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Second Viscosity and the Attenuation of Fourth and Second Sound in Superfluid ^{3}He Near the Transition Temperature

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The relevant coefficient of second viscosity ζ_3 is calculated exactly near the transition in terms of the quasiparticle scattering amplitude in the normal state. ζ_3 is shown to dominate the attenuation of fourth and second sound. Second sound should be well defined only at frequencies below about 1 Hz.

1. INTRODUCTION

A complete hydrodynamic description of the anisotropic superfluid phases of ^{3}He requires the introduction of several a priori unknown transport coefficients.[1,2] Among these the coefficients of second viscosity ζ_1, ζ_2, and ζ_3 are of particular interest since they govern the attenuation of soundlike collective modes involving oscillations of $\mathbf{v}_s - \mathbf{v}_n$. Second viscosity coefficients appear in the momentum conservation law ($\mathbf{j}$ is the momentum density)

$$(\partial/\partial t)\mathbf{j} = \nabla \Pi^r + \nabla \Pi^d \tag{1a}$$

where the dissipative part of the stress tensor is given by

$$\Pi^d_{ij} = \eta_{ijkl}(\nabla_k v^n_l + \nabla_l v^n_k - \tfrac{2}{3}\delta_{kl}\,\mathrm{div}\,\mathbf{v}_n) + (\zeta_1)_{ij}\,\mathrm{div}\,(\mathbf{j} - \rho \mathbf{v}_n) + (\zeta_2)_{ij}\,\mathrm{div}\,\mathbf{v}_n + \xi_{ikj}\nabla_l \phi_{kl} \tag{1b}$$

with ϕ_{kl} defined in Ref. 1, and in the acceleration equation of the superfluid,

$$(\partial/\partial t)m\mathbf{v}_s + \nabla[\mu - m\zeta_3\,\mathrm{div}\,(\mathbf{j} - \rho\mathbf{v}_n) - m(\zeta_1)_{ij}\nabla_i v^n_j] = 0 \tag{2}$$

Here $\mathbf{v}_s$ and $\mathbf{v}_n$ are the velocities of the superfluid and normal components, respectively, μ is the chemical potential, and ρ is the mass density. It is known that the bulk viscosity in the normal state ζ_2 is smaller than the shear viscosity by a factor of $(T/T_F)^2$ and may safely be neglected. It turns out that all the second viscosity coefficients in the momentum conservation law (1) are small, of order $(T/T_F)^2$. Only the scalar coefficient ζ_3 in Eq. (2) is of magnitude comparable to η/ρ^2. This has been shown by Shumeiko[4] for a weak coupling model isotropic Fermi superfluid. Since there is already one experiment in which ζ_3 possibly has been measured[5] and since the observability of second sound depends crucially on the magnitude of ζ_3, it seems worthwhile to perform a realistic calculation of this quantity for superfluid ^{3}He. In this paper the leading term in ζ_3 in the vicinity of the transition is calculated exactly in terms of the normal-state quasiparticle lifetime on the Fermi surface. This parallels the exact calculation of the intrinsic spin relaxation time and orbital relaxation time in the A phase by Pethick and co-workers. In all these cases the transport coefficient diverges like T/Δ near the transition because of the conservation of quasiparticle number or spin in the normal state as opposed to the superfluid state. This divergence makes the solution of the Boltzmann integral equation trivial.

2. SECOND VISCOSITY

In the hydrodynamic regime, the motion of the superfluid condensate is approximately decoupled from the gas of thermal excitations, i.e., the Bogoliubov quasiparticles in the case of a Fermi system. The coefficients ζ_i are a measure of the dissipation brought about by collisions among the quasiparticles in the superfluid if the superfluid density deviates locally from equilibrium. Quantitatively this effect is described by the change in the quasiparticle distribution function $\delta\nu_k(\mathbf{r}, t)$, which obeys the kinetic equation[8]

$$(\partial/\partial t)\delta\nu_k + \nabla_k E_k \cdot \nabla_r \delta\nu_k - \nabla_k \nu_k \cdot \nabla_r \delta E_k = I \tag{3}$$

with the quasiparticle energies in the superfluid state

$$E_k = (\xi_k^2 + |\Delta_k|^2)^{1/2}$$

and in the normal state

$$\xi_k = (k^2/2m^*) - \mu$$

and the deviations from equilibrium of these quantities

$$\delta E_k = (1/E_k)(\xi_k\,\delta\xi_k + \tfrac{1}{2}\,\delta|\Delta_k|^2) + \mathbf{k}\cdot\mathbf{v}_s$$

and

$$\delta\xi_k = (1/N_F m)[F_0\,\delta\rho + (1/k_F{}^2)F_1 \mathbf{k}\cdot\mathbf{j}] - \delta\mu$$

Here F_0 and F_1 are Fermi liquid parameters and $N_F = m^* k_F/\pi^2$ is the density of states at the Fermi level. The equilibrium distribution is given by

$$\nu_k = -\tfrac{1}{2}\tanh(E_k/2T)$$

The change in the particle density is expressed as

$$\delta n = \sum_k [(\xi_k/E_k)\,\delta\nu_k + \nu_k(|\Delta_k|^2/E_k^3)\,\delta\xi_k] \tag{4}$$

The collision integral I conserves energy and momentum but not the number of quasiparticles. The collision integral vanishes if applied to the local equilibrium distribution function

$$\delta\nu_k^l = (\partial\nu_k/\partial E_k)[\delta E_k - \mathbf{k}\cdot\mathbf{v}_n - (E_k/T)\,\delta T] \tag{5}$$

The local values of the chemical potential, velocity of the normal component, and temperature are fixed by the requirement that Eq. (5) give the correct local densities and currents. Therefore the deviation from local equilibrium of the expression (4) for the density has to vanish:

$$\delta n' = 0 = \frac{\partial n}{\partial\mu}\,\delta\mu' + \sum_k \frac{\xi_k}{E_k}\,\delta\nu_k' \tag{6}$$

The primed quantities denote deviations from local equilibrium. In Eq. (6) a term $(\partial n/\partial\Delta)|_\mu\,\delta\Delta'$, describing the deviation of the off-diagonal energy from local equilibrium, has been omitted, since $(\partial n/\partial\Delta)|_\mu$ is of order T/T_F and small. This term contains the component of $\delta\nu_k'$ even in ξ_k.

The terms involving ζ_i in Eq. (2) may be interpreted as a change $\delta\mu'$ in the chemical potential from its local equilibrium value caused by dissipative processes. Equation (6) relates $\delta\mu'$ to the quasiparticle distribution function. The problem of calculating ζ_3 is reduced to finding the component of $\delta\nu_k'$ odd in ξ_k and even in $\mathbf{k}$.

In order to solve the kinetic equation, as a first step the local equilibrium solution (5) is inserted into the left-hand side of Eq. (3), taking into account $\partial\mu/\partial n = (1+F_0)/N_F$ and making use of the continuity equation to eliminate $(\partial/\partial t)\rho$ in favor of $-\operatorname{div}\mathbf{j}$. Also, the negligible quantity $(\partial/\partial t)(\mathbf{v}_n - \mathbf{v}_s)$ may be dropped. Then the explicit expression for the collision integral near T_c, obtained by a Bogoliubov transformation from the matrix collision integral in Ref. 9, is derived. Collecting terms odd in ξ_k and even in $\mathbf{k}$, one obtains

$$\begin{aligned}&\frac{\partial\nu_k}{\partial E_k}\frac{\xi_k}{E_k}N_F^{-1}\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)\\&\quad= -\frac{1}{\tau(\xi)}\,\delta\nu_k' + \frac{1}{\tau_N(0)}\frac{\xi/E}{\cosh(\xi/2T)}\\&\qquad\times\int_{-\infty}^{\infty}\frac{d\xi'}{T}B\left(\frac{\xi-\xi'}{2T}\right)\cosh\frac{\xi'}{2T}\int\frac{d\Omega'}{4\pi}\frac{\xi'}{E'}\,\delta\nu_k'\end{aligned} \tag{7}$$

where

$$1/\tau(\xi)=[1/\tau_N(0)][1+(\xi/\pi T)^2]$$

and

$$B(x)=\frac{1}{\pi^2}\frac{x}{\sinh(x/2)}$$

$\tau_N(0)$ is the quasiparticle lifetime at the Fermi surface in the normal state. The angle dependence of the integral equation (7) is trivially eliminated by defining the isotropic function ψ as

$$\delta\nu_k=-\frac{1}{4T}\psi\left(\frac{\xi_k}{T}\right)\frac{\xi_k/E_k}{\cosh(\xi_k/2T)}\tau_N(0)N_F^{-1}\,\mathrm{div}\,(\mathbf{j}-\rho\mathbf{v}_n) \tag{8}$$

$\psi(x)$ satisfies the integral equation

$$\frac{1}{\cosh(x/2)}=\left[1+\left(\frac{x}{\pi}\right)^2\right]\psi(x)-\int_{-\infty}^{\infty}dx'\,B(x-x')\psi(x')\int\frac{d\Omega_k}{4\pi}\frac{x'^2}{x'^2+D_k^2} \tag{9}$$

where $D_k^2=|\Delta_k|^2/T^2$.

In the limit $D\to 0$, the homogeneous equation is solved by

$$\lim_{D\to 0}\psi(x)=1/\cosh(x/2)$$

taking into account

$$\int dx'\frac{B(x-x')}{\cosh(x'/2)}=\frac{1+(x/\pi)^2}{\cosh(x/2)}$$

One therefore seeks the solution in the form

$$\psi(x)=\frac{\lambda}{\cosh(x/2)}+\psi^{(1)}(x) \tag{10}$$

with the supplementary condition

$$\int dx\,\psi^{(1)}(x)\frac{1+(x/\pi)^2}{\cosh(x/2)}=0 \tag{11}$$

Multiplying Eq. (9) by $1/\cosh(x/2)$ and integrating, one finds

$$\lambda^{-1}=-\frac{1}{4}\int dx\frac{1+(x/\pi)^2}{\cosh^2(x/2)}\left\langle\frac{D^2}{x^2+D^2}\right\rangle=-\frac{\pi}{4}\langle D\rangle+O(D^2) \tag{12}$$

with

$$\langle D\rangle=\int\frac{d\Omega}{4\pi}D_k$$

$\psi^{(1)}(x)$ satisfies the integral equation

$$\frac{1}{\cosh(x/2)} - \frac{4}{\pi^2}\frac{x}{\sinh(x/2)} = \left[1 + \left(\frac{x}{\pi}\right)^2\right]\psi^{(1)}(x) - \int dx'\, B(x-x')\psi^{(1)}(x') \tag{13}$$

and therefore is regular in the limit $D \to 0$.

Inserting Eqs. (12), (10), and (8) into Eq. (6), one finds by comparison with Eq. (2)

$$\zeta_3 = \frac{1}{m}\frac{\partial \mu}{\partial \rho}\tau_N(0)\left(\frac{4}{\pi}\frac{T}{\langle\Delta\rangle} + \alpha\right), \qquad T \leqslant T_c \tag{14}$$

where a rough estimate gives $\alpha = 0.6 \pm 0.1$.

A similar analysis of the part of the kinetic equation even in ξ_k yields an inhomogeneity proportional to div $\mathbf{j}$, with a coefficient smaller by a factor of T/T_F than the one multiplying div$(\mathbf{j} - \rho\mathbf{v}_n)$ in Eq. (7). The second viscosity coefficient ζ_1 therefore is of order $(T/T_F)^2\rho\zeta_3$.

The result, Eq. (14), for ζ_3 is not valid arbitrarily close to T_c, because in the derivation one had to assume $\Delta \gg 1/\tau_N$. For $\Delta\tau_N \lesssim 1$ the energy gap in the quasiparticle spectrum is smeared out by collision effects and the quasiparticle concept is no longer valid in this simple form. The corresponding corrections to the shear viscosity in this gapless regime have been calculated in the framework of a more general theory in Ref. 10. By carrying over the argumentation of Ref. 10 to the present case, it is seen that in the integral expression for λ^{-1}, Eq. (12), Δ^2 has to be replaced by $\Delta^2 + (1/2\tau)^2$ in the denominator, and λ^{-1} is obtained as

$$\lambda^{-1} = -\frac{\pi}{4}\left\langle \frac{\Delta_k^2/T}{[\Delta_k^2 + 1/4\tau^2]^{1/2}} \right\rangle \tag{15}$$

Hence, the second viscosity in the gapless regime is obtained as

$$\zeta_3 = \frac{1}{m}\frac{\partial \mu}{\partial \rho}\frac{2}{\pi}\frac{T}{\langle\Delta^2\rangle}, \qquad \Delta\tau \ll 1 \tag{16}$$

independent of the quasiparticle scattering properties.

3. ATTENUATION OF FOURTH AND SECOND SOUND

Fourth sound is an oscillation of the density of the superfluid component at fixed normal component, i.e., $\mathbf{v}_n = 0$. Solving Eq. (2) simultaneously with the continuity equation

$$(\partial/\partial t)\rho + \text{div}\, \boldsymbol{\rho}_s \cdot \mathbf{v}_s = 0$$

we find the dispersion law of fourth sound as

$$\omega^2 = u_4^2 q^2 (1 - i\omega\tau_4) \tag{17}$$

where

$$u_4^2 = (1/\rho)(\hat{q} \cdot \boldsymbol{\rho}_s \cdot \hat{q}) u_1^2$$

and

$$\tau_4 = \rho\zeta_3/u_1^2 = (4/\pi)(T/\langle\Delta\rangle)\tau_N(0), \; T \lesssim T_c \tag{18}$$

Here $u_1 = [(\rho/m)(\partial\mu/\partial\rho)]^{1/2}$ is the velocity of first sound.

In the A phase an additional damping term appears in Eq. (18) due to the coupling of $\mathbf{v}_s$ to the $\boldsymbol{l}$ vector. This gives rise to a contribution of order $\rho\eta/u_1^2$ in Eq. (18), where η is a transport coefficient in the equation of motion for $\boldsymbol{l}$, introduced by Graham.[1] η may be estimated by comparison with the Cross–Anderson theory of orbital relaxation in terms of their viscosity coefficient μ as

$$\eta \sim (\hbar/2m)^2/\mu \sim 3 \times 10^{-4} (1 - T/T_c)^{-1} \zeta_3$$

Thus η is negligible except very close to T_c.

The damping of fourth sound in an ideal geometry is thus determined by the quasiparticle lifetime at the Fermi surface in the normal state $\tau_N(0)$ and the angular average of the anisotropic gap parameter. While the relative width of the mode $\Delta\omega_4/\omega_4$ diverges as $T \to T_c$ like Δ^{-1}, the absolute attenuation tends to zero $\sim\Delta$.

Fourth sound has been observed by Yanof and Reppy[5] and by Kojima *et al.*[11] Yanof and Reppy quote results for the quality factor of their resonance cavity Q_4, which is directly related to τ_4 by $Q_4 = (\omega\tau_4)^{-1}$. At a pressure of 22.8 bar, reduced temperature $1 - T/T_c = 0.05$, and frequency of 728 Hz, they found $Q_4 = 64$. From Eq. (18), using

$$\langle\Delta\rangle = \frac{\pi}{4}\pi\left(\frac{\Delta C}{C_N}\right)^{1/2} T_c \left(1 - \frac{T}{T_c}\right)^{1/2}$$

$\Delta C/C_N = 1.6$, and $T^2\tau_N(0) = 0.3\,\mu\text{sec}\,\text{mK}^2$, one obtains $Q_4 = 2380$. The quasiparticle lifetime $\tau_N(0)$ has been determined by the relaxation of the wall-pinned spin mode in the B phase,[13] by the relaxation of the $\hat{\boldsymbol{l}}$ vector in the A phase,[14] and by the broadening of the collective mode peak in the sound absorption[10] in ^{3}He-B at $\sim$20 bar. In order to bring down the Q factor to the observed value, some additional damping mechanism, probably normal fluid slippage at the pore surfaces, has to be invoked. The intrinsic Q factor as given by Eq. (18) is inversely proportional to the frequency and temperature independent for a given q (i.e., mode), as found experimentally.

The dispersion law of second sound is found by solving Eqs. (1) and (2) and the entropy conservation law

$$(\partial/\partial t)S + S \operatorname{div} \mathbf{v}_n = (1/T)\nabla \cdot \boldsymbol{\kappa} \cdot \nabla T \tag{19}$$

where $\boldsymbol{\kappa}$ is the thermal conductivity tensor, imposing the constraint

$$\mathbf{j} = \boldsymbol{\rho}_n \cdot \mathbf{v}_n + \boldsymbol{\rho}_s \cdot \mathbf{v}_s = 0$$

The result is

$$\omega^2 = u_2^2 q^2 (1 - i\omega\tau_2) \tag{20}$$

where

$$u_2^2 = (TS^2/\rho C_v)\overline{(\rho_s/\rho_n)}$$

and

$$\tau_2 = \frac{C_v}{S^2 T}\left[\frac{4}{3}\bar{\eta} + \frac{\rho}{C_v}\bar{\kappa}\overline{(\rho_s/\rho_n)}^{-1} + \rho^2 \zeta_3\right] \tag{21}$$

Here we have defined

$$\bar{\kappa} = (\hat{q} \cdot \kappa \cdot \hat{q})$$

$$\bar{\eta} = \sum_{i,j,k,l} \hat{q}_i \hat{q}_j \hat{q}_k \hat{q}_l \eta_{ijkl}$$

and

$$\overline{(\rho_s/\rho_n)} = \hat{q} \cdot \rho_s \cdot \boldsymbol{\rho}_n^{-1} \cdot \hat{q}$$

In Eq. (21) terms involving the coupling to the $\boldsymbol{l}$ vector in the A phase have been neglected, taking into account that the relaxation of the $\hat{\boldsymbol{l}}$ vector is much faster[14] ($\tau^l \sim 10^{-3}$ sec) than the typical time τ_2 of the second-sound mode (see below) in the temperature regime of interest.

In Ref. 8 a more complicated expression for the velocity of second sound was given, involving the tensor entropy σ_{ij}. It was not realized then that σ_{ij} is proportional to the unit tensor if correctly evaluated.[15]

In fact, from Ref. 8

$$\begin{aligned}\sigma_{ij} &= -\frac{1}{T}\sum_k \kappa_i \xi_k \frac{\partial \nu_k}{\partial E_n}\frac{k_j}{m^*} \\ &= \sum_k k_i \frac{\partial}{\partial k_j}[\nu_k \ln \nu_k + (1-\nu_n)\ln(1-\nu_k)] \\ &= \delta_{ij} S\end{aligned}$$

(correcting for a factor of 3 in the definition[8] of σ_{ij}). In Eq. (13) of Ref. 8 a factor of $(\rho c_v)^{-1}$ is missing.

The various contributions to τ_2 in Eq. (21) behave differently in the limit $T \to T_c$. The term involving the thermal conductivity diverges as $1/\Delta^2$, and, multiplied by $u_2^2 \omega q^2$ in Eq. (20), goes over into the thermal diffusion mode of the normal state.

$$\omega_D = -i(\kappa/C_v)q^2$$

The contribution involving the shear viscosity behaves regularly as $T \to T_c$, while the second-viscosity term diverges as Δ^{-1}. In the temperature regime $10^{-3} < 1 - T/T_c < 0.2$, where this theory is thought to be valid, the second-sound attenuation is dominated by the contribution from the second viscosity. At the melting pressure one estimates $\tau_2 \sim (1/10)(1 - T/T_c)^{-1/2}$ sec. In order to observe second sound one would have to work at frequencies as low as

$$f < 1/2\pi\tau_2 \sim 1.5(1 - T/T_c)^{1/2}\ \mathrm{sec}^{-1}$$

In conclusion, further measurement of the attenuation of fourth sound and possibly second sound would provide information on the quasiparticle scattering rate in the normal state and could be compared to similar information obtained from the spin relaxation and orbital relaxation experiments.

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