SCATTERING FUNCTIONS FOR NEUTRINO TRANSPORT

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Abstract. We derive an expression for the scattering functions for electron-neutrino and electron-anti-neutrino Compton scattering in a form suitable for a numerical solution of the neutrino transfer equations. An analytical expression is given for the case of large electron degeneracy. The modification due to possible neutral currents is discussed.

1. Introduction

Current models for supernovae (see Wilson et al., 1974) rely mostly on the coupling between neutrina and matter in the collapsing core for the ejection of the envelope of the star. Neutrina are copiously formed in the shock wave resulting from the bounce of the core. Neutrino transport outwards is mainly governed by ν absorption due to $\nu+n\to p+e$, by 'Compton' scattering on electrons $\nu+e^-\to\nu'+e^{-\prime}$, and by $\nu+n\to\nu+n$ (and coherent nuclear scattering $\nu+A\to\nu+A$). The problem is complicated by the fact that the neutrina are fermions and that their mean free paths λ have been estimated to appreciably exceed the radius of the core for low energies but to be considerably smaller at high energies. Mazurek has also pointed out that the approximation of zero chemical potential of the neutrina (photon-like behavior) commonly used may be inappropriate. Using a LTE diffusion approximation he concludes that appreciable neutrino degeneracy may build up. In view of the low energy neutrino window, however, it appears that scattering cannot be treated in an energy-average diffusion approximation as the down scatter of neutrina is crucial. In additional the time scale for ν interactions $t_\nu\sim\lambda/c$ is of the order of the dynamic time scale so that a stationary state may not arise and neutrina may not be in local thermodynamic equilibrium (LTE). Thus even a multi-energy group diffusion approximation may be insufficient and the solution of a transport equation has to be attempted. The latter is similar to the photon transport equation (Chiu, 1968) since the neutrino velocity is constant and equal to the speed of light c, but differs in the statistics.

The transport equation is then given by

$$\left(\frac{\partial}{\partial t}+c\hat{\mathbf{v}}\cdot\nabla\right)F(\mathbf{v})=\left[\frac{\mathrm{d}F(\mathbf{v})}{\mathrm{d}t}\right]_{\mathrm{em.}}[1-F(\mathbf{v})]-\frac{1}{\tau_a}F(\mathbf{v})+ \\ +[1-F(\mathbf{v})]\int \mathrm{d}\nu'\,\mathrm{d}\Omega_{\nu'}\,R^{\mathrm{in}}(\mathbf{v},\mathbf{v}')\,F_{\nu'\mu'}- \\ -F(\mathbf{v})\int \mathrm{d}\nu'\,\mathrm{d}\Omega_{\nu'}\,R^{\mathrm{out}}(\mathbf{v},\mathbf{v}')\,(1-F_{\nu'\mu'}) \qquad (1)$$

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where

$F(\mathbf{v})$ = neutrino distribution function related to the specific intensity $I(\mathbf{v}) = \frac{h\nu^3}{c^2} F(\mathbf{v})$,

$\left[\frac{\mathrm{d}F(\mathbf{v})}{\mathrm{d}t}\right]_{\text{em.}}$ = the total neutrino emission rate,

$\tau_a(\nu)$ = the absorption life time,

$R^{\{\substack{\text{in}\\ \text{out}}\}}$ = the scattering rates into and out of the beam per unit frequency.

In the case of interest to us (spherical symmetry) the neutrino density depends only on r and μ where the latter is $\mu = \hat{\nu}\cdot\hat{r}$. We have

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right)F_{\nu\mu}(r, t) = \left[\frac{\mathrm{d}F(\mathbf{v})}{\mathrm{d}t}\right]_{\text{em.}}(1 - F_{\nu\mu}) - \frac{1}{\tau_a}F_{\nu\mu} +$$
$$+ (1 - F_{\nu\mu})\int \mathrm{d}\nu'\,\mathrm{d}\mu'\,F_{\nu'\mu'}\int \mathrm{d}\phi\, R^{\text{in}}(\nu, \nu', \cos\theta) -$$
$$- F_{\nu\mu}\int \mathrm{d}\nu'\,\mathrm{d}\mu'(1 - F_{\nu'\mu'})\int \mathrm{d}\phi\, R^{\text{out}}(\nu, \nu', \cos\theta), \tag{2}$$

where

$$\cos\theta = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos\phi.$$

In the present paper we shall not consider the emission and absorption rates, but only the νe scattering rate, which is Compton-like.

2. Electron-Neutrino Scattering

We shall distinguish four-vectors by a tilde under the symbol. Final states are denoted by a prime. The electron and neutrino momenta are, respectively, $\underset{\sim}{P} = (E, \mathbf{p})$ and $\underset{\sim}{\nu} = (\nu, \mathbf{v})$. We adopt atomic units, that is $m_e = \hbar = c = 1$.

The scattering rates are given by

$$R^{\text{out}} = \int \frac{2\,\mathrm{d}^3p}{(2\pi)^3}\,\nu'^2\,\mathrm{d}^3p'\,f(E)[1 - f(E')]W(\underset{\sim}{\nu} \to \underset{\sim}{\nu}', \underset{\sim}{p} \to \underset{\sim}{p}'), \tag{3a}$$

$$R^{\text{in}} = \int \frac{2\,\mathrm{d}^3p'}{(2\pi)^3}\,\nu'^2\,\mathrm{d}^3p\,f(E')[1 - f(E)]W(\underset{\sim}{\nu}' \to \underset{\sim}{\nu}, \underset{\sim}{p}' \to \underset{\sim}{p}), \tag{3b}$$

where

$$f(E) = [1 + \exp(\beta(E - u))]^{-1},$$
$$W(\underset{\sim}{\nu} \to \underset{\sim}{\nu}', \underset{\sim}{p} \to \underset{\sim}{p}') = W(\underset{\sim}{\nu}' \to \underset{\sim}{\nu}, \underset{\sim}{p}' \to \underset{\sim}{p})$$
$$= \frac{1}{(2\pi)^2}\frac{\langle \mathscr{M}^2\rangle}{16\nu\nu' EE'}\,\delta^4(\underset{\sim}{p} + \underset{\sim}{\nu} - \underset{\sim}{p}' - \underset{\sim}{\nu}'), \tag{4}$$

and

$$\langle \mathscr{M}^2 \rangle = \tfrac{1}{2} \sum |M|^2 = 64G^2(\underset{\sim}{p} \cdot \underset{\sim}{\nu})(\underset{\sim}{p}' \cdot \underset{\sim}{\nu}') = 16\pi\sigma_0(\underset{\sim}{p} \cdot \underset{\sim}{\nu})^2 \tag{5}$$

is the average over initial and sum over final spin states of the matrix element squared: i.e.,

$$M = \frac{G}{\sqrt{2}} [\bar{u}_e(\underset{\sim}{p}')\gamma_\alpha(1 - \gamma_5)u_\nu(\underset{\sim}{\nu})][\bar{u}_\nu(\underset{\sim}{\nu}')\gamma_\alpha(1 - \gamma_5)u_e(\underset{\sim}{p})]. \tag{6}$$

The latter corresponds to a point interaction in the conventional charged current theory, and

$$G = 1.0(-5)/M^2(\text{proton}), \qquad \sigma_0 = \frac{4G^2}{\pi} = 1.7(-44)\ \text{cm}^2.$$

The integration over d^3p' can be done with the help of the relation $d^3p'/2E' = d^4p'\ \delta(\underset{\sim}{p}'^2 - 1)$ and the δ^4-function expressing conservation of four-momentum

$$\delta(2\xi) W_{\underset{\sim}{\nu} \to \underset{\sim}{\nu}'} = \nu'^2 \int d^4p'\ W(\underset{\sim}{\nu} \to \underset{\sim}{\nu}', \underset{\sim}{p} \to \underset{\sim}{p}') \times$$
$$\times\ \delta^4(\underset{\sim}{p} + \underset{\sim}{\nu} - \underset{\sim}{p}' - \underset{\sim}{\nu}')\delta(\underset{\sim}{p}'^2 - 1),$$

where

$$\xi = (\underset{\sim}{p} \cdot \underset{\sim}{\nu} - \underset{\sim}{p} \cdot \underset{\sim}{\nu}' - \underset{\sim}{\nu} \cdot \underset{\sim}{\nu}') = [\nu E(1 - \cos\alpha) - \nu' E(1 - \cos\alpha') +$$
$$+\ \nu\nu'(1 - \cos\theta)]. \tag{7}$$

The result is

$$R^{\{\text{out}\}}_{\{\text{in}\}} = \nu'^2 \int \frac{d^3p}{(2\pi)^3} \begin{Bmatrix} f(E)[1 - f(E + \nu - \nu')] \\ f(E + \nu - \nu')[1 - f(E)] \end{Bmatrix} W_{\underset{\sim}{\nu} \to \underset{\sim}{\nu}'}\ \delta(\xi).$$

We have assumed here that the electrons are extremely relativistic ($E = p$). The following algebra could be carried out for the general relation $E = \sqrt{p^2 + 1}$ but it would even be more tedious.

The quantity $\langle \mathscr{M}^2 \rangle$ is given by

$$\langle \mathscr{M}^2 \rangle = 16\pi\sigma_0 E^2 \nu^2 (1 - \cos\alpha)^2.$$

We have introduced the angles by

$$\begin{aligned}
\cos\alpha &= \hat{p} \cdot \hat{\nu}, \\
\cos\theta &= \hat{\nu} \cdot \hat{\nu}', \\
\cos\alpha' &= \hat{p} \cdot \hat{\nu}' = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\beta, \\
d^3p &= E^2\, dE\, d\cos\alpha\, d\beta.
\end{aligned}$$

For convenience we finally write the rates in the form

$$R^{\text{out}}(\nu, \nu', \cos\theta) = \int \mathrm{d}E f(E)[1 - f(E + \nu - \nu')] \times \times h(\nu, \nu', E, \cos\theta), \tag{8}$$

$$R^{\text{in}}(\nu, \nu', \cos\theta) = \int \mathrm{d}E f(E + \nu - \nu')[1 - f(E)] h(\nu, \nu', E, \cos\theta),$$

with

$$h(\nu, \nu', E, \cos\theta) = \left(\frac{1}{2\pi}\right)^4 \sigma_0 E^3 \nu\nu' \int_{-1}^{+1} \mathrm{d}\cos\alpha (1 - \cos\alpha)^2\, 2 \int_0^{\pi} \mathrm{d}\beta\, \delta(\xi). \tag{9}$$

We use the δ-function to integrate over β to get

$$h(\nu, \nu', E, \cos\theta) = \left(\frac{1}{2\pi}\right)^4 \sigma_0 E^3 \nu\nu'\, 2 \int_{-1}^{+1} \frac{\mathrm{d}x (1 - x)^2}{\sqrt{ax^2 + bx + c}}\, \theta(ax^2 + bx + c) \tag{10}$$

with $x \equiv \cos\alpha$.

The θ-function arises from the requirement that $|\cos\beta| \leqslant 1$. The coefficients are given by

$$\begin{aligned} a &= -E^2[\nu^2 + \nu'^2 - 2\nu\nu' \cos\theta] < 0, \\ b &= 2E(\nu - \nu' \cos\theta)[E(\nu - \nu') - \nu\nu'(1 - \cos\theta)], \\ c &= E^2 \nu'^2 \sin^2\theta - [E(\nu - \nu') - \nu\nu'(1 - \cos\theta)]^2. \end{aligned} \tag{11}$$

It is remarkable that the θ-function imposes $x^{\pm}$ as the limits on the integral where $x^{\pm}$ are the roots of the radical. The condition for a real and nonvanishing range is that $b^2 - 4ac = 4E^2\nu'^2 \sin^2\theta\, \nu\nu'(1 - \cos\theta)[2E^2 + 2E(\nu - \nu') - \nu\nu'(1 - \cos\theta)] > 0$. The integral h can be reduced to the form

$$\int_{x_-}^{x_+} \frac{\mathrm{d}x}{\sqrt{ax^2 + bx + c}} = \frac{\pi}{\sqrt{-a}}\, \theta(b^2 - 4ac).$$

Leaping over a barrage of very tedious algebra we obtain

$$h(\nu, \nu', E, \cos\theta) = \frac{\sigma_0}{(2\pi)^3} \frac{\nu'}{\nu} (A + BE + CE^2) \times \times \theta\left[E(E + \nu - \nu') - \frac{\nu\nu'}{2}(1 - \cos\theta)\right],$$

where

$$\begin{aligned} A &= y\nu^2[(\nu - \nu' \cos\theta)^2 - \tfrac{1}{2}\nu'^2 \sin^2\theta], \\ B &= y\nu[2\nu^2 - \nu'^2 + 3\nu\nu' - \nu\nu' \cos\theta - 3\nu'^2 \cos\theta], \\ C &= y[\nu^2 + \nu'^2 + 3\nu\nu' + \nu\nu' \cos\theta], \end{aligned} \tag{12}$$

with

$$y = \frac{\nu^2 \nu'^2 (1 - \cos\theta)^2}{(\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta)^{5/2}}.$$

In general the integrals (8) over E have to be done numerically. When the electrons are sufficiently degenerate, which seems to be the case in the regions where their contribution to neutrino transport is important we can approximate the integral over E by a double Sommerfeld expansion.

When $\nu \neq \nu'$ we have

$$R^{\text{out}}(\nu, \nu', \cos\theta) = \frac{\sigma_0}{(2\pi)^3} \frac{\nu'}{\nu} \times$$
$$\times \begin{cases} \zeta(u) - \zeta(u - \nu + \nu') & \begin{cases} \nu > \nu' \\ u - \nu + \nu' > \varepsilon_0 \end{cases} \\ \zeta(u) - \zeta(\varepsilon_0) & \begin{cases} \nu > \nu' \\ u > \varepsilon_0 > u - \nu + \nu' \end{cases} \\ 0 & \nu' > \nu \quad \text{or} \quad \varepsilon_0 > u, \end{cases} \tag{13}$$

$$R^{\text{in}}(\nu, \nu', \cos\theta) = \frac{\sigma_0}{(2\pi)^3} \frac{\nu'}{\nu} \times$$
$$\times \begin{cases} \zeta(u - \nu + \nu') - \zeta(u) & \begin{cases} \nu' > \nu \\ u > \varepsilon_0 \end{cases} \\ \zeta(u - \nu + \nu') - \zeta(\varepsilon_0) & \begin{cases} \nu' > \nu \\ u - \nu + \nu' > \varepsilon_0 > u \end{cases} \\ 0 & \nu > \nu' \quad \text{or} \quad \varepsilon_0 > u - \nu + \nu', \end{cases} \tag{14}$$

where

$$\varepsilon_0 = \tfrac{1}{2}(\nu' - \nu + \sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta}) \tag{15}$$

and

$$\zeta(x) = Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{3}Cx^3. \tag{16}$$

The β^{-2} term is only approximate when $(u - \nu + \nu') \simeq 0$ or $\varepsilon_0 \approx 0$. When $\nu = \nu'$ we have

$$R^{\left\{\substack{\text{in} \\ \text{out}}\right\}} = \frac{\sigma_0}{(2\pi)^3} \times$$
$$\times \begin{cases} \dfrac{1}{\beta}\left[A + Bu + Cu^2 + \dfrac{\pi^2}{3\beta^2} C\right], & u > \varepsilon_0; \\ 0 & u < \varepsilon_0. \end{cases} \tag{17}$$

Again the β^{-1} contribution is only approximate.

Finally a word of caution when our results are used in a numerical treatment of the transport equation; our rates become infinite (albeit integrably) for elastic forward scattering ($\nu=\nu'$, $\cos\theta=1$) so that a Legendre expansion in $\cos\theta$ would not converge.

Antineutrino electron scattering functions can also be derived *mutatis mutandis* starting with the suitable matrix element

$$\langle \mathcal{M}^2 \rangle = 64G^2(\underset{\sim}{p}'\cdot\underset{\sim}{\nu})(\underset{\sim}{p}\cdot\underset{\sim}{\nu}') = 16\pi\sigma_0[(\underset{\sim}{p}-\underset{\sim}{\nu}')\cdot\underset{\sim}{\nu}]^2. \tag{18}$$

The R^{in} and R^{out} can still be written in the form (14) and (13) with $\xi(x) = A'x + \frac{1}{2}B'x^2 + \frac{1}{3}Cx^3$. A' and B' are related to A and B by an interchange of ν and ν' resulting from the crossing symmetry between the νe and $\bar{\nu}e$ processes

$$B' = -B(\nu\leftrightarrow\nu'), \qquad A' = A(\nu\leftrightarrow\nu').$$

Finally we note that the *positron-neutrino* scattering functions are simply related to the electron scattering function by symmetry of the matrix element so that

$$h(e^+ + \nu) = h(e^- + \bar{\nu}), \qquad h(e^+ + \bar{\nu}) = h(e^- + \nu).$$

3. Neutral Current Theories

Recent theoretical and experimental developments in the theory of weak interactions have raised the possibility of the existence of neutral currents. The results of this paper can readily be extended to include the effects of neutral currents. The most general form of the square of the matrix element (Equation (5)) is given by

$$\langle \mathcal{M}^2 \rangle = 64G^2[\alpha_1(\underset{\sim}{p}\cdot\underset{\sim}{\nu})(\underset{\sim}{p}'\cdot\underset{\sim}{\nu}') + \alpha_2(\underset{\sim}{p}\cdot\underset{\sim}{\nu}')(\underset{\sim}{p}'\cdot\underset{\sim}{\nu}) + \alpha_3(\underset{\sim}{\nu}\cdot\underset{\sim}{\nu}')].$$

The corresponding h functions (Equation (10)) are, therefore, given by

$$h^{NC}(e\nu) = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3.$$

The functions h_1 and h_2 have already been calculated to be $h_1 = h(e\nu)$ and $h_2 = h(e\bar{\nu})$ and h_3 can similarly be evaluated from

$$h_3(\nu,\nu',E,\cos\theta) = \frac{\sigma_0}{(2\pi)^3}\frac{\nu^2\nu'^2(1-\cos\theta)}{(\nu^2+\nu'^2-2\nu\nu'\cos\theta)^{1/2}} \times$$
$$\times\, \theta\left[E(E+\nu-\nu') - \frac{\nu\nu'}{2}(1-\cos\theta)\right].$$

For the case of electron-antineutrino scattering the functions h_1 and h_2 have to be switched by

$$h^{NC}(e\bar{\nu}) = \alpha_1 h_2 + \alpha_2 h_1 + \alpha_3 h_3.$$

In the particular case of Weinberg's model the α coefficients are easily seen to be (Dicus, 1972; Tubbs and Schramm, 1975)

$$\alpha_1 = \tfrac{1}{4}(C_A + C_V)^2, \qquad \alpha_2 = \tfrac{1}{4}(C_A - C_V)^2, \qquad \alpha_3 = \tfrac{1}{4}(C_A^2 - C_V^2),$$

with $C_A = \frac{1}{2}$ and $C_V = \frac{1}{2}(1 + 4\sin^2\theta_w)$. (Note that for point interaction theory $C_V =$

$C_A = 1$.) The currently estimated value of $\sin^2 \theta_w = 0.45 \pm 0.15$, so that the contributions of the second and third terms are of the same order as the $\alpha_1 \simeq 1$ term.

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References

Chiu, H. Y.: 1968, *Stellar Physics*, Blaisdell Publ., p. 171.

Dicus, D. A.: 1972, *Phys. Rev.* **D6**, 941.

Mazwiek, T. J.: 1975, *Astrophys. Space Sci.* **35**, 117.

Tubbs, D. L. and Schramm, D. N.: 1975, *Neutrino Opacities at High Temperatures and Densities*, preprint.

Wilson, J. R., Schramm, D. N., Colgate, S., Bruenn, S. W. and Arnett, D.: 1974, in *Proceedings of the 7th Texas Symposium on Relativistic Astrophysics*.