STATIC CHARGED DUST DISTRIBUTIONS: SOURCES OF PURELY ELECTROMAGNETIC ORIGIN

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Abstract. The present work is a continuation of our earlier work on the charged dust sources of purely electromagnetic origin for static axisymmetric and static spherically-symmetric fields. Here we have extended the above work to the case of generalized static metric and have shown that a static charged dust distribution, irrespective of any symmetry conditions, can be only of purely electromagnetic origin. Incidentally, it follows from this result itself that the 'Weyl–Majumdar–Papapetrou' class of static charged dust sources, which form an important class of astrophysical systems, are also of purely electromagnetic origin.

1. Introduction

The term 'sources of purely electromagnetic origin' emanated from the discovery of the electron. It was believed by Lorentz that 'an extended electron' consists of 'pure charge and no matter'. Such a model like that of an 'extended electron' was called as the 'electromagnetic mass model', in the sense, that all its physical characteristics, viz., the mass, etc., are dependent only on the charge and vanish when this charge vanishes. However, all attempts (Feynman *et al.*, 1964; Rohrlich, 1965; Jackson, 1975) to obtain a mathematical model of such a source in classical electrodynamics, in special relativity or in quantum mechanics were found to be unsatisfactory in some respect or the other. Indeed, no conclusion could be arrived (rather the problem was given up) regarding the existence of such a model. Obviously, with a view to know whether the problem has a possible solution in the realm of general relativity, certain attempts (Tiwari *et al.*, 1984, 1986, 1991; Tiwari and Ray, 1991; Gautreau, 1985; Grøn, 1985, 1986a, b; Ponce de Leon, 1987a, b, 1988) (including one by Einstein (1923) himself) were made by some workers (a historical account of these attempts has been given in one of our papers (Tiwari *et al.*, 1984, 1986)) which also includes a few recent attempts (Tiwari *et al.*, 1991; Tiwari and Ray, 1991), wherein we have investigated axisymmetric and spherically-symmetric static charged dust sources. In these papers we have shown that there exist some static axisymmetric charged dust sources which are of purely electromagnetic origin whereas the static spherically-symmetric charged dust sources can be nothing but only of purely electromagnetic origin.

In the present paper, we have extended this work to the generalized static metric irrespective of any symmetry conditions. We have shown here that there do not exist any finite (bounded) static uncharged dust (irrespective of any symmetry) sources. In the absence of charge, the energy density, the effective gravitational mass, etc., become zero and the underlying space-time becomes flat. In another sense, the entire static charged dust source, irrespective of any symmetry, is purely of electromagnetic origin.

Incidentally, we have also shown that the Weyl–Majumdar–Papapetrou (Weyl, 1917; Majumdar, 1947; Papapetrou, 1947) class of static charged dust sources, which form a very important class of astrophysical systems, also are of purely electromagnetic origin.

2. Static Charged Dust Sources of Purely Electromagnetic Origin

The most general static line element, irrespective of any symmetry, is given by

$$ds^2 = g_{ij}\,dx^i\,dx^j + e^{\omega}(dt)^2 \quad (i, j = 1, 2, 3)\,, \tag{2.1}$$

where g_{ij} and ω are functions of only the space coordinates x^1, x^2, x^3 and are independent of time ($t = x^0$). The signature of the metric (2.1) is assumed to be $(+, -, -, -)$. We also assume that hereafter the greek and the latin indices will take the values 0, 1, 2, 3 and 1, 2, 3, respectively.

The Einstein–Maxwell field equations for charged dust are given by

$$R^{\mu}_{\nu} = -8\pi(T^{\mu}_{\nu} - \tfrac{1}{2}g^{\mu}_{\nu}T)\,, \tag{2.2}$$

$$(\sqrt{-g}\,F^{\mu\nu})_{,\nu} = 4\pi\sigma\sqrt{-g}\,J^{\mu} \tag{2.3}$$

and

$$F_{[\mu\nu,\sigma]} = 0\,, \tag{2.4}$$

where

$$T^{\mu}_{\nu} = \rho u^{\mu}u_{\nu} + (1/4\pi)\,[-F^{\mu\alpha}F_{\nu\alpha} + (\tfrac{1}{4})g^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta}]\,; \tag{2.5}$$

and in which ρ is the proper energy density; σ, the charge density; and u^{μ}, the 4-velocity of matter satisfying the relation

$$u^{\mu}u_{\mu} = 1\,. \tag{2.6}$$

The static condition implies that $u^1 = 0$ and $u^0 = (g_{00})^{1/2}$. The electromagnetic field tensor $F_{\mu\nu}$ and the current 4-vector J^{μ} are defined as

$$F_{\mu\nu} = A_{\mu,\,\nu} - A_{\nu,\,\mu}\,, \tag{2.7}$$

$$J^{\mu} = \sigma u^{\mu}\,, \tag{2.8}$$

where A_{μ} is the 4-potential.

Since there is no magnetic field and the field is static the only surviving component of the 4-potential A_{μ} is A_0, viz., $A_{\mu} = (\varphi, 0, 0, 0)$ where we assume that $A_0 = \varphi$. Thus the only surviving components of the field tensor $F_{\mu\nu}$ are given by

$$F_{0i} = \frac{\partial\varphi}{\partial x^i} = \varphi_i \quad (i = 1, 2, 3)\,. \tag{2.9}$$

Hereafter, subscripts after a variable will denote partial differentiation with respect to coordinates concerned. Furthermore, for the later convenience, to the set of Equations (2.2)–(2.5), we also add the equation $T^{\mu\nu}{}_{;\,\nu} = 0$: viz.,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = -\frac{\sigma}{\rho} F^\mu_\alpha u^\alpha , \tag{2.10}$$

which is derivable from (2.2) (using Equations (2.3)to (2.5)).

The field equations (2.2)–(2.4), corresponding to the metric (2.1), reduce to the form (cf. Synge, 1960, p. 340)

$$\bar{R}_{ij} - e^{-(\omega/2)}\omega_{:\,ij} = 2e^{-\omega}[\varphi_i\varphi_j - \tfrac{1}{2}g_{ij}\Delta_1\varphi] , \tag{2.11}$$

$$e^{\omega}\Delta_2\omega = 2[4\pi\rho - e^{-\omega}\Delta_1\varphi] , \tag{2.12}$$

$$(\sqrt{-g}\, e^{-\omega/2}g^{ik}\varphi_{,k})_{,i} = 4\pi\sigma\sqrt{-g} . \tag{2.13}$$

Similarly, Equation (2.10) takes the form

$$\rho g^{il}e^{-\omega/2}\omega_l = -\sigma g^{il}\varphi_l ; \tag{2.14}$$

where $\bar{R}_{ij}$ is the Ricci tensor corresponding to the 3-space part of the metric (2.1) and

$$\Delta_2\omega = g^{ab}\omega_{:\,ab} , \qquad \Delta_1\omega = g^{ab}\omega_a\omega_b , \qquad \Delta_1\varphi = g^{ab}\varphi_a\varphi_b . \tag{2.15}$$

A colon denotes covariant differentiation with respect to g_{ij}'s of the 3-space part only.

We now assume that in the interior of the charged dust distribution the charge density σ, the electrostatic potential φ, the matter density ρ, and the metric potential are finite, regular, and continuous functions of the coordinates. Accordingly, for a finite region, Equation (2.13) can also be expressed as volume integral over the finite region of the source of distribution. Over this finite region if σ, the charge density, vanishes then the corresponding electrostatic potential φ also will vanish. This, in view of (2.14), will lead to the following three possibilities: viz.,

(i) $\rho = 0 , \quad g^{il}\omega_l \neq 0 ,$

(ii) $\rho \neq 0 , \quad g^{il}\omega_l = 0 ,$

and

(iii) $\rho = 0 , \quad g^{il}\omega_l = 0 .$

In the first case, viz., when $\rho = 0$ ($g^{il}\omega_l \neq 0$), from (2.12), since $\varphi = 0$, we get $\Delta_2\omega = 0$. It may be verified that (Synge, 1960, p. 340)

$$\int_v \Delta_2\omega\, dv = (1/2)\int_v (G^0_0 - G^i_i)e^{\omega/2}\, dv =$$

$$= -(1/2)\int_v (T^0_0 - T^i_i)e^{\omega/2}\, dv . \tag{2.16}$$

In the interior of the region when $\rho = 0$ (σ and φ are already zero) the regularity demands (since $T_0^0 = 0$, $T_i^i = 0$) that ω must be a constant. This also follows directly from (2.12). Thus, Equation (2.11) gives $\overline{R}_{ij} = 0$. Since $\overline{R}_{ij}$ corresponds to the Ricci tensor of 3-space, the vanishing of $\overline{R}_{ij}$ implies that the Riemann curvature tensor of the 3-space also vanishes and, hence, the associated 3-space is flat. With ω and the metric potentials g_{ij} (of the 3-space) being constant the space-time (2.1) becomes flat. Similarly, when we consider the second case: viz., $\rho \neq 0$, $g^{il}\omega_l = 0$, we arrive at the same result ($g^{il}\omega_l = 0$ implies $\Delta_2\omega = 0$, which from (2.12), implies $\rho = 0$). The third possibility is quite trivial. Hence, in the absence of charge, the energy density ρ and the effective gravitational mass, etc., are zero and the underlying space-time becomes flat. Thus a static charged dust, in general, (irrespective of any symmetry) is purely of electromagnetic origin.

3. Weyl–Majumdar–Papapetrou (WMP) Class of Static Charged Dust Sources

Weyl (1917) obtained a class of solutions for source-free axisymmetric Einstein–Weyl fields wherein the metric potential g_{00} and the electrostatic potential φ are related through a functional relation. This study was later extended to the generalized static metric (2.1) by Majumdar (1947) and Papapetrou (1947) independently. It was shown by them that, if there exists any functional relation between the metric potential g_{00} and the electrostatic potential φ, it must be of the form

$$g_{00} = a^2 + b\varphi + \varphi^2 . \tag{3.1}$$

They have further shown that in the special case when (3.1) is expressible as a perfect square: viz.,

$$g_{00} = (a \pm \varphi)^2 , \tag{3.2}$$

then the source-free Einstein–Maxwell equations reduce to a single Laplace equation. This shows that there exists a class of solutions of the source-free Einstein–Maxwell field equations which can be generated from the Laplace equation: viz., from Newtonian potential. Solutions of this class, known as Weyl–Majumdar–Papapetrou (WMP) class of solutions, because of anticipating a very useful astrophysical role, have received a very serious attention and have thoroughly and critically been studied by several people. Hartle and Hawking (1972) have extensively studied the physical character of these solutions and have found that the solutions of WMP class corresponding to monopoles can be analytically extended and interpreted as a system of charged black holes in equilibrium under their gravitational and electrostatic forces. The characteristics of the black hole solutions of this class are quite distinct from the already known black-hole solutions (viz., Schwarzschild and Reissner–Nordström black holes).

The sources of this class (viz., the charged dust sources which produce these fields) are, therefore, equally important. Das (1962), in an attempt to study the sources of WMP class of solutions, has shown that if one assumes the relation (3.1) to be valid inside the static charged dust source and assumes further that the charge density σ and the mass density ρ are equal (in the Newtonian theory the equilibrium of a charged fluid is possible if $\rho = \pm \sigma$) then the relation (3.1) reduces to (3.2). Later, De and Raychaudhury (1968) have shown that for charged dust distribution to be in equilibrium, a functional relation between g_{00} and φ follows directly from the field equations. They have also shown that if one assumes that either the charge density bears a constant ratio to the matter density or if the surfaces of dust distribution be equipotential surfaces (having no hole or pocket of alien matter inside) then the equality of matter and charge densities as well as the relation (3.2) can be deduced directly from the field equations. The generalized static metric (2.1) in that case will reduce to the form (Majumdar, 1947)

$$ds^2 = -f^{-1}(dx^2 + dy^2 + dz^2) + f(dt)^2 , \tag{3.3}$$

and the set of Einstein–Maxwell field equations (2.2) to (2.5), for static charged dust sources, reduce to the single nonlinear Poisson equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho(1 + V)^3 , \tag{3.4}$$

where $f = (1 + V)^{-2}$.

A set of solutions, belonging to WMP class, have explicitly been discussed for a spherically-symmetric case by Bonnor (1960, 1965). A few solutions of this class have been given in our earlier papers (Tiwari *et al.*, 1991; Tiwari and Ray, 1991) also.

What we would like to add here is that, in view of the result obtained in the previous section, the Weyl–Majumdar–Papapetrou class of static charged dust sources (which form an important class of astrophysical systems) are nothing but the sources of purely electromagnetic origin.

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