PLANE SYMMETRIC VACUUM BIANCHI TYPE I COSMOLOGICAL MODEL IN BRANS–DICKE THEORY

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Abstract. The paper presents an exact solution of the vacuum Brans–Dicke field equations for the metric tensor of a spatially homogeneous and anisotropic cosmological model. The Kasner metric is shown as a special case. Some physical properties of the model are discussed.

1. Introduction

The scalar-tensor theories of gravitation gained new interest in cosmology through the discussion by Dehnen and Obregón (1971, 1972) of exact solutions of Brans–Dicke (BD) cosmological equations which has no analogy in Einstein's theory even for the larger values of the parameter ω. Recently Van den Bergh (1980) obtained a complete set of exact solutions of vacuum BD equations for a static spherically-symmetric metric and pointed out that the classical general relativity is more Machian than the BD theory itself.

In this paper an exact solution of the vacuum field equations of BD theory is obtained for the metric tensor of a spatially homogeneous and anisotropic plane symmetric Bianchi type-I cosmological model. The well-known Kasner universe is obtained as a particular case. In the general relativistic case $\omega \to \infty$ the solution passes over to a flat metric.

2. Field Equations and Their Solution

We use here the spatially homogeneous and anisotropic plane symmetric line-element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - A^2(\mathrm{d}x^2 + \mathrm{d}y^2) - B^2\,\mathrm{d}z^2\,, \tag{2.1}$$

where the quantities A and B are functions of time t which measure the expansion or contraction of the model in the transverse (A) and longitudinal (B) direction. We number the coordinates x^1, x^2, x^3, x^4 as x, y, z, t, respectively.

The BD equations in empty space are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -\frac{\omega}{\phi^2}\,(\phi_{;i}\phi_{;j} - \tfrac{1}{2}g_{ij}\phi_{;k}\phi;k - \frac{1}{\phi}\,(\phi_{;i;j} - g_{ij}\Box\phi)\,, \tag{2.2}$$

$$\Box\phi = 0\,. \tag{2.3}$$

Contracting (2.2) we get

$$R_{ij} = -\frac{\omega}{\phi^2}\,\phi_{;i}\phi_{;j} - \frac{1}{\phi}\,\phi_{;i;j}\,, \tag{2.4}$$

where the BD scalar ϕ is the reciprocal of the gravitational constant. Other symbols have their usual meaning. We apply the field equations (2.3) and (2.4) to the metric (2.1).

The field equations may be written as

$$\frac{A_{44}}{A} + \left(\frac{A_4}{A}\right)^2 + \left(\frac{A_4}{A}\right)\left(\frac{B_4}{B}\right) = -\left(\frac{\phi_4}{\phi}\right)\left(\frac{A_4}{A}\right), \tag{2.5}$$

$$\frac{B_{44}}{B} + 2\left(\frac{A_4}{A}\right)\left(\frac{B_4}{B}\right) = \left(\frac{\phi_4}{\phi}\right)\left(\frac{B_4}{B}\right), \tag{2.6}$$

$$\frac{2A_{44}}{A} + \frac{B_{44}}{B} = -\omega\left(\frac{\phi_4}{\phi}\right)^2 - \frac{\phi_{44}}{\phi}\,, \tag{2.7}$$

$$\frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{2A_4}{A} + \frac{B_4}{B}\right) = 0\,; \tag{2.8}$$

where the suffix 4 denotes ordinary differentiation with respect to t.

In order to solve these equations we take u and v the logarithmic derivatives ϕ_4/ϕ and A_4/A of ϕ and A, respectively. Then Equations (2.5) and (2.8) readily give

$$\frac{B_4}{B} = -\left(u + 2v + \frac{v_4}{v}\right), \tag{2.9}$$

$$\frac{B_4}{B} = -\left(u + 2v + \frac{u_4}{u}\right). \tag{2.10}$$

From (2.9) and (2.10) it is obvious to have $u = \mu v$ with μ a constant. Then from (2.9), (2.6), and (2.7) we obtain

$$4v_4 + (6 + \mu^2\omega)v^2 = 0\,; \tag{2.11}$$

which, on integration, gives

$$v = (\alpha t + \beta)^{-1}\,, \tag{2.12}$$

where $\alpha = (6 + \mu^2\omega)/4$ and β an arbitrary constant of integration. We take $\alpha > 0$.

Also, since $u = \mu v$, we have

$$u = \mu(\alpha t + \beta)^{-1}\,. \tag{2.13}$$

Integration of (2.12) and (2.13) yields

$$A = (\alpha t + \beta)^{1/\alpha}\,, \tag{2.14}$$

$$\phi = (\alpha t + \beta)^{\mu/\alpha}\,. \tag{2.15}$$

Now, the Equation (2.9), in conjunction with (2.14) and (2.15) gives

$$B = (\alpha t + \beta)^{1-(\mu+2)/4\alpha} . \tag{2.16}$$

Hence, the metric (2.1) reduces to

$$ds^2 = dt^2 - (\alpha t + \beta)^{2/\alpha}(dx^2 + dy^2) - (\alpha t + \beta)^{2-(\mu+z)/2\alpha} dz^2 . \tag{2.17}$$

However, (2.17) can be written in the form

$$ds^2 = dt^2 - t^{2/\alpha}(dx^2 + dy^2) - t^{2-(\mu+2)/2\alpha} dz^2 \tag{2.18}$$

by a shift of origin of t and by a scale transformation.

For $\mu = 0$, (2.15) gives ϕ constant, and the metric (2.18) reduces to

$$ds^2 = dt^2 - t^{4/3}(dx^2 + dy^2) - t^{-2/3} dz^2 \tag{2.19}$$

which is the Kasner cosmological universe (Misner *et al.*, 1973). Thus the vacuum solution approaches the general relativistic result for $\mu = 0$, independently of the parameter ω. In the case $\omega \to \infty$ (i.e., $\alpha \to \infty$, $\phi = \text{const.}$) we arrive at the metric

$$ds^2 = dt^2 - (dx^2 + dy^2) - t^2 dz^2 , \tag{2.20}$$

which can be reduced to the Galilean form by the transformation

$$t \sinh z = \bar{z} , \qquad t \cosh z = \bar{t} . \tag{2.21}$$

Thus the solution (2.18) has the correct Minkowskian limit as $\omega \to \infty$.

3. Discussion

The metric (2.18) corresponds to a homogeneous but anisotropic space whose total volume increases with increasing t if

$$\omega > \frac{\mu - 12}{\mu^2} , \tag{3.1}$$

and decreases if

$$\omega < \frac{\mu - 12}{\mu^2} . \tag{3.2}$$

For ω to be negative $|\mu| < 12$. There is no expansion or contraction in volume for $\omega = (\mu - 12)/\mu^2$. The linear distances along the two axes (x and y) increase, while they decrease along the third axis (z) according as (3.2) holds.

The 14 invariants of the four-dimensional curvature tensor go to infinity at $t = 0$. Therefore, the moment $t = 0$ is the physical singularity of the model.

The invariant

$$c = \mathrm{d}\,(\log \phi)/\mathrm{d}\,(\log V)\,, \qquad V = A^2 B\,, \tag{3.3}$$

which characterizes the initial velocity of variation for the ϕ-field (that is the gravitational constant $G \sim \phi^{-1}$) with the general expansion of the cosmological models (Ruban and Finkelstein, 1972), has the value

$$c = 4\mu/(8 + \mu^2 \omega) \tag{3.4}$$

when $c = 0$ ($\phi = G^{-1} = \text{const.}$), $\mu = 0$ and the solution (2.18) goes to the Kasner solution (2.19).

For the metric (2.18) we find three non-vanishing components of the torsion coefficients.

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