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Sophisticated Voting Outcomes and Agenda Control *

J. S. Banks

Division of Humanities and Social Sciences, California Institute of Technology, Pasadena, CA 91125, USA

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Abstract. Necessary and sufficient conditions for an alternative to be a sophisticated voting outcome under an amendment procedure are derived. The uncovered set, as first defined by Miller (1980), is shown to be potentially reducible, and conditions are determined for which this reduction equals the set of sophisticated voting outcomes. In addition, simple methods are given for calculating both the uncovered set and its reduction.

1. Introduction

During the last 10 years students of political voting models have become aware of the generic instability of simple majority rule as an aggregate decision process (McKelvey 1979; Schofield 1978; Cohen 1979). The rule rather than the exception was found to be the absence of an "undominated" outcome and the existence of majority voting cycles, implying indeterminate outcomes. Rarely, however, do we find real-world situations which allow for the lack of structure underlying the above-cited results. Further work has focused on the constraints imposed by various structural characteristics (e.g. rules of order, juristictional arrangements), with equilibrium predictions becoming more precise as the structure on the voting process increases (Shepsle 1979; Denzau and Mackay 1981; Denzau and Mackay 1983).

Another avenue of study revolves around the notion of an *agenda* as a means of facilitating the decision problem of voters when faced with a set of alternatives, the definition of agenda being simply an ordering of the alternatives from which pairwise comparisons may be made. The goal here is to determine the influence of the agenda setter over the voting outcome; hence the equilibrium concept involves the subset of alternatives which are outcomes of this process under a given ordering, and under specified assumptions on the voters' behavior. (The assumption here, and in the work

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cited below, is that voters act *sophisticatedly* (Farquharson 1969), implying that they take into account the optimal behavior of others in solving their own optimal decisions.) Miller (1980) derived the uncovered set as a solution for this problem, which in turn led to research by McKelvey (1983) and Shepsle and Weingast (1984) on characteristics of this set. An alternative being in the uncovered set is only a necessary condition, however, for it to be the outcome under any particular ordering; that is, although all voting outcomes are in the uncovered set, not all elements of the uncovered set are voting outcomes. Thus a complete characterization of the equilibrium set of alternatives has not been established.

This paper attempts to fill this void. Necessary and sufficient conditions for an alternative to be a voting outcome are derived using the majority preference relation on the set of alternatives. Also, a further necessary condition is given, one stronger than Miller's and more easily calculable than the above condition, with the property that, in certain situations, the set of alternatives defined by this condition equals the set of sophisticated voting outcomes. (Some of these results have been derived independently, and in a somewhat different context, in Moulin 1984).

2. Notation, Definitions, Assumptions

Consider a set $N = \{1, \ldots, n\}$ of voters, n odd, a finite set $X = \{x_1, x_2, \ldots, x_t\}$ of alternatives, and assume that individual preferences are represented by a simple ordering $P_i \subseteq X \times X$; i.e. all voters have a strict preference over the set X of alternatives. For a set B let $|B|$ denote its cardinality. The *majority preference relation P* is defined as, for any $x_j, x_k \in X$,

$$x_j P x_k \Leftrightarrow |\{i \in N: x_j P_i x_k\}| > \frac{n}{2}.$$

The assumptions on the preferences of voters and the set N of voters guarantee that the majority preference relation P is complete (in Miller's terms, the voting game is a "tournament").

A convenient representation of the majority preference relation is given by the *dominance matrix* $D = [d_{ij}]$ where $d_{ij} = 1$ if and only if $x_i P x_j$, and $d_{ij} = 0$ otherwise. Thus we have a matrix of zeros and ones, as is depicted in Fig. 1:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 1

For this example we see that $x_1 P x_2$, $x_1 P x_4$, and $x_1 P x_5$; $x_2 P x_3$, $x_2 P x_4$, etc. (The zeros on the main diagonal are implied by the irreflexiveness of the majority preference relation.) For any given dominance matrix, define the *score* s_i of alternative x_i as the number of alternatives which x_i defeats; i.e. $s_i = \sum_{j=1}^{t} d_{ij}$.

Define the correspondence $P: X \twoheadrightarrow X$ by

$$P(x_j) = \{x_k \in X: x_k P x_j\};$$

analogously define

$$P^{-1}(x_j) = \{x_k \in X: x_j P x_k\}$$

and set

$$\bar{P}(x_j) = P^{-1}(x_j) \cup \{x_j\}.$$

Definition. An alternative x_j is a *Condorcet Proposal* if $\bar{P}(x_j) = X$.

Given any two alternatives $x_j, x_k \in X$, we say that x_j *covers* x_k if $x_k \in P^{-1}(x_j)$ and $P^{-1}(x_k) \subset P^{-1}(x_j)$. More generally, we have the following:

Definition. For any set $B \subseteq X$ and any $x_j, x_k \in B$, the *covering relation* $C_B \subseteq B \times B$ is defined as:

$x_j C_B x_k$ if and only if
i) $x_k \in P^{-1}(x_j)$, and
ii) $\{P^{-1}(x_k) \cap B\} \subset \{P^{-1}(x_j) \cap B\}$.

Given $B \subseteq X$, we define the *uncovered set* relative to B as:

$$U(B) = \{x_j \in B: \sim [\exists x_k \in B \text{ s.t. } x_k C_B x_j]\}.$$

The following is a generalization of results due to Miller (1980):

Theorem 2.1.
i) Given any set $B \subseteq X$, C_B is transitive;
ii) $x_j \in U(X) \Leftrightarrow \bigcup_{x_k \in \bar{P}(x_j)} P^{-1}(x_k) = X \backslash \{x_j\}$.

Theorem 2.1 i is a direct consequence of the completeness of the majority preference relation, since this implies that the first condition of the covering relation is vacuous; that is, $P^{-1}(x_k) \subset P^{-1}(x_j)$ implies $x_k \in P^{-1}(x_j)$. Since the inclusion relation is transitive, so is the covering relation, and it is in fact a strict partial order or any given subset $B \subseteq X$. Thus, the uncovered set relative to B is the set of maximal elements of B under the order C_B. Theorem 2.1 ii is Miller's "two step" principle which states that, given any alternative $x_k \in X$, an alternative $x_j \in U(X)$ is either majority preferred to x_k or else there exists an alternative $x_i \in P^{-1}(x_j)$ which is majority preferred to x_k.

Notice that the square of the dominance matrix gives a representation of the alternatives which are "reachable" in two steps; since D^2 consists of elements of the form $d_{ij} \times d_{jk}$, an element of D^2 is nonzero if and only if $d_{ij} \neq 0$ and $d_{jk} \neq 0$. That is, x_k is reachable from x_i if and only if there exists an $x_j \in X$ such that $x_i P x_j$ and $x_j P x_k$.

By Theorem 2.1 ii then, the matrix $S = D + D^2$ can be used to calculate the uncovered set. For the example in Fig. 1, we have

$$D^2 = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad S = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Fig. 2

To see whether, for instance, x_1 is in the uncovered set, we merely inspect the first row of the matrix S. If the jth entry in the first row is positive, than either $x_1 P x_j$ or x_1 can reach x_j in two steps. Notice that the main diagonal of S consists of zeros, which is a consequence of the asymmetry and irreflexiveness of the majority preference relation. By Theorem 2.1 ii and Fig. 2, we see that in this example $U(X) = \{x_1, x_2, x_3, x_4\}$ but not x_5, since x_5 does not defeat x_1 and x_5 cannot reach x_1 in two steps.

Before proceeding, we make the following behavioral and procedural assumptions:

Assumption 1. There exists an agenda setter who permutes the set X of alternatives, thus creating an *agenda*. We may characterize this process by letting $T = \{1, 2, \ldots, t\}$ and defining the set of agendas as $\bar{A} = \{(x_{\phi(1)}, \ldots, x_{\phi(t)}) \in X^t : \phi\colon T \to T \text{ and } \phi \text{ is } 1-1\}$. Let $y_i = x_{\phi(i)}$. Then the agenda setter chooses some $A \in \bar{A}$ which assigns to each level of the agenda y_i a unique alternative from X. (Thus, for a fixed agenda, we can compare the y_i's in terms of the majority preference relation, since there is a one-to-one correspondence between alternatives in X and positions in the agenda.)

Assumption 2. Voting follows an *amendment procedure*, where, for a given agenda $A \in \bar{A}$ an aggregate decision rule is arrived at by: i) comparing y_t and y_{t-1} via the majority preference relation (i.e. taking a vote); ii) comparing the preferred alternative to y_{t-2}, etc. After the $t-1$ pairwise comparisons, the remaining alternative is declared the voting outcome. (This procedure is a generalization of Roberts' Rules: let y_1 be the "status quo", y_2 be the "bill", y_3 the "amendment", y_4 the "amendment to the amendment", etc. Roberts' Rules then dictates that the final amendment, say y_t, be voted on against y_{t-1}, the winner vs. y_{t-2}, and so on, with the (possibly amended) bill voted against the status quo last.)

Assumption 3. All voters adopt *sophisticated voting* strategies; the resulting decision we call the *sophisticated voting outcome* (cf. Farquharson 1969).

A complete desciption of sophisticated voting under an amendment procedure can be found in Shepsle and Weingast (1984).

Since each voter is assumed to be able to solve for the other voters' optimal decision rules, from any possible pairwise comparison between alternatives the voters can compute the resulting voting outcome from any decision that might be made. Suppose, returning to the Roberts' Rules procedure, that $t = 3$; i.e. there exists a status quo, a bill, and an amended bill. Let a majority of voters prefer the status quo to the bill. The voters then are able to realize that, if they choose the bill over the amended bill at the first vote, the outcome will be the status quo (since no voter has any incentive to misrepresent their preferences at the final vote). Thus, the first vote is actually between the status quo and the amended bill; since it does not defeat the status quo, the bill cannot be the voting outcome.

The following is a generalization of the above argument:

Definition. Given an agenda $A = (y_1, y_2, \ldots, y_t)$, the *sophisticated equivalent agenda* $A^* = (y_1^*, y_2^*, \ldots, y_t^*)$ is defined as:

i) $y_1^* = y_1$

ii) for $1 < i \leq t$,

$$y_i^* = \begin{cases} y_i & \text{if } y_i P y_j^* \quad \forall j < i \\ y_{i-1}^* & \text{otherwise.} \end{cases}$$

The *reduced form* $A' = (y'_1, y'_2, \ldots, y'_r)$ of the sophisticated equivalent agenda A^* is defined as the truncated version of A^*; for example, given the majority preference relation in Fig. 1 and the agenda $A = (x_5, x_4, x_1, x_3, x_2)$ (i.e. $y_1 = x_5$, $y_2 = y_4$, etc.) we have that $A^* = (x_5, x_4, x_1, x_1, x_1)$, and $A' = (x_5, x_4, x_1)$. It is easily seen that the reduced form contains all the relevant information from the sophisticated equivalent agenda; using again the Roberts' Rules example, it is as if the bill itself did not exist, and the agenda consisted only of the status quo and the amended bill (Note that this is a consequence both of the majority preference relation and of the ordering of the alternatives).

The following is due to Shepsle and Weingast (1984):

Theorem 2.2. *Given an agenda* $A = (y_1, y_2, \ldots, y_t)$,

i) y_t^* *is the sophisticated voting outcome,*

ii) $y_t^* \in \bigcap_{y_i^* \neq y_t^*} P(y_i^*)$.

Theorem 2.2 i simplifies, via the sophisticated equivalent agenda, the calculation of the sophisticated voting outcome for a given agenda, while 2.2 ii gives a condition exemplified by elements of the sophsiticated equivalent agenda. If $|A'| = r$, we can replace y_t^* with y'_r and y_i^* with y'_i in Theorem 2.2 and obtain an analogous result for the reduced form. Theorem 2.2 ii is also equivalent to the following:

$$\forall y_i^* \neq y_t^*, \quad y_i^* \in P^{-1}(y_t^*).$$

That is, the sophisticated voting outcome y_t^* defeats, under the majority preference relation, all other elements $y_i^* \neq y_t^*$ in the sophisticated equivalent agenda.

Thus far we have shown how to compute the sophsiticated voting outcome for any given agenda. Of course, the set of all sophisticated voting outcomes can be calculated by running through all the permutations of the alternatives, solving for the sophisticated equivalent agendas and applying Theorem 2.2 i. Our goal, however, is to achieve a more elegant characterization of the set of outcomes, one which will provide insights into the nature of the problem. It is to this task that we now turn.

3. Solution Sets

Definition. Given a set X of alternatives, let

$$S(X) = \{x_j \in X : \exists \text{ an agenda with } x_j \text{ as the sophisticated voting outcome}\}.$$

$S(X)$ is thus the equilibrium set described in the introduction; all alternatives in $S(X)$ are achievable as outcomes under some agenda, given that voters act sophisticatedly. Miller (1980) states that all sophisticated voting outcomes are in the uncovered set of X; that is, $S(X) \subseteq U(X)$. Unfortunately, this is only a necessary condition; the uncovered set does not completely characterize the set of potential sophisticated voting outcomes for a given set of alternatives [examples are given below]. Also, the uncovered set does not "reduce" in the sense of the dominance relation; that is, the uncovered set of the uncovered set $[U(U(X))]$ is not the proper solution set, as the example in Fig. 1 shows.

Under the given majority preference relation, we see that $U(X) = \{x_1, x_2, x_3, x_4\}$, and that $U(U(X)) = \{x_1, x_2, x_3\}$. But given the agenda $A = (x_5, x_3, x_4, x_2, x_1)$, it can be seen that x_4 is the sophisticated voting outcome.

By Theorem 2.2, an element x_j of the set $S(X)$ must exhibit certain characteristics about an ordering of alternatives $x_k \in \bar{P}(x_j)$ if it is to be a sophsiticated voting outcome under a particular agenda. Now the alternatives in the reduced form of a given sophisticated equivalent agenda constitute a *chain*, call it H, defined as a set together with a linear order (i.e. irreflexive, transitive, complete) on the set. By constuction of the sophisticated equivalent agenda the related linear order is simply the majority preference relation P, and note that the maximal element of the chain is the sophisticated voting outcome of the given agenda. For example, in Fig. 1 and given the agenda described above, the sophisticated equivalent agenda is $A^* = (x_5, x_3, x_4, x_4, x_4)$, the reduced form is $A' = (x_5, x_3, x_4) = H$ and we see that $x_4 P x_3 P x_5$, so that P is a linear order on H.

(Note that the requirement that P be a linear order on H is the same as saying that C_H, the covering relation restricted to the set H, is a linear order on H. Also, note that $H \subseteq \bar{P}(x_j)$ and $x_j \in H$ imply x_j is the maximal element of H, and vice versa.)

Next, define

$$H_j = \{H \subseteq X\colon P \text{ is a linear order on } H \text{ with maximal element } x_j\},$$

and set

$$\bar{H} = \bigcup_{i=1}^{t} H_i;$$

(i.e. $\bar{H}$ is the set of all chains in X). Ordering $\bar{H}$ by the inclusion relation, we have the following:

Definition. $H \in \bar{H}$ is *maximal* in $\bar{H}$ if and only if there exist no $H' \in \bar{H}$ s.t. $H \subset H'$.

That is, H is maximal if it is not possible to add any alternatives to H and still have a chain.

Lemma 3.1. *$H \in H_j$ is maximal in $\bar{H} \Rightarrow$*

$$\bigcup_{x_i \in H} P^{-1}(x_i) = X \backslash \{x_j\}.$$

Proof. Suppose not; then $\exists\, x_k \in X\backslash\{x_j\}$ s.t. $\forall\, x_i \in H$, $x_k P x_i$ (i.e. $x_k \in \bigcap_{x_i \in H} P(x_i)$).

Let $H' = H \cup \{x_k\}$; then H' is a chain (with maimal element x_k) and $H \subset H' \Rightarrow H$ is not maximal. Contradiction. Q.E.D.

The following gives a necessary and sufficient condition for an alternative to be a sophisticated voting outcome.

Theorem 3.1. *$x_j \in S(X)$ if and only if $\exists\, H \in H_j$* s.t. *H is maximal in $\bar{H}$.*

Proof. (nec.) Suppose not; then $x_j \in S(X)$ but $\forall\, H \in H_j$, $\exists$ a maximal $H' \in \bar{H}$ s.t. $H \subset H'$ (by the transitivity of the inclusion relation and the finiteness of the alternative set). Let A be an agenda with x_j as the sophisticated voting outcome; then A' constitutes a chain with maximal element x_j; set $A' = H$ and let H' be maximal with $H \subset H'$;

Claim: $\exists\, x_k \in H' \backslash H$ s.t. $x_k P x_j$.

Proof. If not, then $H'\backslash H \subseteq P^{-1}(x_j)$; since H' is a chain, this implies that x_j is the maximal element of H', which contradicts the assumption that no $H \in H_j$ is maximal, proving the claim.

Now, assuming for the moment that x_k is the only element of $H'\backslash H$ s.t. $x_k P x_j$, we have that H' is a chain, $H \subset H'$, $x_k \in H'\backslash H$ and $x_k P x_j$ which imply that $x_k P x_i$, $\forall x_i \in H$; thus, wherever x_k falls in the agenda, say at $y_k, x_k = y_k = y_k^*$; but $x_k P x_j \Rightarrow x_j \neq y_t^*$. Contradiction.

(Note that the assumption on $H'\backslash H$ was not crucial; if there were more than one such alternative, at least one would have to be in the sophisticated equivalent agenda.)

(suff.) Let $H \in H_j$ be maximal; by Lemma 3.1, $\forall x_k \in X\backslash\{x_j\}$, $\exists x_i \in H$ s.t. $x_i P x_k$; without loss of generality let $H = (x_1, x_2, \ldots, x_j)$; set $x_1 = y_1 = y_1^*, x_2 = y_2 = y_2^*$, etc.; then x_j will be the sophisticated voting outcome under the constructed agenda (ordering alternatives not in H in any fashion following H). Q.E.D.

Corollary 3.1. *In the absence of a Condorcet Proposal* $|S(X)| \geqq 3$.

Proof

i) Suppose $|S(X)| = 1$; let $x_j \in S(X)$; by hypothesis $\exists x_k \in X$ s.t. $x_k P x_j$; set $x_k = y_1 = y_1^*$; then x_j cannot be the sophisticated voting outcome under any agenda of this form. Contradiction.

ii) Suppose $|S(X)| = 2$; let $x_j, x_k \in S(X)$; without loss of generality let $x_j P x_k$; $x_k \in S(X) \Rightarrow \exists x_n \in P^{-1}(x_k)$ s.t. $x_n P x_j$; set $x_j = y_1 = y_1^*$ and $x_n = y_2 = y_2^*$; then x_j is not the sophisticated voting outcome, nor is x_k, since $x_j P x_k$. Contradiction. Q.E.D.

We can now state and prove Miller's Theorem as a corollary to Theorem 3.1:

Corollary 3.2. (*Miller's Theorem*) $S(X) \subseteq U(X)$.

Proof. By Theorem 3.1, $x_j \in S(X) \Rightarrow$ there exists a chain $H \in H_j$ which is maximal in $\bar{H}$.

By Lemma 3.1, this implies that $\bigcup_{x_i \in H} P^{-1}(x_i) = X\backslash\{x_j\}$; but $H \subseteq \bar{P}(x_j) \Rightarrow$ $\bigcup_{x_i \in \bar{P}(x_j)} P^{-1}(x_i) = X\backslash\{x_j\}$, which, by Theorem 2.1 ii implies that $x_j \in U(X)$. Q.E.D.

Another consequence of Theorem 3.1 which relates $U(X)$ and $S(X)$ is the following:

Corollary 3.3. *If* $\bar{P}(x_j) \in H_j$, *then* $x_j \in U(X) \Rightarrow x_j \in S(X)$.

Proof. If $\bar{P}(x_j)$ is a chain, then $x_j \in U(X)$ implies that $\bar{P}(x_j)$ is maximal. Q.E.D.

Consider the following example:

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Fig. 3

Calculating the uncovered set via the method suggested in Sect. 2, we see that $U(X) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ but that $S(X) = \{x_2, x_3, x_4, x_5, x_6\}$. The reader can verify that the following chains are maximal:

$$H = (x_8, x_7, x_3, x_2) \in H_2$$
$$H = (x_8, x_7, x_4, x_3) \in H_3$$
$$H = (x_8, x_7, x_2, x_1, x_4) \in H_4$$
$$H = (x_7, x_9, x_2, x_1, x_5) \in H_5$$
$$H = (x_9, x_8, x_2, x_1, x_6) \in H_6.$$

Because of the cyclical nature of x_7, x_8, x_9, however, $x_1 \notin S(X)$. Given the chain $H = (x_9, x_8, x_2, x_1)$, we could add x_6 and still have a chain; i.e. $x_6 P x_i$, $\forall x_i \in H$. Similarly for the chain (x_7, x_9, x_2, x_1) we could add x_5, and for the chain (x_8, x_7, x_2, x_1) we could add x_4. Since these are the largest possible chains in H_1, we conclude that $x_1 \notin S(X)$.

The derivation of the set $S(X)$ via Theorem 3.1 may seem tedious, since in general there will be more relations to inspect than in the derivation of the uncovered set $U(X)$. This relative difficulty is easily justified, however. Recalling the example in Fig. 3, we see that if we alter one of three particular binary relations, say $x_7 P x_8$ to $x_8 P x_7$, then the elements of $\bar{P}(x_1)$ constitute a chain $H = (x_7, x_9, x_8, x_2, x_1)$, and now applying Corollary 3.3, we see that $x_1 \in S(X)$, while the uncovered set $U(X)$ has remained unchanged. Thus the proper solution set must be sensitive enough to deal with such minor differences in the majority preference relation.

Also, if there is a natural or imposed y_1, e.g. a status quo to be voted on last, the process of computing chains proves to be a more appropriate method. Suppose that x_1 is the status quo, with $x_1 = y_1$ the rule. Define

$$H'_1 = \{H \subseteq X \colon P \text{ is a linear order on } H \text{ with minimal element } x_1\}.$$

Then Theorem 3.1 can be rewritten as:

Theorem 3.1′. *$x_j \in S(X)$ if and only if $\exists H \in \{H'_1 \cap H_j\}$ s.t. H is maximal.*

That is, there must exist a maximal chain H with maximal element x_j and minimal element x_1.

Unfortunately, there do not seem to exist any simple methods for calculating $S(X)$ as there are for $U(X)$. What we do know is the following:

For any subset $B \subset X$, with $|B| = n$, and related dominance matrix D_B, order the elements of B so that their scores restricted to B are in nonincreasing order. Define the *score vector* of B as the n-tuple of scores in B (after re-ordering).

Theorem 3.2 (Moon 1968). *The following are equivalent:*

i) P is transitive on B.
ii) B has score vector $(n-1, n-2, \ldots, 1, 0)$.
iii) Each principal submatrix of D_B contains a row and column of zeros.

Thus it is possible to determine whether a given subset of alternatives constitute a chain by inspecting their scores restricted to the subset and ordered appropriately, or by inspecting the related dominance matrix. To see whether the chain is maximal, we then need to see if we can add any other alternatives to the subset and still maintain the transitivity. If we define D_{x_i} as the dominance matrix restricted to the elements of $\bar{P}(x_i)$ and first calculate maximal chains in H_i, the only way an alternative could be added would be for the alternative to defeat *all* the elements of the chain, and so we need only check to see if we can add any alternatives at the "top" of the dominance matrix; i.e. if B is a chain maximal in H_i then x_i is the maximal element of B, and if $|B| = n$, then $x_j \notin \bar{P}(x_i)$ can be added to B with $B \cup x_j$ transitive if and only if $B \cup x_j$ has score vector $(n, n-1, n-2, \ldots, 1, 0)$.

Nevertheless these conditions are somewhat cumbersome, and we would like to see how far we can reduce the uncovered set before we have to impose them. We proceed now to derive a refinement of the uncovered set and to establish some conditions under which this set equals $S(X)$.

Let $D_u = [d^u_{ij}]$ be the dominance matrix restricted to the alternatives in the uncovered set. We shall say that the uncovered set is *reducible* if it is possible to partition the alternatives into two nonempty subsets, $U^*(X)$ and $U^c(X)$ such that for all $x_i \in U^*(X)$ and all $x_j \in U^c(X)$, $x_i P x_j$; the set is *irreducible* if this is not possible. Let $s^u_i = \sum_{j=1}^{m} d^u_{ij}$. If $U(X)$ is reducible and the scores s^u_i are in nonincreasing order, then D_u has the following partitioned structure:

$$D_u = \begin{bmatrix} D_{u1} & | & 1 \\ - & & - \\ 0 & | & D_{u2} \end{bmatrix} \qquad \text{Fig. 4}$$

where D_{u1} and D_{u2} are the matrices of $U^*(X)$ and $U^c(X)$, respectively (Moon 1968).

If $U(X)$ is reducible and $U^*(X)$ is irreducible, then we can derive the following results:

Lemma 3.2. $x_j \notin U^*(X) \Rightarrow$

i) $\exists x_k \in U^*(X)$ s.t. $x_k C_X x_j$, or

ii) $\forall x_k \in U^*(X)$, $x_k P x_j$.

Proof

1) $x_j \in U(X)$ and $x_j \notin U^*(X) \Rightarrow$ *ii)*, by definition;

2) $x_j \notin U(X) \Rightarrow \exists x_k \in U(X)$ s.t. $x_k C_X x_j$;

   a) if $x_k \in U^*(X)$, then *i)*;

   b) if $x_k \notin U^*(X)$, $x_k C_X x_j$ and $x_i P x_k$, $\forall x_i \in U^*(X) \Rightarrow x_i P x_j$, $\forall x_i \in U^*(X)$. Q.E.D.

The following result shows that we have not eliminated any possible outcomes in our reduction of the uncovered set.

Theorem 3.3. $S(X) \subseteq U^*(X)$.

Proof. Suppose not; then $\exists x_j \in U(X)$, $x_j \notin U^*(X)$ and a maximal chain $H \in H_j$ (by Miller's Theorem we need not consider alternatives not in the uncovered set); $x_j \notin U^*(X) \Rightarrow \forall x_k \in H$, $x_k \notin U^*(X)$; thus, by Lemma 3.2, $x_k \in H \Rightarrow$

a) $x_i P x_k$, $\forall x_i \in U^*(X)$, or

b) $x_i C_X x_k$, for some $x_i \in U^*(X)$.

Case 1. Suppose a) holds $\forall x_k \in H$; then H cannot be maximal, since we can add any $x_i \in U^*(X)$ and still have a chain.

Case 2. Suppose b) holds for some subset $H' \subset H$; let x_m be the maximal element (under the linear order P) of H'; then for some $x_i \in U^*(X)$, $x_i C_X x_m \Rightarrow x_i P x_k \; \forall x_k \in H'$; but $x_i P x_n \; \forall x_n \in H \backslash H' \Rightarrow x_i P x_k \; \forall x_k \in H$ which implies H cannot be maximal.

Q.E.D.

Combining the result of Theorem 3.3 with that of Corollary 3.1, we get:

Corollary 3.4. *In the absence of a Condorcet Proposal,* $|U^*(X)| \geqq 3$.

Figure 5 gives an example of a reducible $U(X)$:

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Fig. 5

In this example $U(X) = \{x_1, x_2, x_3, x_4, x_5\}$, so that

$$D_u = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Fig. 6

Upon inspection of Fig. 6, we see that $U^*(X) = \{x_1, x_2, x_3\}$. Corollaries 3.1 and 3.4 give us that if $|U^*(X)| = 3$, then $U^*(X) = S(X)$. Thus, in this example, we are able to calculate $S(X)$ without having to examine chains of alternatives required to invoke Theorem 3.1. In fact, we can strengthen this sufficiency condition to the following:

Corollary 3.5. If $|U^*(X)| \leqq 4$, then $U^*(X) = S(X)$.

Proof

i) $|U^*(X)| = 1 \Rightarrow x_j \in U^*(X)$ is a Condorcet Proposal; $\Rightarrow x_j \in S(X)$, and $\forall x_k \neq x_j$, $x_k \notin S(X)$.

ii) $|U^*(X)| = 3$: let $x_i, x_j, x_k \in U^*(X)$; by definition they cycle; let $x_i P x_j$, $x_j P x_k$, $x_k P x_i$;

We set $x_i = y_1 = y_1^*$ and $x_k = y_2 = y_2^*$; then x_k will be the sophisticated voting outcome, and using the same argument we see that $x_i, x_j \in S(X)$.

iii) $|U^*(X)| = 4$: let $x_i, x_j, x_k, x_m \in U^*(X)$; by definition each defeats either one or two others;

a) let $x_i P x_j$, $x_i P x_k$, $x_m P x_i$; then either

1) $x_j P x_m$ or
2) $x_k P x_m$;

if 1), set $x_j = y_1 = y_1^*$ and $x_i = y_2 = y_2^*$; then x_i is the sophisticated voting outcome;
if 2), set $x_k = y_1 = y_1^*$ and $x_i = y_2 = y_2^*$; then x_i is the sophisticated voting outcome;

b) let $x_i P x_j$, $x_k P x_i$, $x_m P x_i$; then either

1) $x_j P x_k$ or
2) $x_j P x_m$;

since $x_i \in U^*(X) \subseteq U(X)$, $\exists x_n, x_p \in P^{-1}(x_i)$ s.t. $x_n P x_k$ and $x_p P x_m$; if 1), set $x_p = y_1 = y_1^*$, $x_j = y_2 = y_2^*$, $x_i = y_3 = y_3^*$, if $x_j P x_p$, or $x_j = y_1 = y_1^*$, $x_p = y_2 = y_2^*$, $x_i = y_3 = y_3^*$, if $x_p P x_j$; then x_i is the sophisticated voting outcome; if 2), a similar argument holds.

Repetition of this process proves that $x_i, x_j, x_k, x_m \in S(X)$. Q.E.D.

4. Conclusion

The results of this paper show that there is a valid procedure for calculating the set of potential sophisticated voting outcomes:

i) Derive the uncovered set via the method suggested in Sect. 2. If $|U(X)| = 3$, then $U(X) = S(X)$.
ii) If $|U(X)| > 3$, determine whether $U(X)$ is reducible via the method given in Sect. 3. If it is reducible, find the irreducible set $U^*(X)$. If $|U^*(X)| \leqq 4$, then $U^*(X) = S(X)$.
iii) If i) and ii) fail to determine $S(X)$, begin examining elements of $U(X)$ (or $U^*(X)$ if $U(X)$ is reducible) to find alternatives which satisfy the conditions put forth in Sect. 3 in order to use Theorem 3.1.

In Miller (1980) it was shown that, if we assume *sincere* voting on the part of the voters, then the set of potential voting outcomes was equal to the Condorcet set $C(X)$, defined as the largest subset of X such that for all $x_i \in C(X)$ and all $x_j \notin C(X)$, $x_i P x_j$. But this corresponds to our definitions in Sect. 3 of reducible and irreducible sets, so that the same method used to calculate $U^*(X)$ can be used to determine $C(X)$, where we now apply it to the whole of X. The necessity of an alternative being in $C(X)$ is obvious, while the sufficiency comes from the fact that, since $C(X)$ is irreducible, there exists a cycle which spans $C(X)$; i.e. it is possible to get from any alternative in $C(X)$ to any other alternative in $C(X)$ via the majority preference relation, and also include all other elements of $C(X)$ along the way (Harary and Moser 1966). The agenda setter thus can construct the agenda so that his preferred alternative is voted on last, and assure himself that it will be the final outcome.

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