ON THE PROBLEM OF THE INITIAL STATE IN THE ISOTROPIC SCALAR-TENSOR COSMOLOGY OF BRANS-DICKE

L. E. GUREVICH

Physico-Technical Institute, Leningrad, U.S.S.R.

A. M. FINKELSTEIN

Institute for Theoretical Astronomy, Leningrad, U.S.S.R

and

V. A. RUBAN

Nuclear Physics Institute, Leningrad, U.S.S.R.

(Received 1 December, 1972)

Abstract. In connection with the problem of the initial singularity in the framework of the Brans-Dicke scalar-tensor isotropic cosmology, the dynamics of the early stages of the expansion is studied on the basis of the general analytical solutions for flat models with $P = n\varepsilon$ $(0 \leqslant n \leqslant 1)$. The sourceless scalar field, which plays a role of effective source of the geometry, entirely changes the character of the initial expansion, whereas in the absense of it the dynamics of the models almost does not differ from GR-case. If the connecting parameter $\omega < -6$ then the sourceless scalar field removes the initial singularity for any equations of state, and provides the regular transition form the compression to the expansion through the non-stationary state with $\dot{\varphi} \neq 0$. If ω is positive, the singularity of vacuum nature necessarily exists; the sourceless scalar-field being prevalent over the material sources (at least for $P \leqslant \frac{1}{3}\varepsilon$) near the singularity. For $P > \frac{1}{3}\varepsilon$ parallel with the vacuum singularity the singular initial state with dominant role of gravitating matter is possible. In the course of expansion, the infiuence of gravitating matter and curvature – which as nonessential at the early stage in comparison with sourceless φ-field – becomes dominant, and all models approach the 'Machian' ones which differs slightly from corresponding GR-models. If ω is negative $(\omega < -6)$ the sourceless φ-field slows down the initial expansion and increases essentially the real age of the Universe; it can also influence the primordial nuclear synthesis by diminishing in particular the content of He^4. It is shown that a knowledge of the sign $\tilde{G}$ gives us the information about the nature of the initial Universe state.

Résumé. On considère dans cet article l'influence du champ scalaire sans source sur la dynamique des étapes premières de l'expansion des modèles cosmologiques. On montre, en outre, la possibilité de l'existence de l'état initiale nonsingulier dans la cosmologie scalaire-tensorique de Brans-Dicke.

Zusammenfassung. In diesem Artikel betrachtet man die Einwirkung des quellosen skalaren Feld auf die Dynamik der früheren Stadien der Ausdehnung der kosmologischen Modellen. Man zeigt auch die Möglichkeit der Existenz eines unsingulären ursprünglichen Zustand in der Brans-Dickeschen skalar-tensorischen Kosmologie.

1. Introduction

The Brans-Dicke scalar-tensor gravitation theory (Brans and Dicke, 1961) which is widely discussed at present represents the generalization of the general theory of relativity for the case of variable gravitation parameter G. This parameter G is replaced by a massless scalar φ-field which is proportional to G^{-1}. The existence of such a field has not yet been ascertained. The celestial mechanics effects give only the low limit of the dimensionless connection parameter $|\omega|$ of scalar field and the fundamental tensor of the space-time. The accuracy of the latest observations leads only to the conclusion that the contribution of scalar field to relativistic effects in the weak field cannot

Astrophysics and Space Science **22** (1973) 231–242.

exceed 1–5%, corresponding to $|\omega| \geqslant 6$. However, independently of the value of ω, the very existence of the scalar φ-field leads to some principally new results in cosmology, first of all at the early essentially relativistic stages of expansion of the Universe.

The qualitative analysis by Dicke (1968) and Greenstein (1968) shows that the so-called vacuum stage of the initial expansion with dominating scalar field is possible near the isotropic singularity. Such variable φ-field is not connected with material sources and, consequently, does not satisfy Mach's principle. Morganstern (1971) has obtained several exact solutions in the framework of conformal-transformed version of the scalar-tensor theory (Dicke, 1962) and has confirmed the vacuum nature of initial singularity with its character being determined by the scalar field only. In our article we shall study the isotropic cosmology on the basis of original the version of Brans-Dicke theory in connection with the problem of the existence of the initial singularity.

We have obtained the general analytical solutions for the quasi-euclidean models with the equation of state $P = n\varepsilon$ $(0 \leqslant n \leqslant 1)$ and analyzed in details the dynamics of the early stages of its expansion. It will be shown that two qualitatively different variants of the initial expansion are possible, depending on the initial conditions for the scalar field. In the absence of a sourceless scalar field the dynamics is governed by gravitating matter, so that this special 'machian' models (Nariai, 1969) have singularities (with the infinite energy density ε) and their dynamics differs from that of GR quite insignificantly. The presence of a free scalar field, which is not connected with the matter, changes radically the character of early stages of the Universe's expansion. It plays a role (if we use the interpretation of the conformally-transformed version of the theory) of an effective material source of the geometry similar to the ideal fluid with extreme 'stiff' state-equation $P_\lambda = \varepsilon_\lambda$ (Dicke, 1962). Under the infinite compression free φ-field prevails due to its maximal 'rigidity' under the usual material sources. If ω is positive a true singularity a purely vacuum nature occurs inevitably. In this case, the dynamics of the initial expansion does not depend on the presence and the equation of state of matter (at least, if $P \leqslant \frac{1}{3}\varepsilon$). If ω is negative $(\omega < -\frac{3}{2})$ the scalar field acts effectively as a material source with negative energy and leads to the repulsion in the dynamics (gravitation parameter G also remains positive). So the free scalar field removes the singularity, and provides the transition from compression through a regular minimum to expansion.

The expansion leads to a rapid attenuation of the influence of free φ-field on the dynamics and all models tend to 'machian' regime close to the Friedmann models with $P = n\varepsilon$. Thus Brans-Dicke gravitation theory, in distinction of GRT, admits of the principal possibility of the isotropic nonsingular cosmology if $\omega < -6$.*

* One assertion in the paper Noerdlindger (1968, that $\omega < 0$ contradicts to the minimum action by condition. However, the derivation of this result cannot be correct, for the action must be minimum on the trajectectory in comparison with that on the every curve of the neighbourhood in the sence of generalized coordinates and velocities; and, in his considerations, Noerdlindger uses great values of velocity variation.

2. Basic Equations and General Solutions

At the early stages of the expansion of the Universe, the curvature is not essential; and we can restrict ourselves by the consideration of quasi-euclidean variant of the isotropic model with metric

$$ds^2 = d\tau^2 - R^2(\tau)\{dx^2 + dy^2 + dz^2\}. \tag{1}$$

For this metric the Brans-Dicke field equations (if the scalar field is homogeneous; $\varphi = \varphi(\tau)$) are $(\omega \neq -\frac{3}{2})$

$$3\varphi R_1^1 = \frac{1}{R^3}\frac{d}{d\tau}(\dot{R}^3\varphi) = 3\alpha\{\omega(\varepsilon - P) + \varepsilon\}, \tag{2}$$

$$G_0^0 = 3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{\dot{R}}{R}\frac{\dot{\varphi}}{\varphi} - \frac{\omega}{2}\left(\frac{\dot{\varphi}}{\varphi}\right)^2 = (3 + 2\omega)\alpha\varepsilon/\varphi, \tag{3}$$

$$\Box\varphi = \frac{1}{R^3}\frac{d}{d\tau}(R^3\dot{\varphi}) = \alpha(\varepsilon - 3P), \qquad \alpha = 8\pi/(3 + 2\omega). \tag{4}$$

The frame of reference is co-moving, τ is the proper time $(c=1)$ and point denotes the differentiation with respect to τ. Equation (4) determines the connection of the scalar field with the trace of energy-momentum tensor of material sources. Equation (2)–(4) lead to the usual equations of motion $T_{\cdot i;k}^{k}=0$. For the case of adiabatic expansion of ideal fluid in synchronous-comoving system (1) we obtain

$$\dot{\varepsilon} + 3\frac{\dot{R}}{R}(\varepsilon + P) = 0 \tag{5}$$

where ε and P are energy density and pressure respectively.

In cosmological applications the most important equation of state is the barotropic equation $P = n\varepsilon$ $(0 \leqslant n \leqslant 1)$ which includes the fundamental limiting cases of non-relativistic matter $(n=0)$, radiation and ultrarelativistic gas $(n=\frac{1}{3})$, hypothetical cold hadronic matter in ultradensity state $(n=1)$. For this equation of state one can obtain the general solution of the system of Equations (2)–(5).

Equations (2), (4), and (5) admit of the integrals

$$V^{1+n}\varepsilon = M \tag{6}$$

and

$$\begin{aligned} \varphi V^{-n}\frac{dV}{d\eta} &= 3\alpha M\sigma(\eta + \eta_2), \\ V^{1-n}\frac{d\varphi}{d\eta} &= \alpha M(1 - 3n)(\eta + \eta_1), \qquad n \neq \tfrac{1}{3}, \end{aligned} \tag{7}$$

where η is the new time parameter introduced into the metric (1) by means of relation $d\tau = R^{3n}d\eta = V^n d\eta$, M, η_1, η_2 are the constants of integration and $\sigma = 1 + \omega(1 - n)$.

Equations (6) and (7) lead to the basic equation for the volume factor

$$\frac{V''}{V} + \left\{\frac{(1-3n)(\eta+\eta_1)}{3\sigma(\eta+\eta_2)} - n\right\}\left(\frac{V'}{V}\right)^2 - \frac{1}{\eta+\eta_2}\frac{V'}{V} = 0, \tag{8}$$

where dash denotes the differentiation with respect to η.

The integration of Equation (8) gives the expression for the Hubble parameter

$$\frac{R'}{R} = \frac{\sigma(\eta+\eta_2)}{A\eta^2 + B\eta + C} \tag{9}$$

and relative rate of φ-field variation

$$\frac{\varphi'}{\varphi} = \frac{(1-3n)(\eta+\eta_2)}{A\eta^2 + B\eta + C}, \tag{10}$$

where $2A = (1-3n) + 3\sigma(1-n) = (3/\omega)\left(\sigma - \sqrt{1+\frac{2}{3}\omega}\right)\left(\sigma + \sqrt{1+\frac{2}{3}\omega}\right)$, $B = 3\sigma(1-n)\eta_2 + (1-3n)\eta_1$ and C is the constant of integration.

The constants of integration C, η_1, η_2 are not independent. The connection between them follows from Equation (3) which is the first integral of dynamical Equations (2) and (4). Taking into account that

$$1/\varphi V^{1-n} = \frac{1/\alpha M}{A\eta^2 + B\eta + C}$$

and also Equation (3), we obtain this relation

$$(3+2\omega)\tilde{C} = 3\sigma^2 + 3(1-n)\sigma\tilde{\eta}_1 - \frac{\omega}{2}(1-3n)^2\tilde{\eta}_1^2. \tag{11}$$

Without loss of generality we have chosen $\eta_2 \neq 0$ as a scale of time and have introduced the dimensionless quantities $\tilde{C} = C/\eta_2^2$, $\tilde{B} = B/\eta_2$, $\tilde{\eta} = \eta/\eta_2$.

The character of the solution of Equation (9) and (10) depends essentially on discriminant sign of the dominator $\Delta = \tilde{B}^2 - 4A\tilde{C}$ which we can write as

$$\Delta = \frac{(1-3n)^2\sigma^2}{1+\frac{2}{3}\omega}(\tilde{\eta}_1 - 1)^2, \qquad n \neq \tfrac{1}{3} \tag{12}$$

if we take into account Equation (11). Its absolute value represents the square of a characteristic time (measured in the units of η_2). We connected $|\Delta|$ with the duration of φ-field dominated stage.

If $\Delta > 0\ (\omega > -\frac{3}{2})$ the general solution of Equations (9) and (10) is

$$\begin{aligned} R &= R_0(\tilde{\eta} - \tilde{\eta}_+)^{\omega/[3(\sigma\mp\sqrt{1+\frac{2}{3}\omega})]}(\tilde{\eta} - \tilde{\eta}_-)^{\omega/[3(\sigma\pm\sqrt{1+\frac{2}{3}\omega})]}, \\ \varphi &= \varphi_0(\tilde{\eta} - \tilde{\eta}_+)^{(1\mp\sqrt{1+\frac{2}{3}\omega})/(\sigma\mp\sqrt{1+\frac{2}{3}\omega})}(\tilde{\eta} - \tilde{\eta}_-)^{(1\pm\sqrt{1+\frac{2}{3}\omega})/(\sigma\pm\sqrt{1+\frac{2}{3}\omega})}, \end{aligned} \tag{13}$$

where $\tilde{\eta}_\pm = (-\tilde{B} \pm \sqrt{\Delta})/2A$ and, by definition, $\tilde{\eta}_+ > \tilde{\eta}_-$. Besides we have the next relation between constants of integration

$$8\pi M\eta_2^2 R_0^{3(1-n)}/\varphi_0 = (3+2\omega)/A. \tag{14}$$

This solution has two branches which we may call 'fast' and 'slow' in accordance with two different types of the behavior of φ-field. In the following we shall give detailed analysis of these branches.

The solution (13) has always a singularity where the scale factor R becomes zero and the density of matter becomes infinite. In the case for $\omega \geqslant 0$ the initial singularity takes place at $\tilde{\eta} = \tilde{\eta}_+$ for all values of $n \leqslant 1$ on the 'slow' branch ($\varphi \to \infty$, $G \to 0$) and for $n < (\omega + 1 - \sqrt{1 + \frac{2}{3}\omega})/\omega$ on the 'fast' branch ($\varphi \to 0$, $G \to \infty$). If $n > (\omega + 1 - \sqrt{1 + \frac{2}{3}\omega})/\omega$ the singularity $R = 0$ at the 'fast' branch will be attained only in the future at $\tilde{\eta} \to \infty$ and then $\varphi \to \infty$ $(G \to 0)$.

If $\Delta < 0$ $(\omega < -\frac{3}{2})$ the general solution of Equations (9) and (10) is $(A \neq 0)$

$$R = R_0 \left[(\tilde{\eta} + \tilde{\eta}_-)^2 + \tilde{\eta}_+^2\right]^{\sigma/2A} \exp\left\{\pm \frac{\sqrt{\frac{2}{3}}|\omega| - 1}{A} \operatorname{arctg} \frac{\tilde{\eta} + \tilde{\eta}_-}{\tilde{\eta}_+}\right\}, \tag{15}$$

$$\varphi = \varphi_0 \left[(\tilde{\eta} + \tilde{\eta}_-)^2 + \tilde{\eta}_+^2\right]^{(1-3n)/2A} \times$$
$$\times \exp\left\{\mp 3(1-n) \frac{\sqrt{\frac{2}{3}}|\omega| - 1}{A} \operatorname{arctg} \frac{\tilde{\eta} + \tilde{\eta}_-}{\tilde{\eta}_+}\right\},$$

where $\tilde{\eta}_- = \tilde{B}/2A$, $\tilde{\eta}_+ = \sqrt{|\Delta|}/2A$ and the relation (14) remains valid.

When the parameter $\tilde{\eta}$ and corresponding proper time $\tilde{\tau}$ tend to infinity, the scale factor R does not become zero; so that, in such model, the infinite contraction takes place till a regular minimum R_{min} after which it is followed by the expansion. Thus, in the case $\omega < -\frac{3}{2}$, the singularity with the infinite density of matter is absent in the isotropic cosmology for any physically admissible equations of state.

The particular solution with $\Delta = 0$ $(\tilde{\eta}_1 = 1)$ of Equations (13) and (15) transformed to the proper time τ represents a special class of the cosmological models (Nariai, 1969)

$$R = R_0 (\tau/\tau_0)^{[2 + 2\omega(1-n)]/[4 + 3\omega(1-n^2)]}, \qquad \varphi = \varphi_0 (\tau/\tau_0)^{2(1-3n)/[4+3\omega(1-n^2)]}. \tag{16}$$

These models have always initial singularity $R \to 0$ $(\tau \to 0)$ for all values of $|\omega| > 6$ and if $P \leqslant \varepsilon/3$ the scalar field tends to zero: $\varphi \to 0$ $(G \to \infty)$ for $\omega \geqslant 0$, and tends to infinite: $\varphi \to \infty$ $(G \to 0)$ for $\omega < -6$.

The dynamics of this models (16) is governed mainly by the gravitating matter $(\varepsilon/\varphi \propto \tau^{-2})$ and differs weakly from GR-dynamics. In particular the solution (16) coincides exactly with a corresponding Friedmann solution if $n = \frac{1}{3}$. In the limit of $\omega \to \infty$ the general solutions (13) and (15) tend to the general relativistic solutions (except the limiting case $n = 1$).

It can be noted that the trace T in Equation (4) changes the sign if $P > \frac{1}{3}\varepsilon$ and one may expect that the gravitational attraction will turn into 'anti-gravitation' $(G < 0)$. The simple analysis of the relation (14) shows this expectation to be true. The gravitational parameter $G \propto \varphi_0^{-1}$ changes its sign at $n > (1 + \omega - \sqrt{1 + \frac{2}{3}\omega})/\omega$ that corresponds to $n \gtrsim \frac{4}{5}$ if $\omega > 6$. Equation (17) shows that the very variance of the parameter G does not change qualitatively the dynamics of the isotropic models expansion in the scalar-tensor gravitation theory.

The principal distinction between the general relativistic and scalar-tensor cosmologies arises in the general case ($\Delta \neq 0$) if one repudiates Mach's principle in Dicke's sense (Dicke, 1964), which excludes the free scalar field not being connected with the material sources. The condition $\Delta \neq 0$ ($\tilde{\eta}_1 \neq 1$) is the criterion of the existence of such a vacuum field, so that Mach's principle means a highly special choice of the initial conditions for φ-field. The important role of the free scalar field is conditioned by its being a specific material substratum (the 'scalar fluid' in the homogenious case) with extreme 'stiff' state equation $P_\lambda = \varepsilon_\lambda$, which determines the geometry of the space-time parallel with the material sources and modifies the dynamics of the early stages of isotropic models expansion according to the general solutions (13) and (15). The sign of the effective energy-density of φ-field is determined by the quantity $(2\omega+3)$. Therefore, its energy density and pressure at $\omega < -\frac{3}{2}$ becomes negative. This very negative contribution of free scalar φ-field accounts for the removal of the initial singularity in the isotropic scalar-tensor cosmology of Brans-Dicke.

3. The Model with Nonrelativistic Matter ($P=0$)

We shall begin the analysis of the general solutions of Equations (9) and (10) with the simple case of the equation of state $P=0$ ($n=0$) when the parameter η coincides exactly with the proper time τ. In addition to its significant role for the description of the late stages of the Universe expansion (when the rest-mass of the nonrelativistic matter dominates in the sources), the solution with $P=0$ is also of the interest from the methodological point of view, as it shows all characteristic features of more complicated models with $P \neq 0$. Besides, the analysis of the properties of strong-interaction hadronic matter shows (cf. Hagedorn, 1970) that the case $P=0$ may be a good approximation for the description of the state of superdense hot stages near singularity.

In the general case depending on the sign of the connecting parameter ω two principal different types of the initial expansion of the cosmological models may be realized, with singularity ($\omega > -\frac{3}{2}$) and without singularity ($\omega < -\frac{3}{2}$).

If $\omega > -\frac{3}{2}$ the solution for $P=0$ looks like ($\omega \neq -\frac{4}{3}$)

$$R = R_0 (\tilde{\tau} - \tilde{\tau}_+)^{(\omega + 1 \pm \sqrt{1+\frac{2}{3}\omega})/(3\omega+4)} (\tilde{\tau} - \tilde{\tau}_-)^{(\omega+1 \mp \sqrt{1+\frac{2}{3}\omega})/3\omega+4} \tag{17}$$

$$\varphi = \varphi_0 (\tilde{\tau} - \tilde{\tau}_+)^{(1 \pm 3\sqrt{1+\frac{2}{3}\omega})/(3\omega+4)} (\tilde{\tau} - \tilde{\tau}_-)^{(1 \mp 3\sqrt{1+\frac{2}{3}\omega})/(3\omega+4)}$$

according to Equation (15).

Both branches of the solution (17) (for $\omega > 0$) possess the initial singularity at the time moment $\tilde{\tau} = \tilde{\tau}_+$, near which the behavior of the scale factor and the scalar field is characterized by the asymptotic expressions of type

$$R \propto \tau^{(\omega+1 \pm \sqrt{1+\frac{2}{3}\omega})/(3\omega+4)} \to 0, \qquad \varphi \propto \tau^{(1 \mp 3\sqrt{1+\frac{2}{3}\omega})/(3\omega+4)}, \tag{18}$$

φ becoming infinity ($G \to 0$) for the upper ('fast') branch of Equation (18) and φ becoming zero ($G \to \infty$) for the low ('slow') one. The character of this singularity is fully determined with the general vacuum solution for isotropic flat metric (1) (cf. O'Hanlon and Tupper, 1972)

$$R = R_0\,(\tau/\tau_0)^{1/3(1+c)}, \qquad \varphi = \varphi_0\,(\tau/\tau_0)^{c/(1+c)},$$
$$c = \frac{1 \pm \sqrt{1 + \frac{2}{3}\omega}}{\omega} = \frac{\mathrm{d}\ln\varphi}{\mathrm{d}\ln V}. \tag{19}$$

If we move the singularity, the effective energy density of the free scalar field increases much faster $((\dot\varphi/\varphi)^2 \infty\, \tau^{-2})$ then the contribution of the gravitating matter taking into account the variation of $G(\varepsilon/\varphi \infty\, \tau^{-1})$. That is why the free φ-field prevails in the dynamics of the initial expansion.

The choice between two different regimes of model behavior corresponding to the 'fast' and 'slow' branches of the solution (17) (for $\omega > 0$) is dertemined by the sign of relative velocity of the free φ-field variance with the general expansion in the initial singular state

$$c = \left(\frac{\mathrm{d}\ln\varphi}{\mathrm{d}\ln V}\right)_{\tilde\tau = \tilde\tau_+} = \frac{1 \pm \sqrt{1 + \frac{2}{3}\omega}}{\omega}. \tag{20}$$

At the vacuum stage, with the prevalent role of φ-field, the gravitation decreases during the expansion if $c > 0$ and increases if $-1 < c < 0$.

If $\omega < -\frac{3}{2}$, the solution of Equation (15) for $P = 0$ is

$$R = R_0\left[(\tilde\tau + \tilde\tau_-)^2 + \tilde\tau_+^2\right]^{(\omega+1)/(3\omega+4)} \exp\left\{\pm \frac{\sqrt{\frac{2}{3}|\omega| - 1}}{3\omega + 4}\,\mathrm{arctg}\,\frac{\tilde\tau + \tilde\tau_-}{\tilde\tau_+}\right\},$$
$$\varphi = \varphi_0\left[(\tilde\tau + \tilde\tau_-)^2 + \tilde\tau_+^2\right]^{1/(3\omega+4)} \exp\left\{\mp \frac{3\sqrt{\frac{2}{3}|\omega| - 1}}{3\omega + 4}\,\mathrm{arctg}\,\frac{\tilde\tau + \tilde\tau_-}{\tilde\tau_+}\right\}. \tag{21}$$

The characteristic property the cosmological models of such types $(\Delta \neq 0)$ consists in their absence of initial singularities. When the proper time varies from $-\infty$ up to $+\infty$ the models contract to the regular minimum (at $\tilde\tau = -1$)

$$R_{\min} = \tilde R_0\,(\tilde\tau_1 - 1)^{2(1+\omega)/(3\omega+4)} \times$$
$$\times \exp\left\{\pm \frac{2\sqrt{\frac{2}{3}|\omega| - 1}}{3\omega + 4}\,\mathrm{arctg}\,\frac{\sqrt{\frac{2}{3}|\omega| - 1}}{\omega + 1}\right\}, \tag{22}$$

and they expand again after 'rebound'.

Two branches of the solution (21) describe the cosmological models which do not differ qualitatively in the dynamics of the expansion, and in which the scalar field may only decrease (the gravitational parameter G only increases).

It should be emphasized that φ-field has a single maximum at $\tau = -\tau_1$; and, hence, the minimal value of G is attained at the epoch not coinciding with the moment of the 'bouncing' from the regular minimum $R_{\min}$. The difference between the two branches of solution (22) manifests itself in the fact that G, attains its minimal value before or after the moment of 'rebound'. Consequently, in the nonsingular variants the Universe may not be in the static state when both variation velocity $\dot{R}$ and $\dot{\varphi}$ becomes zero.

Although the sourceless scalar field (corresponding to the criterion $\Delta \neq 0$) is a decisive factor in the dynamics of the early stages, and even removals of the singularity, it cannot have purely vacuum character for negative values of $\omega < -\frac{3}{2}$, and exists only in the presence of material sources.

After the completion of the scalar-dominated stage of the duration $\tau \sim \sqrt{|\Delta|} \sim \sim (\tilde{\tau}_1 - 1)$, the influence of the free φ-field weakens quickly during the expansion in comparison with that of the gravitating matter; and, independently on the sign of ω, the general solutions (17) and (21) approach asymptotically ($\tilde{\tau} \gg \tilde{\tau}_+$) to the 'Machian' regime (16) with $n=0$

$$R = R_0 (\tau/\tau_2)^{2(1+\omega)/(3\omega+4)}, \qquad \varphi = \varphi_0 (\tau/\tau_2)^{2/(3\omega+4)}. \tag{23}$$

4. The Model with the Radiation ($P=\frac{1}{3}\varepsilon$)

The degenerate case $n=\frac{1}{3}$ requires a special consideration. The model with $P=\frac{1}{3}\varepsilon$ is of a particular interest for a description of the 'hot' Universe at the radiative stage when the primordial radiation prevails in the material sources. Besides, this limiting equation of state may be used as the asymptotic limit near the singularity for the majority of the relativistic types of the interactions.

The scalar φ-field is not generated by the matter with $P=\frac{1}{3}\varepsilon$

$$\Box \varphi = \frac{1}{R^3} \frac{d}{d\tau} (R^3 \dot{\varphi}) = 0, \qquad \dot{\varphi} R^3 = \tau_1 = \text{const.},$$

and it may be only sourceless ($\tau_1 \neq 0$). After introducing the parameter $d\eta = V^{-1/3}\, d\tau$ in the Equations (2) and (4) we obtain the first integrals

$$V^{2/3} \frac{d\varphi}{d\eta} = 8\pi M \eta_1 ; \qquad \varphi \frac{dV}{d\eta} = 8\pi M (\eta + \eta_2)\, V^{1/3} ; \tag{24}$$

η_1 and η_2 being the constants of integration. Similarly to the general case (see Equation (8), we have

$$\frac{R'}{R} = \frac{\tilde{\eta} + 1}{\tilde{\eta}^2 + (2 + 3\tilde{\eta}_1)\,\tilde{\eta} + \tilde{\eta}_3} ; \qquad \frac{\varphi'}{\varphi} = \frac{3\tilde{\eta}_1}{\tilde{\eta}^2 + (2 + 3\tilde{\eta}_1)\,\tilde{\eta} + \tilde{\eta}_3} ; \tag{25}$$

where the constants $\tilde{\eta}_3$ and $\tilde{\eta}_1$ are connected by

$$\tilde{\eta}_3 = 1 + 3\tilde{\eta}_1 - \tfrac{3}{2}\omega\tilde{\eta}_1^2 . \tag{26}$$

Thus the discriminant of the denominator Δ in Equation (25) according to Equation (26) is of the form

$$\Delta = 3\tilde{\eta}_1^2 (3 + 2\omega) . \tag{27}$$

If the free field is absent ($\Delta = 0$, $\eta_1 = 0$) the corresponding 'Machian' model coincides entirely with general relativistic Friedmann model filled by the radiation: i.e., $\varphi = \text{const.}$, $R \sim \tau^{1/2}$.

In the presence of sourceless φ-field the character of the initial expansion depends essentially on the sign of ω as in the general case.

If $\omega > -\frac{3}{2}$ the solution of the Equation (25) is

$$\begin{aligned} R &= R_0 (\tilde{\eta} - \tilde{\eta}_+)^{1/2 \pm 1/2\sqrt{1+\frac{2}{3}\omega}} (\tilde{\eta} - \tilde{\eta}_-)^{1/2 \mp 1/2\sqrt{1+\frac{2}{3}\omega}}, \\ \varphi &= \varphi_0 (\tilde{\eta} - \tilde{\eta}_+)^{\mp 1/\sqrt{1+\frac{2}{3}\omega}} (\tilde{\eta} - \tilde{\eta}_-)^{\pm 1/\sqrt{1+\frac{2}{3}\omega}}, \end{aligned} \tag{28}$$

where $\eta_\pm = 1 + \frac{3}{2}\eta_1 \pm \frac{1}{2}\sqrt{\Delta}$, $\tilde{\eta}_+ > \tilde{\eta}_-$, $8\pi \mathrm{M}\eta_2^2/3\varphi_0 R_0^2 = 1$.

This solution has inevitably an initial singularity at $\tilde{\eta} = \tilde{\eta}_+$ which is characterized by the universal asymptotic expressions (19) in terms of the proper time τ.

If $\omega < -\frac{3}{2}$, the solution of Equation (25) has no the singularity and assumes the form

$$R = R_0 [(\tilde{\eta} + \tilde{\eta}_-)^2 + \eta_+^2]^{1/2} \exp\left\{ \pm \frac{1}{\sqrt{\frac{2}{3}|\omega| - 1}} \operatorname{arctg} \frac{\tilde{\eta} + \eta_-}{\tilde{\eta}_+} \right\}, \tag{29}$$

$$\varphi = \varphi_0 \exp\left\{ \mp \frac{2}{\sqrt{\frac{2}{3}|\omega| - 1}} \operatorname{arctg} \frac{\tilde{\eta} + \tilde{\eta}_-}{\tilde{\eta}_+} \right\},$$

where $\tilde{\eta}_+ = \frac{1}{2}\sqrt{|\Delta|}$, $\tilde{\eta}_- = 1 + \frac{3}{2}\eta_1$, $8\pi M\eta_2^2/3\varphi_0 R_0^2 = 1$.

It may be noted that the solutions (28 and (29) can be obtained from the general formulae (13), (15) by a substitution for Δ from Equation (27).

The dynamics of the initial expansion of the models with $P = \frac{1}{3}\varepsilon$ during the scalar-dominated stage does not differ qualitatively from the case $P = 0$. In the process of the expansion the influence of gravitating matter (not essential at early stages) becomes dominant and leads asymptoticaly ($\tilde{\eta} \gg \tilde{\eta}_+$) the models with $P = \frac{1}{3}\varepsilon$ – i.e., to the general relativistic regime with $G \to \text{const.}$, $R \sim \tau^{1/2}$.

A consideration of common properties of the solutions with $P = 0$ and $P = \frac{1}{3}\varepsilon$ leads us to some conclusions concerning the dynamics of the expansion for more realistic models of the Universe containing a mixture of matter and radiation. Such considerations are important in connection with the problem of primordial nuclear synthesis and relict radiation for the 'hot' variant of the initial state of the Universe.

It is evident that the dynamics of such a complex model at the early scalar-dominated stage has a universal vacuum character and is almost insensitive to the presence of both matter and radiation. After a transition to the 'Machian' regime, where the influence of sourceless scalar field becomes negligible, the expansion velocity is determined mainly by the dominating contribution of radiation, although the matter with $P = 0$ is essential for the generating of φ-field. The resulting behaviour of a 'hot' Universe model filled by a mixture of matter and radiation is intermediate between the 'Machian' regimes with $P = \frac{1}{3}\varepsilon$ and $P = 0$ on a radiative stage, if the influence of free scalar field may be neglected. Such dynamics does not differ qualitatively from that of Lemaitre-Gamov model in the general relativistic cosmology.

The characteristic feature of isotropic cosmological models with sourceless scalar field ($\Delta \neq 0$) is the essential difference of the initial expansion velocities for different signs of connection parameter ω. If $\omega > 0$ the sourceless φ-field during the stage of its domination accelerates the expansion in comparison with the general relativistic case, and slows it down for $\omega < -\frac{3}{2}$. Therefore, it can influence the primordial nuclear synthesis in 'hot' model of the Universe, increasing or decreasing of the content of helium.

If $\omega > 6$ and the nuclear synthesis occurs at the vacuum stage of expansion, the content of helium then produced depends on the duration of this scalar-dominated stage. If the duration is sufficiently prolonged the expansion is so fast that the helium does not form (Dicke, 1968; Greenstein, 1968).

In the nonsingular isotropic cosmology ($\omega < -6$) the problem of primordial chemical composition requires quite another approach. In particular, the primordial nuclear synthesis could not have taken place at all if the temperature at the moment of 'rebound' was too low ($T < 10^9$ K). In the case of high temperature at the moment of 'rebound' ($T > 10^{12}$ K) and of the nuclear synthesis at the scalar-dominated stage, the content of primordial helium would be smaller than in the case of 'hot' general relativistic model ($He^4/H^1 \leqslant 25$–30% for mass). This is due to the fact that slowed-down initial expansion tends to the thermodynamical equilibrium (at $T < 10^{12}$ K) and, therefore, after the stage of 'hardering' the number of remaining neutrons is very small.

5. Conclusions

The hypothetical scalar field appears in Brans-Dicke gravitation theory in two quite different aspects. On the one hand, it replaces the Newtonian gravitational constant and, on the other, it plays the role of additional material source of the geometry. If the scalar field is always connected with the material sources, and the sourceless scalar field does not exist at all, the isotropic scalar-tensor cosmology does not differ qualitatively from that of the general relativity. On the contrary, the existence of a scalar field not connected with matter can radically change the dynamics of the early stage of the Universe expansion. If connection parameter ω is positive ($\omega > 6$), then the singularity $R \to 0$, $\varepsilon \to \infty$ is inevitable, and vacuum stage of the initial expansion with two possible asymptotical branches ('fast' and 'slow') of behaviour exists. This stage is mainly determined by the free field, and does not depend on the presence of matter for $P \leqslant \frac{1}{3}\varepsilon$.*

The homogenious free φ-field plays, in some respects, the role of an effective source analogous to a fluid with extremely 'stiff' equation of state $P_\lambda = \varepsilon_\lambda$. Its contribution near the singularity rises proportionally to τ^{-2} and, therefore, it dominates as compared with the usual material sources ($\varepsilon/\varphi \sim \tau^{-1-3nA}$, $A = (1 + \omega = \sqrt{(1+\frac{2}{3}\omega)/(4+3\omega)})$ and curvature ($K \sim R^{-2} \sim \tau^{-2A}$).

If $P > \frac{1}{3}\varepsilon$ the vacuum stage remains only the 'fast' branch. On the 'slow branch' the

* Near the singularity we have, in the general case,

$$R \infty \, (\tilde{\eta} - \tilde{\eta}_+)^{\frac{1}{3}\omega/(\sigma \pm \sqrt{1+\frac{2}{3}\omega})}, \qquad \varphi \infty (\tilde{\eta} - \tilde{\eta}_+)^{(1+\sqrt{1+\frac{2}{3}\omega})/(\sigma \mp \sqrt{1+\frac{2}{3}\omega})},$$

$$\tau/\tau_0 \infty (\tilde{\eta} - \tilde{\eta}_+)^{(1+\omega \mp \sqrt{1+\frac{2}{3}\omega})/(\sigma \mp \sqrt{+\frac{2}{3}\omega})}.$$

behaviour of the model is more complex; for the gravitation of matter near the singularity can become essential if $\eta > (1+\omega-\sqrt{1+\frac{2}{3}\omega})/\omega$ and the asymptotics of the initial expansion can become 'Machian' (23).

If ω is negative ($\omega < -6$) the free φ-field contribution to the sources is negative; and this fact leads to the effective repulsion in the dynamics. The negative energy density of this 'scalar fluid' increases due to its maximal 'rigidity' faster than that of the gravitating matter. This is why the free φ-field always removes the singularity for any equations of state and leads to a regular transition from the phase of contraction of the Universe to its expansion. Surprisingly, the free scalar field removes the singularity even for the models filled by the matter with extremely 'stiff' equation of state.

In the course of expansion the influence of gravitating matter and curvature, which were nonessential at the early stages in comparison with the sourceless φ-field, becomes dominant; and all models approach to the 'Machian' ones, which differ only slightly from corresponding general relativistic models.

It is worth to mention that, for $\omega < -6$, the Hubble velocity of the expansion at the scalar-dominated stage may be smaller by several orders than in the GRT. If this stage is prolonged enough, the deceleration of the Hubble velocity may lead to a considerable increase in the real age of the Universe. If ω is positive, the more rapid expansion decreases the age of the Universe only insignificantly.

If the duration of the scalar-dominated stage was long enough, the influence of the free φ-field would appear even at the present epoch which, in turn, would lead to more rapid variation of the gravitation constant as compared with the 'Machian' regime with $P=0$ where $|\dot{G}/G| < 10^{-11}$ yrs^{-1} according to Dicke (1970), and Morganstern (1971). Experimentally determined upper limit of the variation of G of: $|\dot{G}/G| < 10^{-10}$ yr^{-1} (Shapiro *et al.*, 1971) puts the restriction on the possible duration of the scalar-dominated stage which can not be longer than 10^8–10^{-9} yr if ω is positive.

Thus, the extremely small secular variation of gravitation constant allowed by the modern observation can modify the dynamics of the Universe expansion essentially at the early stages. In accordance with Equations (4) and (14) the sign of velocity variation of the gravitation constant $\dot{G}$ at the late 'Machian' stages is determined by the sign of ω, and does not depend on the curvature. Consequently, the data concerning the behaviour of the gravitation constant at the present epoch gives us the insight into the nature of the initial state. If the gravitation parameter G increases at present and the sourceless scalar field exists, the singularity of the Universe has been absent in the past.

References

Brans, C. and Dicke, R.: 1961, *Phys. Rev.* **124**, 925
Dicke, R.: 1962, *Phys. Rev.* **125**, 2163
Dicke, R.: 1964, in H. Chiu and H. Hoffmann (eds.), *Gravitation and Relativity*, New York-Amsterdam.
Dicke, R.: 1968, *Astrophys. J.* **152**, 1.
Dicke, R.: 1970, *Gravitation and the Universe,* Amer. Philosophical Soc., Philadelphia.
Greenstein, G.: 1968, *Astrophys. Space Sci.* **2**, 155

Hagedorn, R.: 1970, *Astron. Astrophys.* **5**, 184
O'Hanlon, J. and Tupper, B.: 1970, *Nuovo Cimento* **137**, 305.
Morganstern, R.: 1971a, *Phys. Rev.* **D4**, 278, 282, 954.
Morganstern, R.: 1971b, *Nature Phys. Sci.* **232**, 109.
Morganstern, R.: 1972, *Nature Phys. Sci.* **237**, 70.
Nariai, H.: 1969, *Progr. Theor. Phys.* **42**, 544.
Noerdlindger, P.: 1968, *Phys. Rev.* **170**, 1175
Shapiro, I., Smith, W., Ash, M., Ingalls, R., and Pettengill, G.: 1971, *Phys. Rev. Letters* **26**, 27.