THERMAL AND ELECTRICAL CONDUCTIVITIES OF CRYSTALS IN NEUTRON STARS AND DEGENERATE DWARFS

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Abstract. Thermal and electrical conductivities due to electron scattering on phonons are calculated for degenerate cores of white dwarfs and envelopes of neutron stars for wide ranges of density, temperature and ion charge. In the stellar zones, in which $T \geq \hbar\omega_{pi}(Z^{1/3}e^2/\hbar v_F)$ (ω_{pi} is the ion plasma frequency and v_F the Fermi velocity of electrons), the main contribution into scattering comes from the Umklapp processes. In the zones with lower T, the Umklapp processes are frozen out, that results in a sharp growth of electrical and thermal conductivities. This, for instance, should make nuclear burning more stable in such zones.

(1) To study the many processes in neutron stars and degenerate dwarfs (cooling, evolution of magnetic field, nuclear burning of accreted matter, etc) one should know the transport properties of stellar matter. This paper presents calculations of electron thermal and electrical conductivities for degenerate envelopes of neutron stars and cores of white dwarfs at temperatures below the melting temperature, T_M, of ion crystals. Without an allowance for zero-point quantum oscillations of ions, T_M is determined by the condition (Pollock and Hansen, 1973) $Z^2e^2/k_BT_Ma \approx 150$ – i.e., $T_M \approx 1.5 \times 10^5 Z^{5/3}(\rho_6/\mu_e)^{1/3}$ K; where $a = (3/4\pi n_i)^{1/3}$, $\mu_e = A/Z$, A and Z are the mass and charge numbers of the ions, n_i the ion number density, ρ the mass density of the matter, and $\rho_6 = \rho/10^6$ g cm^{-3}. At rather high densities, $\rho \geq 10^{12}$ g cm^{-3}, when neutrons begin to drip from nuclei (e.g., see Baym and Pethick, 1975), μ_e becomes equal to the total number of nucleons per proton instead of to A/Z. Evaluation of T_M with an allowance for zero-point ion oscillations was performed by Mochkovitch and Hansen (1979) on the basis of the Lindemann criterion. These authors found that zero-point oscillations become significant at $\rho > 400\ Z^6A^4$ g cm^{-3} and suppress the onset of crystallization. As ρ reaches $6.1 \times 10^4\ Z^6A^4$ (for odd A) or $1.7 \times 10^4 Z^6A^4$ g cm^{-3} (for even A) T_M falls down to 0, and at higher ρ no stable lattice is possible. The density dependence of T_M is shown in Figure 1 for helium and carbon ions, ^{4}He and ^{12}C.

We shall consider the densities $\rho \gg Z^2$ g cm^{-3} at which electrons are ideal ($Ze^2/a \ll k_BT_F$, T_F being the degeneracy temperature). At such densities ions are usually fully ionized. They form Coulomb crystals (≡ ion Wigner crystals) on the uniform electron-charge-density background. Vibration properties of these crystals are similar to those of electron Wigner crystals (with an ion-charge back-

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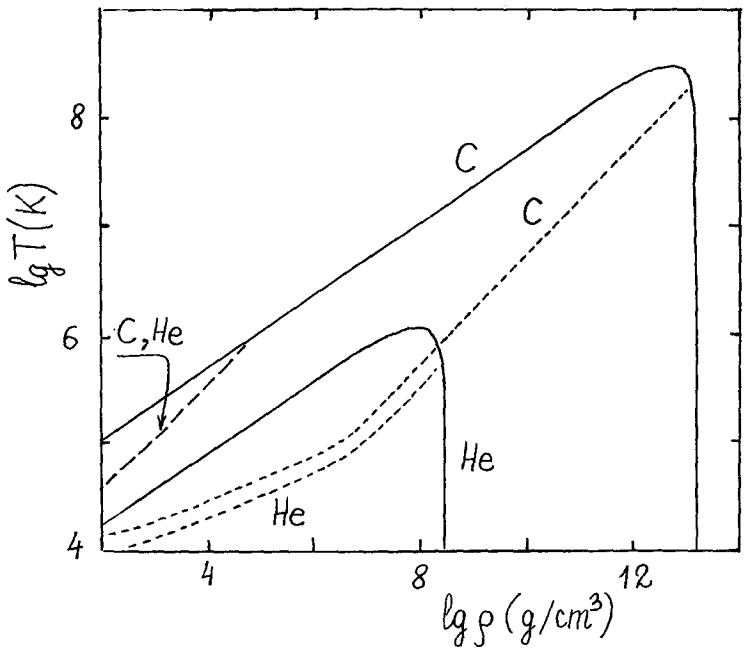

Fig. 1. A density-temperature diagram for the matter of neutron star envelopes and degenerate dwarf cores. Solid lines are the crystallization curves (Mochkovitch and Hansen, 1979) for carbon and helium ions, ^{12}C and ^{4}He. The dashed line corresponds to $k_B T = \hbar\omega_{pi}$ (i.e., $t \equiv k_B T/\hbar\omega_{pi} = 1$, see Section 2), and the dots show the offset of the Umklapp processes ($t = Z^{1/3}e^2/\hbar v_F$, see Section 3).

ground) theoretically investigated in detail in solid state physics (Cohen and Keffer, 1955; Carr, 1961; Coldwell-Horsefall and Maradudin, 1960; also see Albers and Gubernatis, 1981, for references). Also, the thermodynamics and vibration spectrum of Coulomb crystals were studied directly for astrophysical conditions (Pollock and Hansen, 1973; Mochkovitch and Hansen, 1979).

It is widely recognized that the crystals under discussion (with one species of ions) have a bcc lattice (for bcc being most stable, see Coldwell-Horsefall and Maradudin, 1960; Pollock and Hansen, 1973). In degenerate dense stellar matter, heat and charge are mainly transported by electrons, the most important scattering mechanism of electrons at $T < T_M$ being the scattering on phonons. Transport properties of such a matter have been studied in a number of papers, of which the works by Flowers and Itoh (1976) and Yakovlev and Urpin (1980) seem to be the most detailed ones. For references to other works and their critical analysis, see Yakovlev and Urpin (1980). As a rule, various authors have considered heavy ions ($Z \gg 1$) and rather high temperatures of crystals (higher than the Debye temperature). This paper considers arbitrary Z, including $Z = 1$, and arbitrary temperatures, including very low ones; the effect of the magnetic field will be neglected. Astrophysical applications of the obtained results will be discussed in Section 4.

(2) The thermal and electrical conductivities, κ and σ, of the relativistic degenerate electron gas are conveniently described (Flowers and Itoh, 1976) in terms of the effective electron collision frequencies ν_κ and ν_σ:

$$\kappa = \frac{\pi^2 k_B^2 T n_e}{m_* \nu_\kappa} \approx 4.11 \times 10^{15}\, x^2 \beta T_\sigma \left(\frac{10^{16}\,\mathrm{s}^{-1}}{\nu_\kappa}\right) \frac{\mathrm{ergs}}{\mathrm{cm\ s\ deg}}, \tag{1}$$

$$\sigma = \frac{e^2 n_e}{m_* \nu_\sigma} \approx 1.51 \times 10^{22}\, x^2 \beta \left(\frac{10^{16}\,\mathrm{s}^{-1}}{\nu_\sigma}\right) \mathrm{s}^{-1},$$

where n_e is the number density of electrons, $x = \hbar k_F / mc \approx (\rho_6/\mu_e)^{1/3}$, $\beta = v_F/c = x(1+x^2)^{-1/2}$, with $k_F = (3\pi^2 n_e)^{1/3}$, v_F and $m_* = m(1+x^2)^{1/2}$ being the momentum, velocity and relativistic (effective) mass of electrons on the Fermi surface, respectively.

To calculate ν_κ and ν_σ for electron scattering on phonons we shall use the variational method well-known in solid state physics (see Ziman, 1962). The deviation of the electron distribution function, $f(\mathbf{p})$, from the equilibrium Fermi-Dirac function $f_0 = \{\exp[(\epsilon - \mu)/k_B T] + 1\}^{-1}$ will be set equal to the simplest trial function which is $(\partial f_0/\partial \epsilon)(\mathbf{p}\dot{\mathbf{E}})$ for ν_σ($\mathbf{E}$ the electric vector) and $(\partial f_0/\partial \epsilon) \times (\epsilon - \mu)\mathbf{p}\nabla T)$ for ν_κ. At temperatures above the Debye temperature Θ (see below) this approach leads to exact result, and at $T \leqslant \Theta$ it gives negligibly small errors for ν_σ and the errors $< 20\%$ for ν_κ (see Ziman, 1962). This accuracy is quite satisfactory for astrophysical applications.

We will use the formalism of extended Brillouin zones in the free-electron approximation to describe electron states. In this case, the Fermi surface as well as dispersion relation are the same as for an ideal electron gas (see below). Since the ions are fully ionized, the potential of electron interaction with lattice vibrations has a simple form

$$V(r) = 4\pi Z e^2 i \int \frac{\mathrm{d}\mathbf{q}}{q^2 + k_{TF}^2} \sum_I (\mathbf{q}\boldsymbol{\xi}_I)\, e^{iq(r - r_I)}, \tag{2}$$

where $\boldsymbol{\xi}_I$ is a small displacement of an ion numerated by $I = 1, 2, \ldots$ from its equilibrium position, $\boldsymbol{\tau}_I$, in a lattice. Equation (2) takes into account the static screening of the ion charge by electrons with the screening length $k_{TF}^{-1} = v_F/\sqrt{3}\omega_{pe}$, where $\omega_{pe} = (4\pi e^2 n_e/m_*)^{1/2}$ is the electron plasma frequency. Subsequent calculations are similar to those in solid state physics (Ziman, 1962). The only difference is in considering relativistic electrons and rather peculiar spectrum of phonons. The result is

$$\nu_{\sigma,\kappa} = \frac{e^2}{\hbar v_F} \frac{k_B T}{\hbar} F_{\sigma,\kappa} \approx 0.96 \times 10^{15}\, T_\sigma \beta^{-1} F_{\sigma,\kappa}\ \mathrm{s}^{-1}, \tag{3}$$

where we introduce the convenient dimensionless functions

$$F_{\sigma,\kappa} = \frac{2}{t^2 S^2} \int \frac{\mathrm{d}S\, \mathrm{d}S'}{(q^2 + k_{TF}^2)^2} \left[1 - \left(\frac{\beta q}{2k_F}\right)^2\right] \sum_s [\mathbf{q}\mathbf{e}_s(\mathbf{k})]^2 (e^{z_s} - 1)^{-2} e^{z_s} g_{\sigma,\kappa}, \tag{4}$$

in which

$$t = \frac{k_B T}{\hbar \omega_{pi}} \approx \frac{0.187 T_\sigma}{x^{3/2}} \left(\frac{A}{2Z}\right)^{1/2}, \qquad z_s \equiv \frac{\hbar \omega_s(k)}{k_B T}, \tag{5}$$

$$g_\sigma = q^2, \qquad g_\kappa = q^2 - \frac{q^2 z_s^2}{2\pi^2} - \frac{3k_F^2 z_s^2}{\pi^2},$$

$\omega_{pi} = (4\pi Z^2 e^2 n_i / m_i)^{1/2}$ is the ion plasma frequency and s numerates the phonon modes. In Equation (4) integration is carried out over all the possible positions of the electron momenta before and after scattering, $\mathbf{p}$ and $\mathbf{p}'$, on the Fermi surface (see Figure 2 for illustration) and $S = 4\pi k_F^2$ is the area of the Fermi surface. Furthermore, $\mathbf{q} = \mathbf{p}' - \mathbf{p}$, and $\mathbf{k}$ and $\mathbf{e}_s(\mathbf{k})$ denote the momentum and polarization vector of a phonon excited or absorbed by an electron. According to quasi-momentum conservation rule in electron-phonon scattering, $\mathbf{k}$ lies in the first Brillouin zone and is given by $\pm\mathbf{k} = \mathbf{q} - \mathbf{K}$, where $\mathbf{K}$ is the vector of the

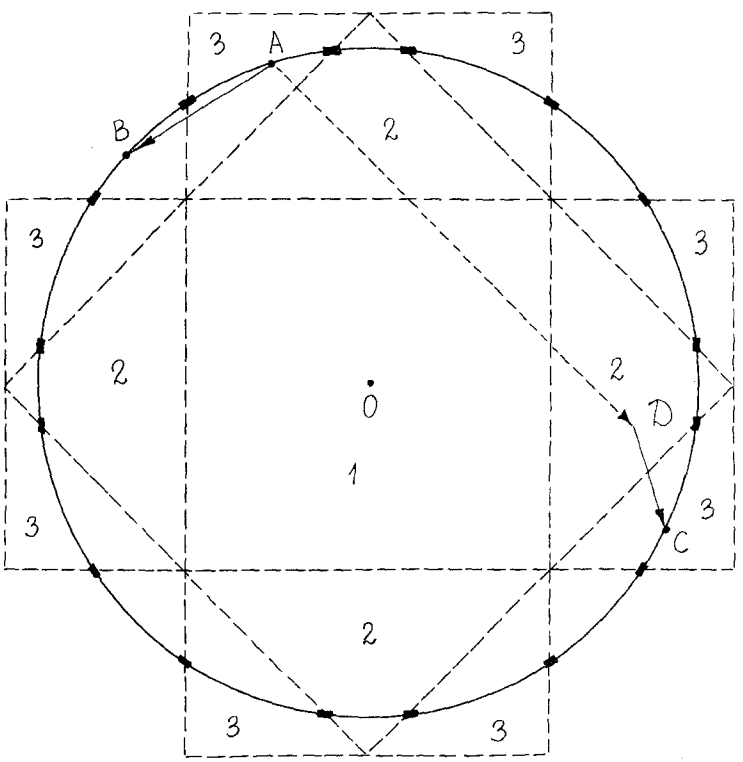

Fig. 2. A schematic picture of electron scattering on phonons in a simple square two-dimensional lattice at $Z = 5$ (for illustration). Dot-lines show the boundaries of the first three Brillouin zones 1, 2, and 3. The solid circle is the Fermi surface, heavy short lines indicate the places where the Fermi surface is distorted due to the Bragg reflection of electrons. The transition $A \to B$ corresponds to normal process whereas the transition $A \to C$ corresponds to the Umklapp process. The vectors **AB** and **DC** are the phonon quasi-momenta for these transitions, and **AD** is the corresponding reciprocal cell vector for the Umklapp process.

reciprocal cell containing the end of the vector **q**. Equation (4) takes into account one-phonon transitions only which is not very accurate at $T \approx T_M$ (Flowers and Itoh, 1976). It should be noted that the expressions for ν_κ and ν_σ used by Flowers and Itoh (1976) as well as by Yakovlev and Urpin (1980) are not as general as Equations (3) and (4), and may be derived from (3) and (4) by making use of approximate treatment of the Umklapp processes described by Ziman (1962, Chapter 9, §5).

Three phonon modes, $s = 1, 2, 3$, are available in Coulomb crystals (Cohen and Keffer, 1955; Carr, 1961). Two of them ($s = 1, 2$) are acoustic, and the third ($s = 3$) is optical. Near the centre of the Brillouin zone (at $k \ll k_D$, $k_D = (6\pi^2 n_i)^{1/3} = (9\pi/2)^{1/3} a^{-1}$ being the radius of the sphere whose volume is equivalent to the volume of the Brillouin zone) the acoustic modes are transverse but the optical one is longitudinal. At $k \sim k_D$ the modes are purely transverse and longitudinal for several specific directions only.

Since ions are fully ionized under the astrophysical conditions, the number of free electrons per one ion is equal to Z. If $Z = 1$, **q** lies in the first Brillouin zone at any orientation of **p** and **p**′. In this case **k** = **q** in Equation (4), and only normal processes contribute into scattering (Figure 2). If, however, $Z \geqslant 2$, **q** 'jumps out' of the first zone at some orientations of **p** and **p**′ that corresponds to Umklapp-processes (Figure 2). The number of reciprocal cells, occupied by **q**, may be roughly estimated as $(k_F/k_D)^2 \sim Z^{2/3}$, i.e., increases with Z.

If $Z \gg 1$, the integrals (4) may be analytically evaluated. Let us first fix **p** and integrate over dS'. For this purpose let us define the reciprocal lattice with the centre in the end of the vector **p**, where **q** = 0 (point A in Figure 2). The Fermi surface will be separated into many ($\sim Z^{2/3}$) pieces which lie in different reciprocal cells. The main contribution into the integrals comes from the pieces distant from the central one. The vector **q** changes only slightly, while integrating over the surface of a distant piece, whereas the quantities which depend upon **k** change rather sharply. Therefore, **q** in integrand may be replaced by a constant equal to some mean vector that characterizes the position of a given piece on the Fermi surface. Then the integral equals the surface area of the piece multiplied by the value of the rapidly oscillating function of **k** averaged over this area. Taking into account the variety of distant pieces, the above averaged value may be replaced by the value averaged over the volume of the first Brillouin zone. It is easy to show that this replacement is equivalent to setting, in Equation (4),

$$\sum_{s=1}^{3} [\mathbf{e}_s(\mathbf{k})\mathbf{q}]^2 z_s^n (e^{z_s} - 1)^{-2} \to t^2 q^2 G_{(t)}^{(n)},$$

$$G_{(t)}^{(n)} = (3 V_B t^2 \pi^n)^{-1} \sum_s \int \mathrm{d}\mathbf{k} z_s^n (e^{z_s} - 1)^{-2} e^{z_s}, \tag{6}$$

where $n = 0$ or 2, and integration is carried out over the first Brillouin zone (for bcc lattice, the first zone has the shape of rhombic dodecahedron; its volume is

denoted by V_B). With replacement (6) integration over dS and dS′ in Equation (4) is easily carried out. In this case the integration region over dS′ should be restricted by the values $q > q_{\min} \sim k_D$ corresponding to the distant parts of the Fermi surface which give the main contribution into the integrals. Since $k_{TF}/k_D \sim Z^{1/3}(e^2/\hbar v_F)^{1/2} \ll 1$, the screening of the ion charge may be neglected. Small q's do not contribute to the integrals in almost all the terms in which one can set $q_{\min} = 0$. Only in one term (in F_κ at small q), does the integrand behave as dq/q. Therefore, this term depends on $q_{\min}$, although only slightly. Finally, we obtain

$$F_\kappa = (2-\beta^2)G^{(0)}(t) + (3\lambda - \beta^2 - 1)G^{(2)}(t),$$
$$F_\sigma = (2-\beta^2)G^{(0)}(t); \qquad \lambda = \ln\left(\frac{2k_F}{q_{\min}}\right). \tag{7}$$

These equations were derived by Yakovlev and Urpin (1980) by the different method from simplified expressions for ν_κ and ν_σ. For evaluating, Yakovlev and Urpin (1980) put $q_{\min} = k_D$ that gives $\lambda = \frac{1}{3}\ln(4Z)$. Numerical calculations performed in this work (see below) show that a more accurate result is

$$\lambda \approx \tfrac{1}{3}\ln(4Z) + 0.4 = \tfrac{1}{3}\ln Z + 0.86. \tag{7a}$$

Moreover, Yakovlev and Urpin (1980) derived the asymptotic expressions of $G^{(0)}(t)$ and $G^{(2)}(t)$ at $t \gg 1$ and $t \ll 1$, and proposed the following analytic formulae for arbitrary t which fit the main terms of the asymptotic expressions

$$G^{(0)}(t) = u_{-2}\left[1 + \left(\frac{3u_{-2}}{\pi^2 c_2 t}\right)^2\right]^{-1/2} \approx 13(1 + 0.017t^{-2})^{-1/2}, \tag{8}$$
$$G^2(t) = (\pi t)^{-2}\left[1 + t^{-2}\left(\frac{15}{4\pi^4 c_2}\right)^{2/3}\right]^{-3/2} \approx 0.1t^{-2}(1 + 0.0116t^{-2})^{-3/2},$$

where $u_{-2} \approx 13$ and $c_2 \approx 30$ are the numerical constants which characterize the phonon spectrum (see Yakovlev and Urpin, 1980, for details). In this work we calculated $G^{(0)}(t)$ and $G^{(2)}(t)$ numerically from initial Equations (6) with the exact spectrum of phonons (see below) at $t > 0.01$. The simple fitting formulae (8) appeared to give quite satisfactory accuracy (the error $< 10\%$ even at $t \sim 1$).

If Z is small (~ 1), simple analytic evaluation of the integrals (4) seems to be impossible. Therefore, the integration was carried out numerically by Monte Carlo method. The phonon frequencies $\omega_s(\mathbf{k})$ and polarization basic vectors $\mathbf{e}_s(\mathbf{k})$ in this integration were determined by interpolating the values found by Carr (1961) at 512 points of the Brillouin zone. This appears to be quite accurate everywhere in the Brillouin zone except in the small vicinity $ka \ll 1$ near its centre. Thus, in a certain small vicinity (whose sizes were varied to check the accuracy of computation), the phonon spectrum was determined directly from the dispersion relations which, at $ka \ll 1$, are simplified and acquire the form (Cohen and Keffer, 1955; Carr 1961)

$$\left[k_i k_j - (ka)^2(\alpha k_i k_j + \beta_0 k^2 \delta_{ij} + \gamma_{ijlm} k_l k_m) - \frac{\omega_s^2}{\omega_{pi}^2} k^2 \delta_{ij}\right] e_{ij} = 0, \tag{9}$$

where α, β_0, and γ_{ijlm} are dimensionless constants evaluated by Cohen and Keffer (1955). By substituting the values of these constants Carr (1961) found that the phonon frequencies at $ka \ll 1$ are given by

$$\frac{\omega_{1,2}^2}{\omega_{pi}^2} = \left(\frac{ka}{\pi}\right)^2 [0.583 - 1.010(Q \pm \sqrt{Q^2 - 3P})], \tag{10}$$

$$\frac{\omega_3^2}{\omega_{pi}^2} = 1 - 1.17\left(\frac{ka}{\pi}\right)^2 + 2.02Q\left(\frac{ka}{\pi}\right)^2, \tag{10}$$

$$P = k_x^2 k_y^2 k_z^2 k^{-6}, \qquad Q = (k_x^2 k_y^2 + k_y^2 k_z^2 + k_x^2 k_z^2) k^{-4}.$$

We determined the polarization basic vectors from Equation (9) by substituting the frequencies (10).

The above-mentioned data on phonons do not take into account electron response to lattice vibrations. According to Pollock and Hansen (1973) this response becomes important only at small $k \sim k_{TF} \ll k_D$. Its effect is that the mode $s = 3$ changes the dispersion relation from (10) to the acoustic relation $\omega_3(\mathbf{k}) = \omega_{pi} k / K_{TF}$. To account for this effect, the values $\omega_3(\mathbf{k})$ in the computer program were multiplied by $k(k^2 + k_{TF}^2)^{-1/2}$.

Computations were performed for different Z and $t \geq 0.01$. The results appeared unexpectedly well fitted (with error $< 15\%$) by Equations (7) and (8) at any $Z \geq 2$ (for the case $Z = 1$, see Section 4), if in (7) $q_{min} = k_D/1.49$ and, thus, $\lambda = \frac{1}{3} \ln Z + 0.86$. It is remarkable that in Equations (7) and (8) only λ depends on Z, and yet very weakly (logarithmically). It should be noted that the Debye temperature of Coulomb crystals is given by (Carr, 1961) $\Theta = 0.45\hbar\omega_{pi}/k_B$. Thus, at $t = 1$, the temperature $T = 2.2\Theta$. Therefore, the extreme cases $t \gg 1$ and $t \ll 1$ correspond to temperatures much higher and lower than the Debye temperature, respectively. The validity of Equations (7) and (8) down to $Z = 2$ may be explained by the major role of the Umklapp processes under astrophysical conditions (Flowers and Itoh, 1976; Yakovlev and Urpin, 1980). The importance of the Umklapp process is mainly due to the absence (at $k > k_{TF}$) of the longitudinal acoustic mode in the phonon spectrum. In particular, at $t \ll 1$, only the phonons with frequencies $\omega_s \lesssim \omega_{pi} \ll \omega_{pi}$ may participate in scattering. In Coulomb crystals (at $k > k_{TF}$) these are the acoustic phonons ($s = 1$, 2) with $k_{1,2} \lesssim k_D t \ll k_D$. But these phonons are traverse and, thus, contribute only in Umklapp processes. Strictly speaking, at $k \lesssim k_{TF}$ the mode $s = 3$ turns into an acoustic one and contributes towards normal processes, but it is easy to verify that, in the presence of Umklapp processes, this contribution is insignificant.

(3) We used the free-electron approximation in the above calculations. It is well-known, however, that electrons are not free near the boundaries of Brillouin zones due to Bragg reflection. Near the boundaries, the dispersion relation of electrons, $\epsilon(\mathbf{k})$ differs from that for free electrons, and at the boundaries $\epsilon(\mathbf{k})$ contains energy gaps whose widths, $\Delta\epsilon$, are easily estimated in the weak coupling approximation (see, e.g., Kittel, 1976). The estimate is $\Delta\epsilon \sim V_k$, where $V_k \sim Ze^2 n_i k^{-2}$ is the Fourier component of the Coulomb lattice potential. The

distance to the Brillouin zone boundary, at which the effect of the gap is significant, is given by $\Delta k \sim \Delta\epsilon/\hbar v_F$.

Thus, at $Z \geqslant 2$, electrons on the Fermi surface are not free in the bands (Figure 2) of the width Δk near the intersections of the Fermi surface with Brillouin zone boundaries (at $Z = 1$ the intersections are absent). In these bands the shape of the Fermi surface is distorted. Taking $k = k_F$ we obtain $\Delta k \sim k_F e^2/\hbar v_F$. Since $e^2/\hbar v_F \ll 1$ at the conditions under consideration, the bands are very narrow (in fact, this is because the typical electron-ion interaction energy is much smaller than the kinetic energy of electrons).

Brillouin zones are restricted by the planes which pass through the middle of reciprocal lattice vectors and are perpendicular to these vectors. That is why the intersections of the Fermi surface with the boundaries of Brillouin zones are the circles of the length $2\pi(4k_F^2 - k^2)^{1/2}$. The number of intersections is roughly estimated as $(k_F/k_D)^3 \sim Z$ (while estimating we drop the numerical factors ~ 1). The above calculations are valid, provided the total surface area of the distorted bands on the Fermi surface, $k_F \Delta k Z$, is much smaller than the area of the whole surface. Using the above value of Δk, we find that the validity condition of the calculations coincides with the standard Born-approximation condition $Ze^2/\hbar v_F \ll 1$.

It should be emphasized, however, that the presence of distorted bands on the Fermi surface plays an important role at low temperature. This is because, with decreasing T, the places of the Fermi surface, between which the Umklapp processes proceed effectively, become more and more narrow and tighten to the intersection lines with the boundaries of Brillouin zones (Figure 2). When the widths of these places, δk, become smaller than the widths of the distorted bands, the Umklapp processes are switched off ('frozen out'). Then the main contribution to scattering comes from normal processes. Taking into account that at low T the acoustic phonons $s = 1, 2$, for which $\delta k \sim k \sim tk_D$, are most important for Umklapp processes we conclude that Umklapp processes are switched off at $t \sim Z^{1/3} e^2/\hbar v_F = Z^{1/3}/137\beta$.

At $t \ll Z^{1/3} e^2/\hbar v_F$, only normal processes on phonons $s = 3$ with $k \sim k_{TF} t \ll k_D$ contribute to scattering. For these phonons, one can set $(\mathbf{e}_3\mathbf{q})^2 = q^2$, $\omega_3(\mathbf{k}) = \omega_{pi} k/k_{TF}$ in Equation (4). The contribution of the phonons $s = 1, 2$ is negligibly small because they are almost transverse ($(\mathbf{e}_{1,2}\mathbf{q})^2 \sim q^6 k_D^{-4}$, see Equation (9)). Finally, we obtain from Equation (4)

$$F_\sigma = 120\zeta(5)\left(\frac{k_{TF}}{k_F}\right)^2 t^4, \qquad F_\kappa = \frac{360}{\pi^2}\zeta(5) t^2, \tag{11}$$

where $\zeta(5) = 1.03$ is the value of the Riemann zeta-function, and $(k_{TF}/k_F)^2 = 4e^2/\pi\hbar v_F \approx 0.0093\beta^{-1}$. Notice that formally F_σ and F_κ from Equation (11) correctly describe the contribution of normal processes at $t < Z^{1/2}(e^2/\hbar v_F)^{3/4}$ and $t < Z^{1/3}(e^2/\hbar v_F)^{1/2}$, respectively.

(4) We discover that, at $Z \geqslant 2$, electron thermal and electrical conductivities

due to scattering on phonons, are given by Equations (7) and (8) if $t \gg Z^{1/3}e^2/\hbar v_F$ and by Equation (11) if $t \ll Z^{1/3}e^2/\hbar v_F$. Thus, the transport coefficients are different at three temperature intervals.

At temperatures above the Debye temperature ($t \geq 1$), according to Equations (7) and (8),

$$F_\kappa = F_\sigma = (2-\beta^2)u_{-2} \approx 13(2-\beta^2)\,, \tag{12}$$

i.e., σ and κ-temperature dependences are $\sigma \propto T^{-1}$ and $\kappa \propto T^0$. In this case the Wiedemann-Franz rule is valid, $\kappa/\sigma = \pi^2 k_B^2 T/3e^2$.

At temperatures below the Debye temperature but above the Umklapp process switch-off temperature, when $Z^{1/3}e^2/\hbar v_F \ll t \ll 1$, in accordance with (7) and (8),

$$F_\sigma = \frac{\pi^2}{3}(2-\beta^2)c_2 t, \qquad F_\kappa = \frac{\pi^2}{5}(2-3\beta^2+4\lambda)c_2 t; \qquad c_2 \approx 30\,, \tag{13}$$

i.e., $\sigma \propto T^{-2}$ and $\kappa \propto T^{-1}$. The Wiedemann-Franz rule is violated only quantitatively in this temperature interval but not qualitatively. This is due to a major contribution of the Umklapp processes under astrophysical conditions (Flowers and Itoh, 1976; Yakovlev and Urpin, 1980) and is rather unusual for 'terrestrial' metals (cf. Ziman, 1962).

Finally, at temperatures $t \ll Z^{1/3}e^2/\hbar v_F$, when the Umklapp processes are switched off, Equations (11) become valid. In this case $\sigma \propto T^{-5}$ and $\kappa \propto T^{-2}$, just as in terrestrial metals.

An exception should be made for proton crystals ($Z = 1$), where Umklapp processes are absent. However, this case requires no special evaluation for it is described by Equations (11). This is because the protons solidify at very low temperatures (Mochkovitch and Hansen, 1979) when Equations (11) are already of use for describing the contribution of normal processes.

For crystallized helium ions (Mochkovitch and Hansen, 1979), $Z = 2$, the parameter t is always < 1 (Figure 1), so that the case (12) of high-temperature phonons is not realized. For heavier ions, all three cases are possible (Figure 1). Let us emphasize that, according to Equations (13) and (11), while passing from the region where $t \gg Z^{1/3}e^2/\hbar v_F$ into the region where $t \ll Z^{1/3}e^2/\hbar v_F$ (the latter region usually lies deeper in the neutron star envelopes than the first one), the thermal and electrical conductivities suffer rather sharp increases due to the offset of the Umklapp processes. This effect may play an important role in thermonuclear burning of matter in neutron stars and degenerate dwarfs (see, e.g., Lamb and Lamb, 1978). The burning regime is extremely sensitive to thermal conduction. Low conductivity is unable to carry away the released nuclear energy and leads to the nuclear explosion. On the contrary, high thermal conductivity makes nuclear burning stable. The activity of bursters is usually explained by the explosive burning of accreted matter. According to the above results, the conditions for stable burning are more favourable in layers where

$t \ll Z^{1/3}e^2/\hbar v_F$ than in layers where $t \gg Z^{1/3}e^2/\hbar v_F$. It is interesting to notice that if the burning occurs in the intermediate layers, at $t \sim Z^{1/3}e^2/\hbar v_F$, the heat conduction inside the neutron star will be much larger than outside. Thus, the main fraction of the released heat will propagate inside the neutron star and heat its core.

The above results may also be of use for investigating some other processes in neutron stars and degenerate dwarfs, for instance, the cooling of neutron stars (for references see Glen and Sutherland, 1980; Van Riper and Lamb, 1981; Yakovlev and Urpin 1981; Nomoto and Tsuruta, 1981), the diffusion of the magnetic field (e.g., see: Ewart *et al.*, 1976), and the onset of superfluidity in the neutron star cores (see Greenstein, 1975, for references).

It should be noticed that in not very pure crystals at sufficiently low temperatures, the electron scattering on impurities may be more important than on phonons. This mechanism is investigated in detail by Flowers and Itoh (1976) and Yakovlev and Urpin (1980). Also, the phonon thermal conductivity may become significant at low temperatures; its rough estimation is given by Yakovlev and Urpin (1980).

Recently, Flowers and Itoh (1981) derived simple fitting formulae for their values of thermal conductivity computed earlier (Flowers and Itoh, 1976). It should be noted that their values of the thermal and electrical conductivities for scattering on phonons are three to five times larger than our values (7). This difference has been analysed in detail by Yakovlev and Urpin (1980) and is mainly because Flowers and Itoh (1976) did not use a very accurate approximation of the phonon spectrum, especially near the centre of the Brillouin zone. The present work is based on exact phonon spectrum.

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References

Albers, R. C. and Gubernatis, J. E.: 1981, *Phys. Rev.* **23B**, 2782.
Baym, G. and Pethick, C. J.: 1975, *Ann. Rev. Nuc. Sci.* **25**, 27.
Carr, W. J.: 1961, *Phys. Rev.* **122**, 1437.
Cohen, M. N. and Keffer, K.: 1955, *Phys. Rev.* **99**, 1128.
Coldwell-Horsefall, R. A. and Maradudin, A. A.: 1960, *J. Math. Phys.* **1**, 395.
Ewart, G. M., Guyer, R. A., and Greenstein, G.: 1976, *Astrophys. J.* **206**, 218.
Flowers, E. and Itoh, N.: 1976, *Astrophys. J.* **206**, 218.
Flowers, E. and Itoh, N.: 1981, *Astrophys. J.* **250**, 750.
Glen, G. and Sutherland, P.: 1980, *Astrophys. J.* **239**, 671.
Greenstein, G.: 1975, *Astrophys. J.* **200**, 281.
Kittel, C.: 1976, *Introduction to Solid State Physics*, John Wiley and Sons, New York, London, Sydney, and Toronto.

Lamb, D. Q. and Lamb, F. K.: 1978, *Astrophys. J.* **220**, 291.
Mochkovitch, R. and Hansen, J. P.: 1979, *Phys. Rev.* **73A**, 35.
Nomoto, K. and Tsuruta, S.: 1981, *Astrophys. J.* **250**, L19.
Pollock, E. M. and Hansen, J. P.: 1973, *Phys. Rev.* **8A**, 3110.
Van Riper, K. A. and Lamb, D. Q.: 1981, *Astrophys. J.* **244**, L13.
Yakovlev, D. G. and Urpin, V. A.: 1980, *Astron. Zh.* **57**, 526 (*Soviet Astron.* **24**, 303).
Yakovlev, D. G. and Urpin, V. A.: 1981, *Pis'ma v Astron. Zh.* **7**, 157.
Ziman, J. M.: 1960, *Electrons and Phonons*, Oxford Univ. Press.