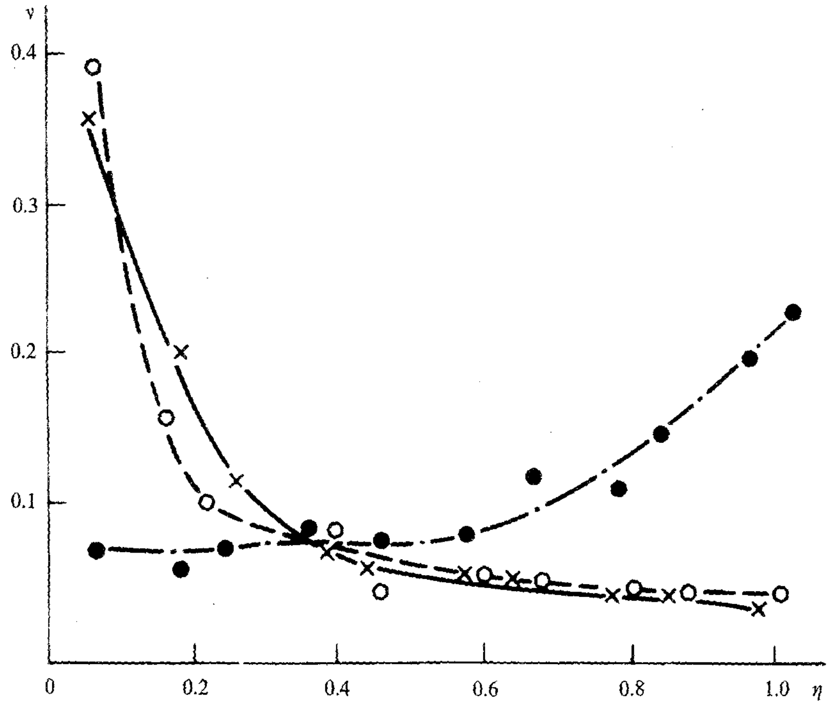


Fig. 11a.

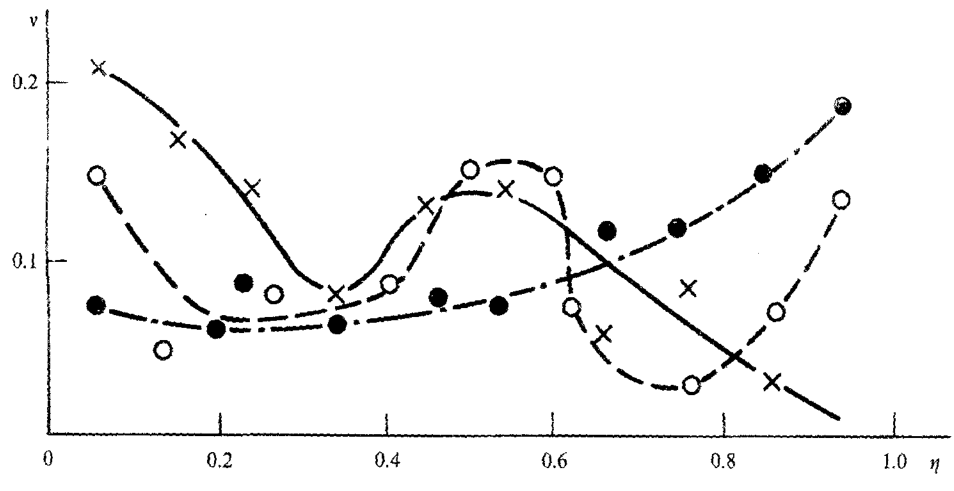


Fig. 11b.

Fig. 11. Comparisons between observations and simulations. (a) The case of triple stars. (b) The case of galaxy triplets. Computer simulations are shown as solid lines and observations by light dashed lines. Heavy dashed lines with filled circles denote randomly chosen configurations.

TABLE XI

The life times $T(\rho^*)$ for actual triple systems

| Quantities | Solar neighbourhood | Open cluster | Globular cluster | Metagalaxy | Galaxy cluster |
|---|---|---|---|---|---|
| $\nu$ | 0.12 $pc^{-3}$ | 10 $pc^{-3}$ | 200 $pc^{-3}$ | 0.02 $Mpc^{-3}$ | 2 $Mpc^{-3}$ |
| $D$ | – | 5 pc | 20 pc | – | 5 Mpc |
| $\rho^*$ | 1.3 pc | 0.2 pc | 0.08 pc | 2 Mpc | 0.4 Mpc |
| $\rho^*, d$ | 130 | 20 | 8 | 40 | 8 |
| $\bar{T}(\rho^*)/\tau$ | 160 | 100 | 70 | 130 | 70 |

and $d$, the mean size of the triple systems. Hence, the quoted *lower* values $\langle T^* \rangle \approx 100\tau$ for isolated triple systems can be considered as an average lifetime of actual stellar and galactic systems.

## 5. Suggestions for Further Studies

We may consider the present sample of $\approx 3 \times 10^4$ simulations as representing data for a complete statistical study of dynamical evolution and escape for isolated triple systems with negative total energy. Further investigations at the Leningrad Observatory (by Anosova, Orlov, and Zavalov) are planned along the following lines:

(1) A detailed study of strongly interacting states (close triple approaches).

(2) A study of dynamical evolution and escape of bound and unbound triple systems in the presence of external regular and irregular fields.

(3) Investigation of the dynamical states of observed triple stars, taking into account errors in astrometric and astrophysical data.

(4) Statistical and analytical generalizations of the earlier results obtained for simulations of triple systems with negative and positive energy.

## Acknowledgements

The author wishes to express her deep appreciations to Professor T. A. Agekian and Dr V. V. Orlov for their collaboration and constant interest in this work, as well as valuable discussions of the results. The author is also extremely grateful to the staff of the Leningrad University Computer Centre for the extensive use of their facilities over a 20 year period, amounting to a total computer time of more than three years. It is a pleasure to thank Dr Sverre J. Aarseth, Institute of Astronomy, Cambridge, U.K. for his interest in the author's work, as well as for discussions of results and editing of the English text.

## References

Aarseth, S. J.: 1974, *Astron. Astrophys.* **35**, 237.
Agekian, T. A.: 1954, *Astron. Zh.* **31**, 544.

Agekian, T. A. and Anosova, J. P.: 1967, *Astron. Zh.* **44**, 1261; *Soviet Astron.* **2**, 1006.
Agekian, T. A. and Anosova, J. P.: 1968, *Astrofizika* **4**, 31.
Agekian, T. A. and Anosova, J. P.: 1974, in Y. Kozai (ed.), *The Stability of the Solar Systems and of Small Stellar Systems*, D. Reidel Publ. Co., Dordrecht, Holland, p. 247.
Agekian, T. A. and Anosova, J. P.: 1977, *Trans. Astron. Obs. Leningr. Univ.* **33**, 52.
Agekian, T. A. and Martynova, A. I.: 1973, *Messenger of Leningr. Univ.* **1**, 122.
Agekian, T. A., Anosova, J. P., and Orlov, V. V.: 1983, *Astrofizika* **19**, 111; (*Astrophysics* **19**, 66).
Ambartsumian, V. A.: 1937, *Astron. Zh.* **14**, 207.
Ambartsumian, V. A.: 1951, *Lect. Acad. Sci. Armenian SSR* **13**, 97.
Anosova, J. P.: 1968, *Trans. Astron. Obs. Leningr. Univ.* **25**, 100.
Anosova, J. P.: 1969a, *Trans. Astron. Obs. Leningr. Univ.* **26**, 88.
Anosova, J. P.: 1969b, *Astrofizika* **5**, 161; *Astrophysics* **5**, 81.
Anosova, J. P.: 1969c, *Trans. Zh. Astron. Obs. Acad. Sci. USSR* **77**, 126.
Anosova, J. P.: 1977, *Messenger of Leningr. Univ.* **13**, 158.
Anosova, J. P.: 1984, in *Astrophysical Problems*, Saransk, p. 75.
Anosova, J. P. and Orlov, V.: 1983a, *Trans. Astron. Obs. Leningr. Univ.* **38**, 142.
Anosova, J. P. and Orlov, V.: 1983b, in *Star Clusters and Problems of Stellar Evolution*, Sverdlovsk, p. 121.
Anosova, J. P. and Orlov, V.: 1984, *Trans. Astron. Obs. Leningr. Univ.* **39**, 101.
Anosova, J. P. and Orlov, V.: 1985, *Trans. Astron. Obs. Leningr. Univ.* **40**, 65.
Anosova, J. P. and Poloahenzev, D. D.: 1978, *Trans. Astron. Obs. Leningr. Univ.* **34**, 128.
Anosova, J. P. and Zavalov, N. N.: 1981, *Trans. Astron. Obs. Leningr. Univ.* **36**, 109.
Anosova, J. P. and Zavalov, N. N.: 1986, *Astron. Zh.*, in press.
Anosova, J. P., Bertov, D. I., and Orlov, V. V.: 1981, *Astrofizika* **20**, 327 (*Astrophysics* **20**, 177).
Birkhoff, G. D.: 1927, *Dynamical Systems*, American Mathematical Society, p. 247.
Black, D. C.: 1982, *Astron. J.* **87**, 1333.
Golubev, V. G.: 1967, *Lect. Acad. Sci. USSR* **174**, 767.
Griffith, J. S. and North, R. D.: 1974, *Celes. Mech.* **8**, 473.
Harrington, R. S.: 1972, *Celes. Mech.* **6**, 322.
Heggie, D. C.: 1975, *Monthly Notices Roy. Astron. Soc.* **173**, 729.
Hohl, J. and Watt, J.: 1976, in J. Hohl and J. Watt (eds.), *Modern Numerical Methods for the Solution of Ordinary Differential Equations*, Oxford, p. 312.
Karachentsev, I. D. and Karachentseva, V. E.: 1981, *Astrofizika* **17**, 5; *Astrophysics* **17**, 1.
Karachentsev, I. D. and Karachentseva, V. E.: 1982, *Astrofizika* **18**, 5; *Astrophysics* **18**, 1.
Karachentseva, V. E., Karachentsev, I. D., and Sherbanovskii, A. L.: 1979, *Izvest. Special Astrofiz. Obs. USSR* **11**, 3.
Kuznetsova, E. A. and Orlov, V. V.: 1983, Trudy XII Stud. Conf., Leningrad Univ.
Marchal, C.: 1974, *Celes. Mech.* **9**, 381.
Marchal, C., Yoshida, J., and un-Yi-Sui: 1984, *Celes. Mech.* **33**, 193.
Monaghan, J. J.: 1976, *Monthly Notices Roy. Astron. Soc.* **176**, 63.
Standish, E. M.: 1971, *Celes. Mech.* **4**, 44.
Standish, E. M.: 1972, *Astron. Astrophys.* **21**, 185.
Standish, E. M.: 1976, *Celes. Mech.* **14**, 493.
Szebehely, V.: 1971, *Celes. Mech.* **4**, 116.
Szebehely, V.: 1972, *Celes. Mech.* **6**, 84.
Szebehely, V.: 1973, in B. D. Tapley and V. Szebehely (eds.), *Recent Advances in Dynamical Astronomy*, D. Reidel Publ. Co., Dordrecht, Holland, p. 75.
Szebehely, V.: 1979, *Astron. Astrophys.* **78**, 349.
Szebehely, V. and Zare, K.: 1977, *Astron. Astrophys.* **58**, 145.
Tevzadze, G. A.: 1962, *News Acad. Sci. Armenian SSR* **15**, 67.
Von Hoerner, S.: 1960, *Z. Astrophys.* **50**, 184.
Wielen, R.: 1974, *Astron. Rechen-Inst. Heidelberg*, No. 77, 326.
Yoshida, J.: 1972, *Publ. Astron. Soc. Japan* **24**, 391.