BOUNDED MOTION IN A GENERALIZED TWO-BODY PROBLEM

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Abstract. In this paper we prove the existence of ring-type bounded motion in an isolated system consisting of a massive point particle and a homogeneous cube. We study the case of planar motion where the particle moves in a symmetry plane of the cube and we use a rotating frame of reference with its center at the mass center of the cube and its axes coinciding with the symmetry axes of the cube. We prove that, for negative values of the total energy and properly chosen values of the total angular momentum, the relative distance of the bodies has an upper and a lower bound – i.e., the regions of possible motion lie inside an annulus around the cube (motion inside a ring or an island).

1. Introduction

The existence of bounded motions in the gravitational N-body problem is a property with obvious physical interest. In the case of two-point bodies bounded motions do exist for negative energies and they are of the ring type. In the case of N point bodies ($N > 2$) the integrals of energy and angular momentum cannot assure the existence of bounded motion (Bozis, 1976; Loks and Sergysels, 1985). However, if one replaces at least one of the particles with an extended body the possibility of bounded motion arises. In fact, in the case $N = 3$, if we replace one particle by a homogeneous sphere it can be proved that bounded motions exist (Bozis and Michalodimitrakis, 1982).

Here we start with the problem of two-point bodies and we generalize it by replacing one of them by a homogeneous cube. We prove that the integrals of energy and angular momentum can assure the existence of ring-type bounded motions. These motions require negative energies.

2. Equations of Motion and First Integrals

Let K be the mass center of a homogeneous cube of mass M and edge $2a$. Let also $Kxyz$ be a rectangular system of Cartesian axes coinciding with the symmetry axes of the cube. We consider a point particle Σ, of mass m, which moves in the symmetry plane $z = 0$ of the cube. The cube K and the particle Σ form an isolated system of two bodies moving in a plane under their mutual gravitational attraction. We also consider an inertial rectangular system $GXYZ$ of Cartesian axes with its origin at the mass center G of the system (K, Σ) and its $Z = 0$ plane coinciding with the $z = 0$ plane (Figure 1).

The configuration of the system is completely specified by the coordinates x, y of Σ relative to the Kxy axes and by the angle φ between the positive x and X semi-axes. We choose x, y, φ as generalized coordinates.

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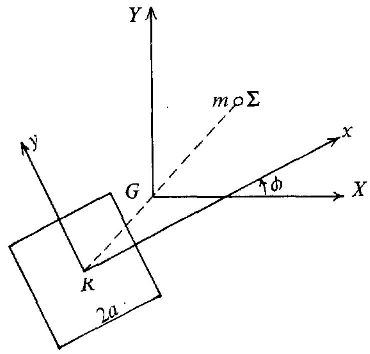

Fig. 1. The inertial axes $GXYZ$ and the rotating axes $Kxyz$. G is the mass center of the system.

The absolute velocities of K and Σ are equal to

$$\bar{U}_K = -m(M+m)^{-1}[(\dot{x} - \dot{\varphi}y)\bar{i} + (\dot{y} + \dot{\varphi}x)\bar{j}],$$

$$\bar{U}_\Sigma = M(M+m)^{-1}[(\dot{x} - \dot{\varphi}y)\bar{i} + (\dot{y} + \dot{\varphi}x)\bar{j}].$$

Introducing the moment of inertia $I = 2Ma^2/3$ of the cube with respect to the z axis and the reduced mass $\mu = Mm/(M+m)$, the kinetic energy of the system can be written in the form

$$T = \tfrac{1}{2}I\dot{\varphi}^2 + \tfrac{1}{2}\mu[(\dot{x} - \dot{\varphi}y)^2 + (\dot{y} + \dot{\varphi}x)^2].$$

The potential energy of the system is equal to $G\sigma mV$ where (cf. MacMillan, 1973)

$$\begin{aligned} V = & -(x+a)(y+a)A_1 + (x+a)(y-a)A_2 - 2a(x+a)A_3 - \\ & -(x-a)(y-a)A_4 + (x-a)(y+a)A_5 + 2a(x-a)A_6 - \\ & -2a(y+a)A_7 + 2a(y-a)A_8 + (x+a)^2(B_1 - B_2) - \\ & -(x-a)^2(B_3 - B_4) - (y+a)^2(B_5 - B_6) - (y-a)^2(B_7 - B_8) - \\ & -a^2(B_9 - B_{10} + B_{11} - B_{12}), \end{aligned}$$

with

$$A_1 = ln\,\frac{\rho_{11}+a}{\rho_{11}-a}, \quad A_2 = \ln\frac{\rho_{12}+a}{\rho_{12}-a}, \quad A_3 = \ln\frac{\rho_{11}+\rho_{12}+2a}{\rho_{11}+\rho_{12}-2a},$$

$$A_4 = \ln\frac{\rho_{22}+a}{\rho_{22}-a}, \quad A_5 = \ln\frac{\rho_{21}+a}{\rho_{21}-a}, \quad A_6 = \ln\frac{\rho_{22}+\rho_{21}+2a}{\rho_{22}+\rho_{21}-2a}, \tag{1a}$$

$$A_7 = \ln\frac{\rho_{11}+\rho_{21}+2a}{\rho_{11}+\rho_{21}-2a}, \quad A_8 = \ln\frac{\rho_{22}+\rho_{12}+2a}{\rho_{22}+\rho_{12}-2a},$$

$$B_1 = \tan^{-1}\frac{a(y+a)}{\rho_{11}(x+a)}, \quad B_2 = \tan^{-1}\frac{a(y-a)}{\rho_{12}(x+a)},$$

$$B_3 = \tan^{-1} \frac{a(y+a)}{\rho_{21}(x-a)}, \qquad B_4 = \tan^{-1} \frac{a(y-a)}{\rho_{22}(x-a)},$$

$$B_5 = \tan^{-1} \frac{a(x-a)}{\rho_{21}(y+a)}, \qquad B_6 = \tan^{-1} \frac{a(x+a)}{\rho_{11}(y+a)},$$

$$B_7 = \tan^{-1} \frac{a(x+a)}{\rho_{12}(y-a)}, \qquad B_8 = \tan^{-1} \frac{a(x-a)}{\rho_{22}(y-a)}, \tag{1b}$$

$$B_9 = \tan^{-1} \frac{(x+a)(y-a)}{a\rho_{12}}, \qquad B_{10} = \tan^{-1} \frac{(x+a)(y+a)}{a\rho_{11}},$$

$$B_{11} = \tan^{-1} \frac{(x-a)(y+a)}{a\rho_{21}}, \qquad B_{12} = \tan^{-1} \frac{(x-a)(y-a)}{a\rho_{22}}.$$

The distances ρ_{ij} $(i, j = 1, 2)$ are given by

$$\begin{aligned} \rho_{11}^2 &= (a+x)^2 + (a+y)^2 + a^2, \qquad \rho_{12}^2 = (a+x)^2 + (a-y)^2 + a^2; \\ \rho_{21}^2 &= (a-x)^2 + (a+y)^2 + a^2, \qquad \rho_{22}^2 = (a-x)^2 + (a-y)^2 + a^2. \end{aligned} \tag{2}$$

The Lagrangian of the system is equal to

$$L = T - G\sigma mV.$$

Since $\partial L/\partial\varphi = 0$ and $\partial L/\partial t = 0$, the integrals of angular momentum and energy exist and are equal, respectively, to

$$I\dot{\varphi} + \mu(x\dot{y} - \dot{x}y) + \mu\dot{\varphi}(x^2 + y^2) = \text{const.} \equiv c, \tag{3}$$

$$\tfrac{1}{2}I\dot{\varphi}^2 + \tfrac{1}{2}\mu[(\dot{x} - \varphi y)^2 + (\dot{y} + \dot{\varphi}x)^2] + G\sigma mV = \text{const.} \equiv h. \tag{4}$$

The corresponding Lagrangian equations are

$$\begin{aligned} \ddot{x} &= -\mu^{-1} G_\sigma mV_x + \dot{\varphi}(2\dot{y} + x\dot{\varphi}) + I^{-1} yG\sigma m(xV_y - yV_x), \\ \ddot{y} &= -\mu^{-1} G\sigma mV_y - \dot{\varphi}(2\dot{x} - y\dot{\varphi}) - I^{-1} xG_\sigma m(xV_y - yV_x), \\ \dot{\varphi} &= I^{-1}(xV_y - yV_x)G\sigma m, \end{aligned} \tag{5}$$

where the partial derivatives V_x and V_y are given by

$$\begin{aligned} V_x = {} & a(-A_1 + A_5 - A_2 + A_4 - 2A_3 + 2A_6 + 2B_1 - 2B_2 + 2B_3 - 2B_4) + \\ & + 2x(-B_3 + B_4 + B_1 - B_2) + y(-A_1 + A_5 + A_2 - A_4) \end{aligned} \tag{5a}$$

and

$$\begin{aligned} V_y = {} & a(-A_1 + A_2 + A_4 - A_5 - 2A_7 + 2A_8 - 2B_5 + 2B_6 + 2B_7 - 2B_8) + \\ & + x(-A_1 + A_2 - A_4 + A_5) + 2y(-B_5 + B_6 - B_7 + B_8). \end{aligned} \tag{5b}$$

Introducing the dimensionless variables ξ, n, τ by the transformation

$$x = a\xi , \qquad y = an , \qquad t = \left(G\sigma \frac{M+m}{M}\right)^{-1/2} \tau \equiv b\tau ,$$

the equation and the integrals of motion assume, respectively, the form

$$\begin{aligned}
\ddot{\xi} &= -V_\xi + \dot{\varphi}(2\dot{n} + \xi\dot{\varphi}) + 1.5kn(\xi V_n - nV_\xi) , \\
\ddot{n} &= -V_n - \dot{\varphi}(2\dot{\xi} - n\dot{\varphi}) - 1.5k\xi(\xi V_n - nV_\xi) , \\
\ddot{\varphi} &= 1.5k(\xi V_n - nV_\xi)
\end{aligned} \tag{6}$$

and

$$\frac{2}{3k}\dot{\varphi} + (\xi\dot{n} - \dot{\xi}n) + \dot{\varphi}(\xi^2 + n^2) = L , \tag{7}$$

$$\frac{2}{3k}\dot{\varphi}^2 + (\dot{\xi}^2 + \dot{n}^2) + 2\dot{\varphi}(\xi\dot{n} - \dot{\xi}n) + \dot{\varphi}^2(\xi^2 + n^2) + 2V(\xi, n) = E , \tag{8}$$

where $k = m/(M + m)$ and dots denote differentiation with respect to the normalized time τ. The new constants L and E are related to the old ones (c and h) by the relations

$$L = bc(ka^2M)^{-1} , \qquad E = 2hb^2(M + m)(mMa^2)^{-1} .$$

The function $V(\xi, n)$ in (8) can be obtained from the function $V(x, y)$ in (1) by the substitution $x = \xi$, $y = n$, $a = 1$. Using polar coordinates r, θ we write Equations (7) and (8) in the form

$$\left(\frac{2}{3k} + r^2\right)\dot{\varphi} + r^2\dot{\theta} = L , \tag{7a}$$

$$\left(\frac{2}{3k} + r^2\right)\dot{\varphi}^2 + 2r^2\dot{\theta}\dot{\varphi} + \dot{r}^2 + r^2\dot{\theta}^2 + 2V = E . \tag{8a}$$

3. Curves and Surfaces of Zero Velocity

Eliminating $\dot{\varphi}$ between (7) and (8) we find that

$$\left(\frac{2}{3k} + \xi^2\right)\dot{\xi}^2 + \left(\frac{2}{3k} + n^2\right)\dot{n}^2 + 2\xi n\dot{\xi}\dot{n} =$$

$$= (E - 2V(\xi, n))\left(\frac{2}{3k} + \xi^2 + n^2\right) - L^2 . \tag{9}$$

It is easily seen that the first member of (9) is a positive definite quadratic function of

$\dot{\xi}$ and $\dot{n}$. Therefore, we must have

$$(E - 2V(\xi, n))\left(\frac{2}{3k} + \xi^2 + n^2\right) - L^2 \geq 0\,. \tag{10}$$

Inequality (10) defines in the $0\xi n$ plane the regions of possible motion. The equality sign in (10) corresponds to the boundaries of these regions – i.e., to the curves of zero velocity (hereafter referred to as ZVC).

The ZVC depend on the choice of the values of the constants E and L. If $E < 0$ it follows from (10) that V cannot become zero. Since V is equal to zero in the case of infinite separation of K and Σ only, it follows that, for $E < 0$, the particle Σ cannot escape to infinity, i.e., the motion of Σ relative to $Kxyz$ is bounded.

Eliminating $\dot{\theta}$ between (7a) and (8a) we find that

$$\dot{r}^2 + 2V + \frac{2}{3k}\left(1 + \frac{2}{3kr^2}\right)\dot{\varphi}^2 + \frac{L^2}{r^2} - \frac{4L\dot{\varphi}}{3kr^2} = E\,. \tag{11}$$

Since $\dot{r}^2 \geq 0$ it follows that

$$F(\xi, n, \dot{\varphi}) \geq 0\,, \tag{12}$$

where

$$F(\xi, n, \dot{\varphi}) = -\frac{2}{3kr^2}\left(r^2 + \frac{2}{3k}\right)\dot{\varphi}^2 + \frac{4L}{3kr^2}\,\dot{\varphi} + (E - 2V) - \frac{L^2}{r^2}\,. \tag{13}$$

Equation $F(\xi, n, \dot{\varphi}) = 0$ defines in the ξ, n, $\dot{\varphi}$ space a zero velocity surface (hereafter referred to as ZVS) for which we note that:

(i) The values of $\dot{\varphi}$ along a ZVS are finite. In fact, for $\dot{\varphi} \to \infty$ (or $\dot{\varphi} \to -\infty$) we should have

$$-\frac{2}{3k}\left(1 + \frac{2}{3kr^2}\right) \to 0\,,$$

which is impossible.

(ii) The domain of definition of (13) on the $0\xi n$ plane can be found by eliminating $\dot{\varphi}$ between (13) and $\partial F/\partial\dot{\varphi} = 0$. We find that

$$(E - 2V)\left(r^2 + \frac{2}{3k}\right) - L^2 = 0\,, \tag{14}$$

i.e., the above domain is bounded by the ZVC.

(iii) The intersection of a ZVC with the plane $\dot{\varphi} = 0$, i.e., the curve

$$(E - 2V)r^2 - L^2 = 0 \tag{15}$$

lies in the interior of (14).

(iv) The intersection of a ZVS with the plane $\dot{\varphi} = \dot{\varphi}_0$ is the curve

$$-\left(\frac{2}{3k} + \frac{4}{9k^2r^2}\right)\dot{\varphi}_0^2 + \frac{4L}{3kr^2}\,\dot{\varphi}_0 + (E - 2V) - \frac{L^2}{r^2} = 0\,. \tag{16}$$

For $r \to \infty$ we have $V \to 0$ and (16) gives

$$\dot{\varphi}_0 = \pm (1.5kE)^{1/2}\,. \tag{17}$$

Therefore, for $E \geq 0$ the surface (13) extends to infinity asymptotically to the planes (17). For $E < 0$ it is obvious that the surface (13) is bounded in the space ξ, n, $\dot{\varphi}$.

4. Ring-type and Island-type Bounded Motions

We shall prove that for negative E and properly chosen values of L the regions of possible motion lie inside a circular ring encircling the cube.

From (1) and (2) we find that

$$V(A_1) = -8\ln\frac{\sqrt{6}+1}{\sqrt{6}-1} - 4\ln\frac{\sqrt{6}+\sqrt{2}+2}{\sqrt{6}+\sqrt{2}-2} + 8\tan^{-1}\frac{\sqrt{6}}{12} + 4\tan^{-1}\frac{\sqrt{6}}{3}$$

and

$$V(A_2) = -4\ln 10 + 8\tan^{-1}\tfrac{1}{3} + \tan^{-1}\tfrac{4}{3}\,,$$

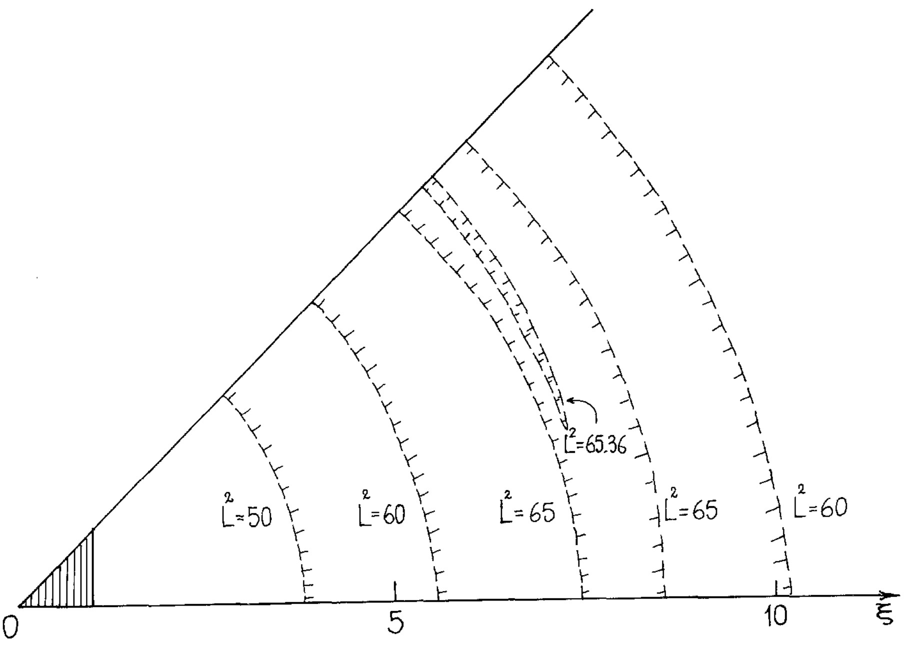

Fig. 2. Evolution of the regions of possible motion with increasing L for $E = -1$ and $k = 0.5$ (only the lower half of the first quadrant is shown).

where A_1 and A_2 are, respectively, the points $(1, 0)$ and $(1, 1)$. We note that

$$V(A_1) < V(A_2) < 0 .$$

More generally, we prove that

$$V(A_1) \leq V(\xi, n) \Bigg| \begin{array}{l} 1 \leq \xi < \infty , \\ \xi \geq n \geq 0 . \end{array} \tag{18}$$

In fact, the function $V(\xi, 0)$ is monotonically increasing since the force $F_\xi = -\mathrm{d}V(\xi, 0)/\mathrm{d}\xi$ on the ξ axis is attractive ($F_\xi < 0$). Therefore,

$$V(A_1) < V(\xi, 0) \mid 1 < \xi < \infty .$$

On the other hand we have

$$V(\xi_0, 0) \leq V(\xi_0, n) \mid \xi \geq n \geq 0 ,$$

for every $\xi_0 \in (1, \infty)$. In fact, if for $\xi = \xi_0$ there exists a $n \neq 0$ such that $0 > V(\xi_0, 0) > V(\xi_0, n)$ then we could find a $n = n_0 \neq 0$ such that (because $V(\xi_0, \infty) = 0$)

$$\left[\frac{\mathrm{d}V(\xi_0, n)}{\mathrm{d}n}\right]_{n = n_0} = 0 \quad \text{or} \quad \left[\frac{\partial V(\xi, n)}{\partial n}\right]_{\substack{\xi = \xi_0 \\ n = n_0}} = -F_n(\xi_0, n_0) = 0 .$$

But this is impossible. Therefore, the inequality (18) is valid.

Let $V_\sigma = V_\sigma(\xi)$ and $F_\sigma = F_\sigma(\xi)$ be, respectively, the potential and the algebraic value of the force on the ξ axis inside the cube. Although we do not know the expressions for V_σ and F_σ we do know that

$$F(A_1) \leq F_\sigma(\xi) \leq 0 \mid 0 \leq \xi \leq 1 \quad \text{and} \quad F_\sigma(0) = 0 .$$

We shall prove that

$$V_\sigma(0) < V(A_1) .$$

We introduce the functions

$$V^*(\xi) = \begin{cases} V_\sigma(\xi) \mid 0 \leq \xi < 1 , \\ V(\xi, 0) \mid 1 \leq \xi < \infty \end{cases}$$

and

$$F^*(\xi) = \begin{cases} F_\sigma(\xi) \mid 0 \leq \xi < 1 , \\ F(\xi, 0) \mid 1 \leq \xi < \infty . \end{cases}$$

We note that:

(i) $F^*(\xi) \leq 0 \mid 0 \leq \xi < \infty$.

(ii) The function $F^*(\xi)$ is decreasing for $0 \leq \xi < 1$ and increasing for $1 \leq \xi < \infty$.

(iii) The function $F^*(\xi)$ is not necessarily continuous at $\xi = 1$.

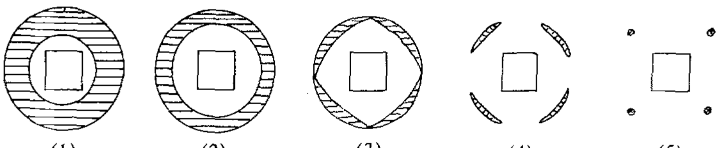

Fig. 3. Schematic evolution of the regions of possible motion in the case of Figure 2.

(iv) The function $V^*(\xi)\,|\,0 \le \xi < \infty$ is continuous because $V^*(\xi) = -\int_{\infty}^{\xi} F(\xi)\,\mathrm{d}\xi$ and $F(\xi)$ has one point of discontinuity (at $\xi = 1$) at most.

We also note that for $\xi \neq 1$ we have $\mathrm{d}V^*(\xi)/\mathrm{d}\xi = -F^*(\xi) \ge 0$, i.e., $V^*(\xi)$ is an increasing function of ξ. Therefore, $V^*(0) < V^*(A_1)$ and according to (18) we must have

$$V_\sigma(0) < V(A_1) < V(\xi, n)\,. \tag{19}$$

Let us choose E so that

$$0 > E > 2V_\sigma(0)\,, \tag{20}$$

then

$$\frac{E - 2V_\sigma(0)}{E - 2V(\xi, n)} > 1\,. \tag{21}$$

Let us also choose L, so that

$$L^2 \ge 2(E - 2V_\sigma(0))\left(1 + \frac{1}{3k}\right). \tag{22}$$

Then the inequality (10) becomes

$$(E - 2V)r^2 \ge L^2 - \frac{2}{3k}\,(E - 2V)\,,$$

or

$$r^2 \ge \frac{1}{E - 2V}\left\{2(E - 2V_\sigma(0)) + \frac{2}{3k}\,(E - 2V_\sigma(0)) - \frac{2}{3k}\,(E - 2V)\right\} =$$

$$= \frac{1}{E - 2V}\left\{2(E - 2V_\sigma(0)) + \frac{4}{3k}\,(V - V_\sigma(0))\right\} > 2\,\frac{E - 2V_\sigma(0)}{E - 2V} > 2\,.$$

From the above we conclude that if E and L are chosen so that (20) and (22) are verified, the particle Σ cannot escape to infinity or approach the cube closer than $\sqrt{2}$ (ring-type bounded motion).

Figure 2 shows the evolution of the ZVC with increasing L for $E = -1$ and $k = 0.5$. We note that the regions of possible motion are almost circular rings whose width decreases as L increases. However, it turns out that the width along the diagonal of the cube decreases at a slightly lower rate than that along the ξ or n axes. In this way, the moment the L takes the value corresponding to the equilibrium points on the ξ or n axes, the width along these axes becomes zero, i.e. the inner and the outer ZVC meet at the equilibrium points of the ξ and n axes. If we further increases L, the permissible ring brakes at these equilibrium points and four islands of permissible motion are formed. These islands shrink (as L increases) and finally they reduce to points which, obviously, are equilibrium points on the diagonal of the cube. The above evolution of the regions of possible motion is shown schematically in Figure 3.

5. Conclusions

The main conclusion of this paper is the fact that, for properly chosen values of the energy and angular momentum of an isolated system consisting of a homogeneous cube and a massive point particle moving in a fixed plane under their mutual gravitational attraction, there exist ring-type bounded motions. The corresponding regions of possible motion take the form of almost circular rings around the cube or of islands around the diagonal equilibrium points.

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