THE INFLUENCE OF A NET CHARGE ON THE CRITICAL MASS OF A NEUTRON STAR

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Abstract. We derived the set of equations determining the structure of a spherically symmetric charged star within the framework of general relativity (modified Oppenheimer-Volkoff equations). The equations have been solved for a completely degenerate Fermi gas with a charge density assumed to be proportional to the matter density. It is shown that the presence of a net charge does not affect the existence of a critical mass. The value, however, could be substantially altered, in some cases doubled.

1. Introduction

The possibility that a self-gravitating system could contain a net charge has been discussed in the context of stars by Eddington (1926), the universe by Lyttleton and Bondi (1959) and in a slightly different version (electrically polarized universe) by Bally and Harrison (1978) among others. The presence of such a net charge could result from the escape of some electrons from the system, and/or from the inequality of the electronic and proton charge and/or from the non-zero charge on a neutron. The first possibility would give rise to a charge-to-mass ratio of ~ 100 coulombs per solar mass (Bally and Harrison, 1978), while a limit of $\delta q \simeq 10^{-20-22} q_e$ can be imposed on the last two possibilities from precision experiments (see, e.g., Hughes 1964). It seems that the immediate physical consequences of such a net charge are always found to be extremely small, but nonetheless the picture of an electric field playing a role in gravitationally-bound systems is, in principle, intriguing and interesting.

In this short note we address ourselves to the specific problem of the effect of a net charge on the structure of a degenerate configuration, for example, a neutron star. In particular, we want to find out if the additional pressure support coming from the electrostatic field would affect the concept and existence of a critical mass. It seems manifestly and intuitively clear that significant changes are not to be expected if the magnitude of the field is limited by the arguments in the last paragraph. However, we are taking up an issue of principle here by studying how the presence of various amount

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of net charge would affect the structure of the system, without worrying about the fact that *presently* known mechanisms cannot give us the required amount for interesting results. It is in this spirit that the subsequent calculations should be taken, as also should be the case with other work on the same topic (e.g., Bally and Harrison, 1978; Mehra, 1982).

2. The Structural Equations

For a static, spherically-symmetric system, the metric and the energy-momentum tensor can be written in the form

$$ds^2 = c^2 \mathrm{d}t^2 \mathrm{e}^{\nu(r)} - \mathrm{e}^{\lambda(r)} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \tag{1}$$

and

$$T^\mu_\nu = (p + \rho c^2) v^\mu v_\nu + p\delta^\mu_\nu + \frac{1}{4\pi}(F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\mu_\nu), \tag{2}$$

where all the symbols have their usual meanings. The fluid four-velocity in the present case can be written as

$$v_r = v_\theta = v_\phi = 0\,; \quad v_0 = \mathrm{e}^{-\nu/2}\,; \tag{3}$$

and the electromagnetic field tensor $F^{\mu\nu}$ can be obtained from the four potential A_μ given by

$$A_0 = \phi(r)\,; \quad A_\mu = 0\,, \quad \forall \mu \neq 0\,. \tag{4}$$

With the help of Equations (1)–(4), we can then obtain the following structural equations for the system:

$$\mathrm{e}^{-\lambda}\left(\frac{1}{r^2} - \frac{1}{r}\frac{\mathrm{d}\lambda}{\mathrm{d}r}\right) - \frac{1}{r^2} = -\frac{8\pi G}{c^4}\left(\rho c^2 + \frac{u^2}{8\pi}\right), \tag{5}$$

$$\mathrm{e}^{-\lambda}\left(\frac{1}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = \frac{8\pi G}{c^4}\left(p - \frac{u^2}{8\pi}\right), \tag{6}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r}(p + \rho c^2) + \frac{u}{8\pi}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r}\right), \tag{7}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{2u}{r} = 4\pi\rho_e \mathrm{e}^{\lambda/2}\,; \tag{8}$$

where $u(r) \equiv \exp((\lambda + \nu)/2)(\mathrm{d}\phi/\mathrm{d}r)$, and $\rho_e(r)$ is the charge density measured in proper co-ordinates. We note that Equations (5) and (6) are obtained from the Einstein field equations, Equation (7) is derived from the vanishing four-divergence of T^μ_ν (i.e., $T^\mu_{\nu;\mu} = 0$) and Equation (8) from the general relativistic Maxwell equation, where the four-current j_μ is related to ρ_e by $\rho_e = -v_\mu(j^\mu/c) = \exp(\nu/2)(j^0/c)$.

3. General Solutions

The set of Equations (5)–(8) can be solved for the unknowns λ, ν, p, ρ, u and ρ_e if they are supplanted by an equation of state and a charge distribution law. We will defer a discussion of this to the next section while we study here the properties of the equations in general. Equation (8) can be readily reduced to a quadrature for u: i.e.,

$$u(r) = \frac{1}{r^2} \int_0^r 4\pi r^2 \rho_e \, e^{\lambda/2} \, dr . \tag{9}$$

It is obvious that $u(0) = 0$ and for $r \geq R$, the 'radius' of the star;

$$u(r) = \frac{Q}{r^2}, \tag{10}$$

where

$$Q = \int_0^R 4\pi r^2 \rho_e \, e^{\lambda/2} \, dr . \tag{11}$$

Similarly, Equation (5) can be formally solved for λ to yield

$$e^{-\lambda} = 1 - \frac{2GB(r)}{rc^2}, \tag{12}$$

where

$$B(r) = \int_0^r 4\pi r^2 \left(\rho + \frac{u^2}{8\pi c^2} \right) dr . \tag{13}$$

With the aid of Equation (11), we can write Equation (22) as

$$e^{-\lambda} = 1 - \frac{2GB}{rc^2} + \frac{GQ^2}{r^2 c^4}, \tag{14}$$

where

$$B = \int_0^\infty 4\pi r^2 \left(\rho + \frac{u^2}{8\pi c^2} \right) dr .$$

It is obvious from Equation (12) that $\lambda(0) = 0$. We also remark that B and Q represents the total 'gravitating' mass (both material and electromagnetic) and the total charge of the star as measured by a distant observer.

Combining Equations (5) and (6), and using $\lambda(\infty) = \nu(\infty) = 0$, we obtain

$$\nu(r) = -\lambda(r) - \int_r^\infty \frac{8\pi G r\, e^\lambda}{c^4} (p + \rho c^2)\, dr\,;$$

from which we immediately have

$$\nu(r) = -\lambda(r)\,, \quad \text{for } r \geq R \tag{15}$$

and

$$\nu(0) = -\int_0^\infty \frac{8\pi G r\, e^\lambda}{c^4} (p + \rho c^2)\, dr\,. \tag{16}$$

With the above expressions for λ, ν and u, we can write the 'hydrostatic' equation of equilibrium (7) as

$$\frac{dp}{dr} = \frac{-G\left[B(r) + 4\pi r^3\left(\frac{p}{c^2} - \frac{u^2}{8\pi c^2}\right)\right]}{r^2\left(1 - \frac{2GB(r)}{rc^2}\right)} (\rho + p/c^2) + \rho_e u\, e^{\lambda/2}\,. \tag{17}$$

A comparison with the Oppenheimer–Volkoff equation for the uncharged case of the form

$$\frac{dp}{dr} = \frac{-G(M(r) + 4\pi r^3 p/c^2)}{r^2\left(1 - \frac{2GM(r)}{rc^2}\right)} (\rho + p/c^2)\,, \tag{18}$$

shows that the presence of a net charge (or electrostatic field) contributes not just an increase in repulsion *but also* an increase in the 'mass' or the gravitating source as defined by Equation (13). This is important in the discussion of the critical mass.

4. Numerical Solution: Application to Neutron Stars

For the purpose of numerical integration, it is convenient to rewrite Equations (5), (6), (7) and (8) in the following way:

$$\frac{d\lambda}{dr} = \frac{8\pi G}{c^2} r\, e^\lambda \left(\rho + \frac{u^3}{8\pi c^2}\right) - \left(\frac{e^\lambda - 1}{r}\right), \tag{19}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{8\pi G}{c^2} r\, e^{\lambda}\left(\frac{p}{c^2} - \frac{u^2}{8\pi c^2}\right) + \left(\frac{e^{\lambda} - 1}{r}\right), \tag{20}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\left[\frac{4\pi G r\, e^{\lambda}}{c^4}\left(p - \frac{u^2}{8\pi}\right) + \left(\frac{e^{\lambda} - 1}{2r}\right)\right](p + \rho c^2) + \rho_e u\, e^{\lambda/2}; \tag{21}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{2u}{r} + 4\pi\rho_e\, e^{\lambda/2}. \tag{22}$$

Equation (20) can be decoupled from the rest of the equations, which can then be solved for a given $p_c = p(0)$ (we recall $\lambda(0) = u(0) = 0$) if we are also given $p = p(\rho)$ and $\rho_e = \rho_e(\rho)$.

We now consider the case of a self-gravitating completely degenerate fermion system described by

$$\rho = \frac{K}{c^2}(\cosh\xi - \xi), \tag{23}$$

$$p = K\left(\frac{1}{3}\cosh\xi - \frac{8}{3}\cosh\frac{\xi}{2} + \xi\right), \tag{24}$$

where

$$K = \frac{m_0^4 c^3}{32\pi^2 h^3}, \quad \xi = 4\sinh^{-1}\frac{p_F}{m_0 c},$$

and p_F, m_0 are the Fermi momentum and the rest mass of the fermions, respectively. We furthermore assume that

$$\rho_e = \alpha\rho, \tag{25}$$

where α is a constant. Equation (21) can now be re-written as an equation for ξ, of the form

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}r} = \Bigg[&-\left\{\frac{4\pi G r\, e^{\lambda}}{c^4}\left[K\left(\frac{1}{3}\sinh\xi - \frac{8}{3}\sinh\frac{\xi}{2} + \xi\right) - \frac{u^2}{8\pi}\right] + \frac{e^{\lambda} - 1}{2r}\right\} \times \\ &\times \left(4\sinh\xi - 8\sinh\frac{5}{2}\right) + \\ &+ 3\alpha u\, e^{\lambda/2}(\sinh\xi - \xi)/c^2\Bigg] \Big/ \left(\cosh\xi - 4\cosh\frac{\xi}{2} + 3\right).\end{aligned} \tag{26}$$

Equation (26) has been solved numerically together with Equations (19) and (22) for $\xi(r)$, $\lambda(r)$ and $u(r)$ for various values of ξ_0 (and hence, ρ_0). Similarly, the mass $m(r)$

defined by

$$\frac{dm}{dr} = 4\pi r^2 \rho , \tag{27}$$

with $m(0) = 0$ is also evaluated. With the solution for $\lambda(r)$, $\xi(r)$ and $u(r)$, we can then solve Equation (20) for $v(r)$, using $v(0)$ as given by Equation (16).

The results are readily separable into two different classes, depending on whether α is smaller or larger than a critical value $\alpha_c \simeq 0.00025$. For $\alpha > \alpha_c$, no stable equilibrium configuration can be obtained. It seems that the increase in the pressure gradient cannot compensate for the corresponding increase in the effective mass. For $\alpha < 0.00025$, on the other hand, a stable equilibrium configuration was obtained, and for a given ξ_0 (hence ρ_0), the set of equations can be integrated outwards to a finite value of r at which $\xi = 0$. The resultant mass M as given by Equation (27) is plotted as a function of ρ_0 for various values of α in Figure 1. We remark that while B represents the total 'gravitating'

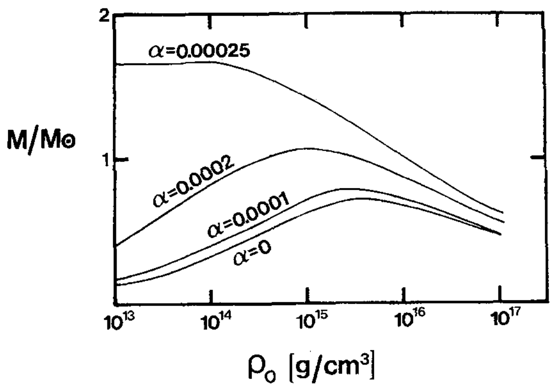

Fig. 1. Mass versus central density for various values of α.

mass, we nonetheless plot M vs ρ_0 here, as this will afford the closest comparison with the non-charged case. If can be seen that for a given ρ_0, $M(\alpha \neq 0) > M(\alpha = 0)$, an expected result in view of the additional pressure support. However, the property that M increases with ρ_0 up to a maximum value M_{cri} at ρ_c and then decreases with ρ_0 still remains as long as $\alpha < \alpha_c$. Thus, the concept of a critical mass and the associated transition of the equilibrium configuration from a stable to an unstable one (see, e.g., Weinberg, 1972) are not changed by the presence of a net charge as long as $\alpha < \alpha_c$. Furthermore, the value of M_{cri} increases with α, and M_{cri} $(\alpha = 0.00025) \sim 2\,M_{\mathrm{cri}}$ $(\alpha = 0)$, as shown in Figure 2. We also plot the run of λ, v and ρ with r for various values of α in Figures 3 and 4. It can be seen that the electrostatic repulsion results in a more gradual density gradient, a feature particularly obvious in the outer layers, with a resultant increase in the radius of the configuration.

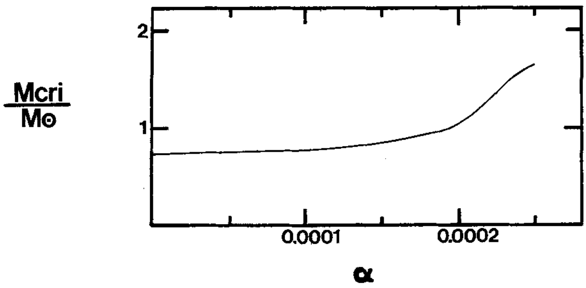

Fig. 2. Critical mass as a function of α.

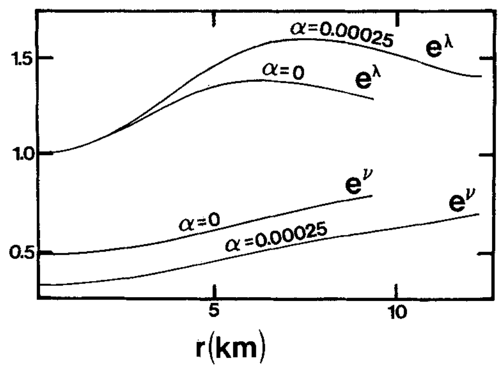

Fig. 3. The metric coefficients as a function of radius for various values of α.

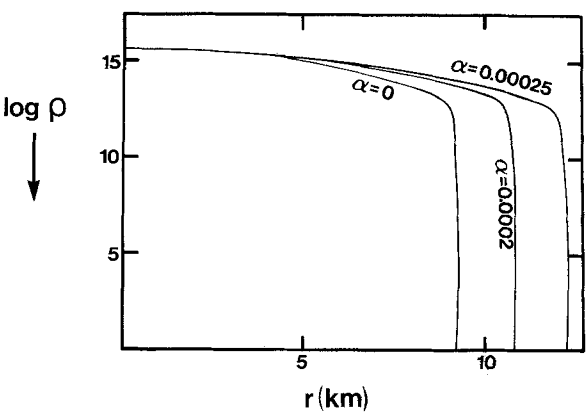

Fig. 4. The density profile for various values of α.

5. Summary and Conclusions

We observe the following in connection with the structure of a selfgravitating completely degenerate, ideal fermion system carrying a net charge density $\rho_e = \mathrm{d}\rho$; ρ being the 'baryonic' mass density:

(1) For $\alpha < \alpha_c = 0.00025$, stable configurations can exist. For the same central density, the values for the mass and the radius are larger than corresponding ones for the non-charged case. These effects are understandable in terms of the electrostatic repulsion.

(2) For a given $\alpha < \alpha_c$, there still exists a critically maximum mass for the equilibrium configuration, although its value could be drastically changed (indeed doubled for $\alpha \simeq \alpha_c$). Thus the associated increase in effective mass manages to compensate for the increase in pressure gradient, both effects being results of the same electrostatic field.

(3) For $\alpha > \alpha_c$, no stable equilibrium configuration can be found.

While the above results are interesting, and certainly correct, in principle, it is still appropriate to relate α to values imposed by *presently* known charge-inequality mechanisms. Thus we find $\alpha \sim 100$ coulombs/solar mass if the net charge comes from the escape of some electrons, while $\alpha \sim 10^{-12}$ if we have $q_e \neq q_p$, or $q_n \neq 0$. It is comforting to realize that the concept of a critical mass for a completely degenerate configuration is not changed even when α has a value 100 million times larger than the one presently established.

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