NUMERICAL STUDY OF A RANDOM DYNAMICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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Abstract. The classical situation of dynamical systems with two degrees of freedom is completely modified by the introduction of a stochastic element which provides a slow dissolution of the zone of stability. Using a Fokker-Planck method it has been shown that the Gambler's ruin problem is a rather good approximation of the diffusion process. Furthermore the orbits exhibit a *C*-system character even during the diffusion process.

1. Introduction

The study of dynamical systems with n degrees of freedom can be reduced to the study of a $2n-2$ measure preserving mapping, using the method of surface of section (Hénon and Heiles, 1964). On the other hand a second important reason to study such a diffeomorphism is that the same phenomena and problems occur as in the qualitative theory of differential equations (Smale, 1967). For $n=2$ the 'surface of section' has two dimensions and the corresponding mapping T which maps an intersecting point into the next one is an area preserving mapping which can be easily represented. Numerical studies indicate that the set of points corresponding to a given trajectory, i.e. P_0. $P_1=T(P_0)$, $P_2=T(P_1)$,... $P_n=T(P_{n-1})=T^n(P_0)$..., sometimes lies on a one-dimensional manifold (invariant curve), and sometimes fills a two-dimensional region. Complicated intermediate structures are found (Hénon, 1969). However, such dynamical systems are idealizations of real problems. Hence, the introduction of stochastic perturbations is quite natural (Sinaï, 1970; Sulem and Frisch, 1972; Frisch *et al.*, 1973).

The present paper deals with an area preserving mapping but in which a stochastic element has been introduced. Hence, we have researched how the classical situation of dynamical systems with two degrees of freedom was modified. In Section 2 we give some properties of the mapping T and its features. In Section 3 we give graphical displays of the diffusion process. We define and estimate the diffusion time in Section 4 and in Section 5 its variations with the magnitude of the random term. In Section 6 we study the ergodic character of the diffusion using the variations of the eigenvalues of the linear tangential mapping as indicators of stochasticity.

2. The Mapping

We consider the mapping T_ε of the (x, y) plane over itself defined by

$$T_\varepsilon \begin{cases} x_1 = x_0 + a \sin (x_0 + y_0), \\ y_1 = x_0 + y_0 + \varepsilon. \end{cases} \quad (\text{mod } 2\Pi) \tag{1}$$

where ε is a constant. If we make the change of variables

$$\begin{cases} X = x + \varepsilon, \\ Y = y - \varepsilon, \end{cases}$$

we get in the new axes XY the mapping T_0 hence the topology of T_ε is the same as that of T_0 within a translation of vector $(-\varepsilon, \varepsilon)$.

Figures 1 and 2 display typical sets of points for the mapping T_0. The initial conditions and values of the parameter a are presented in Table I. N is the total number of points plotted for each orbit. Figure 1 exhibits all the characteristics and well-known features of problems with two degrees of freedom, i.e. invariant curves and islands which correspond to the existence of isolating integrals, and also wild zones, sometimes called 'ergodic' where the points seem to fill a broad region in the plane, and which correspond to the non-existence of isolating integrals. On the other hand,

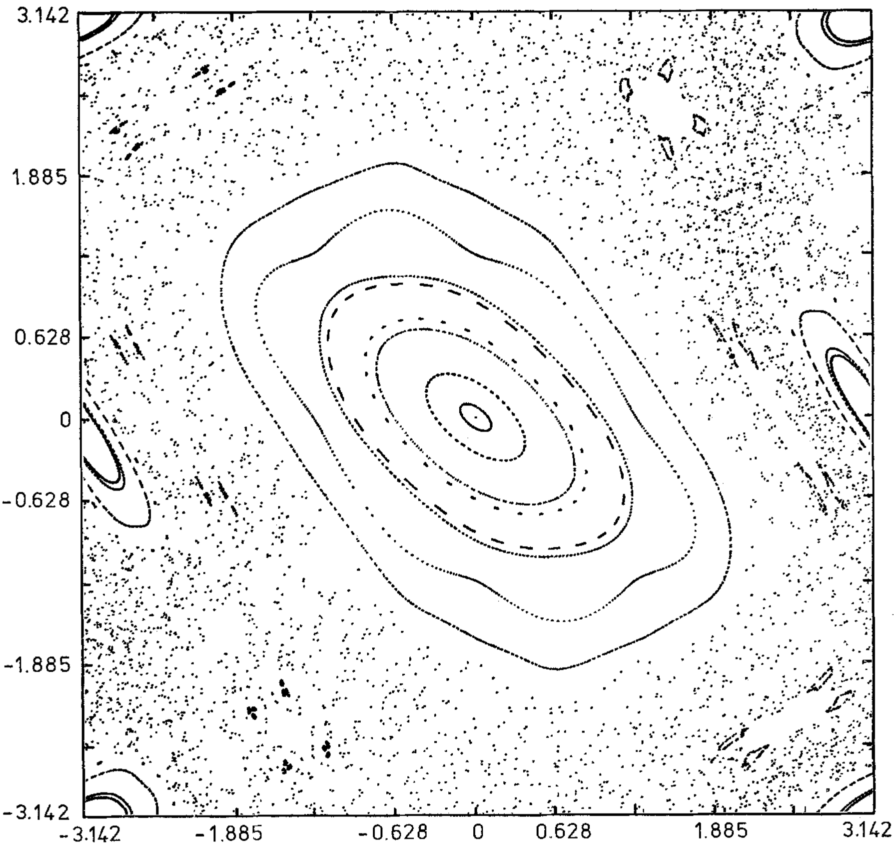

Fig. 1. The area preserving mapping T_0 for $a_1 = -1.3$.

TABLE I

Data for Figures 1 and 2

Fig.	a	x_0	y_0	N
1	−1.3	2.8274	−3.1416	700
		2.6000	−3.1416	700
		2.5133	−3.1416	700
		3.0000	0	700
		2.9845	0	700
		2.8274	0	1000
		2.5133	0	900
		2.1991	0	800
		1.8860	0	800
		1.5708	0	700
		1.2566	0	500
		1.0000	0	400
		0.9425	0	400
		0.7000	0	200
		0.6283	0	300
		0.3142	0	200
		0.1000	0	10
		0	0	3
		−2.0741	−1.7318	700
		−1.7880	−2.3114	700
		0.1969	−2.2867	700
2	−0.3	±3.1400	0	10
		±3.0000	0	700
		±2.8000	0	700
		±2.7000	0	1000
		±2.6500	0	400
		±2.6000	0	700
		+2.4000	0	700
		±2.2000	0	700
		±2.0000	0	700
		±1.8000	0	500
		±1.6000	0	500
		±1.4000	0	500
		±1.2000	0	500
		±1.0000	0	500
		±1.9500	0	500
		±0.8500	0	500
		±0.8000	0	500
		±0.6000	0	500
		±0.4000	0	500
		±0.2000	0	200
		0	0	1

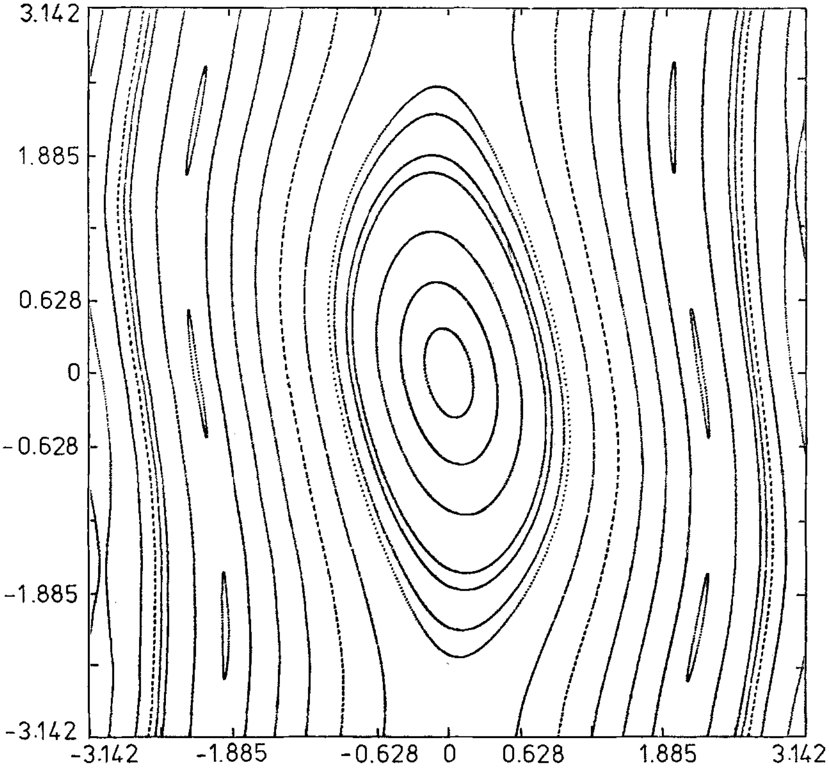

Fig. 2. The area preserving mapping T_0 for $a_1 = -0.3$.

on Figure 2, isolating integrals seem to exist everywhere: this is a case very close to an integrable case. All the points are either on *libration* curves (associated with the stable invariant point (0, 0)), or on *circulation* curves (associated with the unstable invariant point (Π, 0)).

In this article we study a random mapping $\mathscr{T}_\omega$ for which $\omega = (\varepsilon_1 \varepsilon_2 \ldots)$ is an infinite sequence where each of the ε_k can take the value ε' or ε with equal probabilities. Let $T_{\varepsilon'}$ and T_ε be two area preserving mappings defined by the Equations (1). Let

$$\mathscr{T}^n_\omega = T_{\varepsilon n} T_{\varepsilon n-1} \ldots T_{\varepsilon 1}, \tag{2}$$

denote the random product of n mappings $T_{\varepsilon'}$ or T_ε. This problem, apparently an academic one, has in fact connections with various physical problems: the study of a plane wave propagating in a semi-infinite random medium (Sulem and Frisch, 1972; Frisch *et al.*, 1973); the problem of an artificial satellite at a low altitude, etc., ...

3. Graphical Displays of the Diffusion Process

In order to study the fluctuations of the invariant curves, the points $P_n = \mathscr{T}^n_\omega(P_0)$ ($n = 1$ to N) have been plotted for various values of N. Figure 3 shows the results for

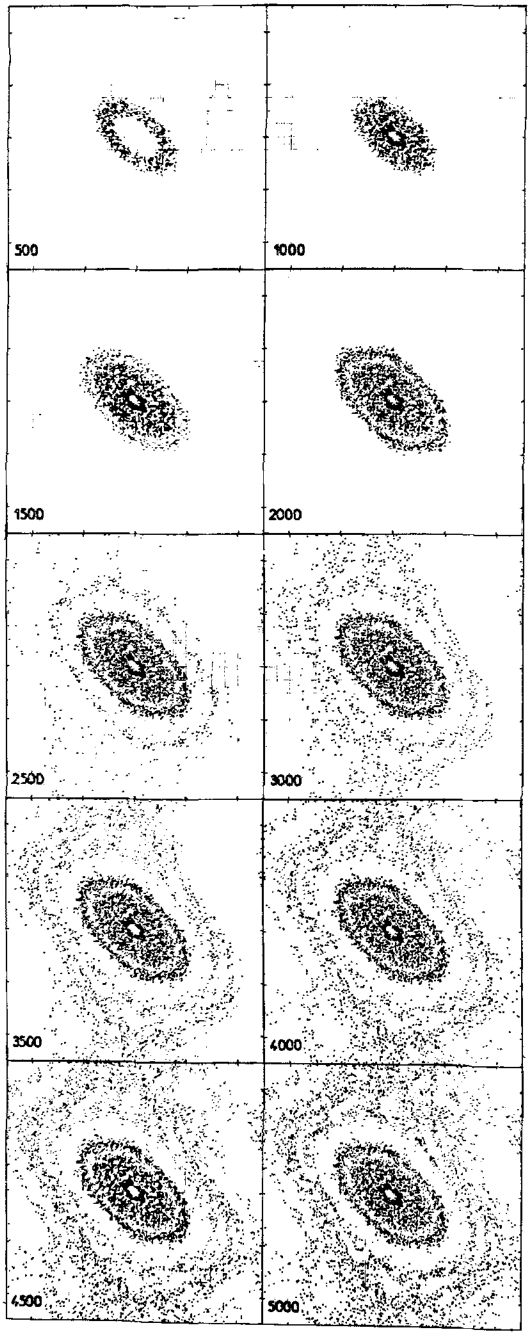

Fig. 3. An orbit of the random mapping $\mathscr{T}_\omega$ where $a_1=-1.3$, $x_0=0.4$, $y_0=0.4$, $\varepsilon'=0$, $\varepsilon=0.05$ for various values of N and a given realization of ω.

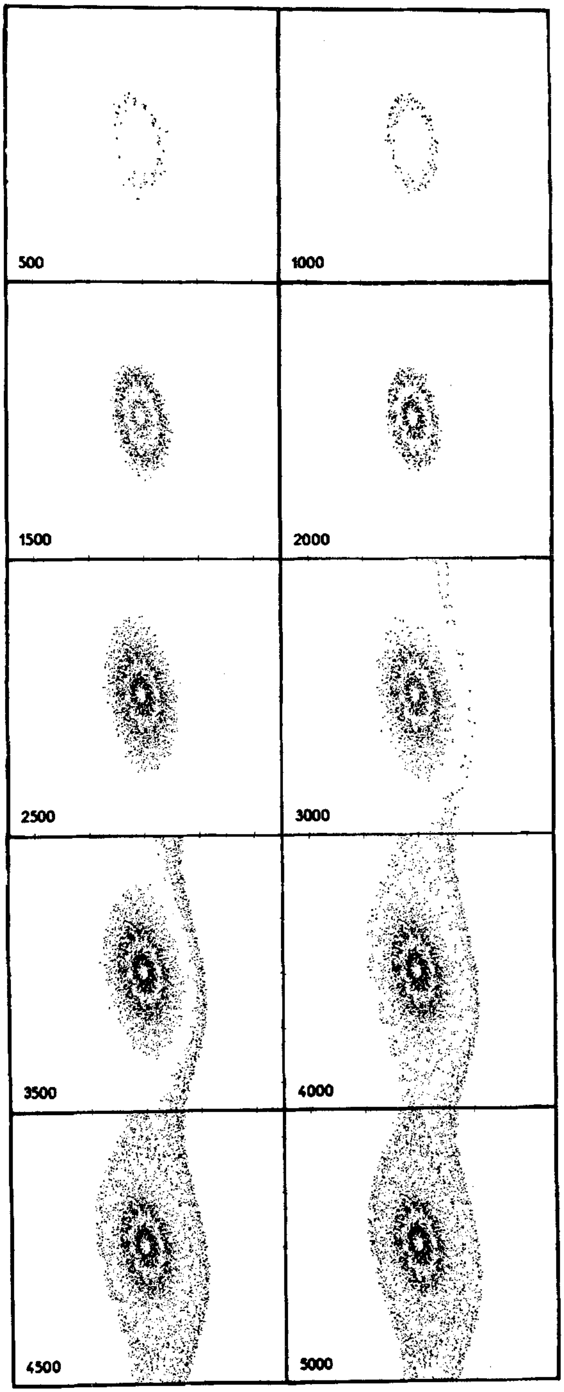

Fig. 4. An orbit for the same initial conditions as for Figure 3 but for $a_1 = -0.3$.

$a = -1.3$, $\varepsilon' = 0$, $\varepsilon = 0.05$ and the initial conditions $x_0 = 0.4$, $y_0 = 0.4$, which correspond to a zone of invariant curves for both $T_{\varepsilon'}$ and T_ε. For N less than 2000, the points lie on strips including the invariant curves. The width of these strips increases with N. For N greater than 2000, the points are scattered and 'ergodicity' appears. Figure 4 shows the same experiments for $a = -0.3$. Again the points lie on strips including the invariant curves surrounding the origin, but for N greater than 2000 the points still remain in a band along the y axis. The phenomenon of diffusion is much slower than in the preceding case. This is due to the topology of the invariant curves for $a = -0.3$. Indeed as we no longer have an ergodic zone but some invariant curves nearly parallel to the y axis, the effect of the perturbation which is also on y is strongly weakened.

4. Definition and Estimation of the Diffusion Time

In order to have some quantitative information about this random walk, a measure D_j of the dimension of the invariant curve has been computed here, as given by

$$D_j = \sum_{m=j-99}^{m=j} (x_m^2 - a(y_m^2 + x_m y_m))/100. \tag{3}$$

The quadratic term in this expression is constant, in the vicinity of the origin, in the linear approximation, in the case $\varepsilon' = \varepsilon = 0$.

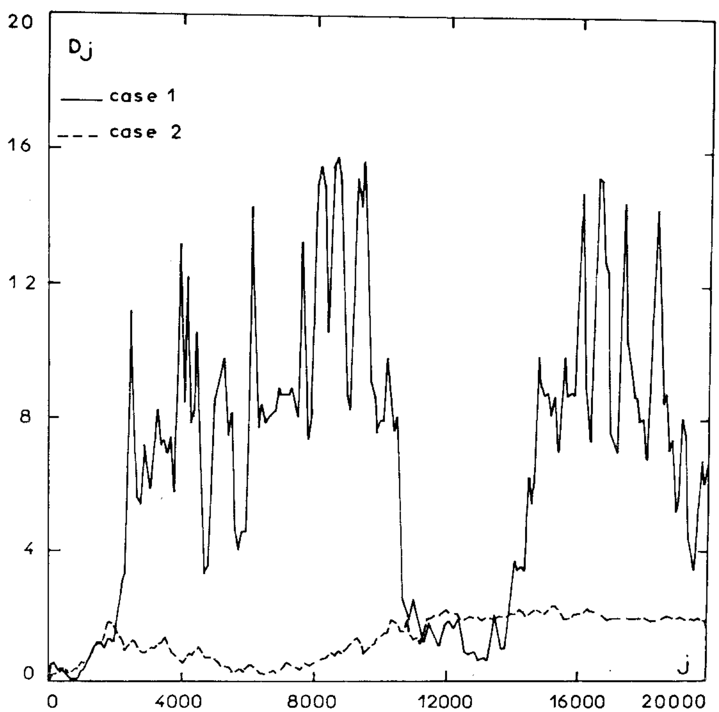

Fig. 5. D_j, a measure of the dimension of the curves, against j.

Figure 5 shows the variations of D_j for the same case as for Figures 3 (case 1) and 4 (case 2). The variations of D_j are quite large and in good agreement with the graphical results given by Figure 3. Indeed we observe, for the values of j, corresponding in Figures 3 and 4 to the scattering of the points, a sudden increase in the value of D_j. We call 'diffusion time' the number of iterations of the mapping $\mathcal{T}_\omega$ which are necessary for the points (x_n, y_n), starting in the integrable zone (or the libratory zone) of T_0, to reach the wild (ergodic) zone (or the circulatory zone). In other words, 'diffusion time' is the time necessary for the disappearance of isolating integrals in the case 1. For estimating this time, we use the sudden change in the value of the measure D_j. As soon as D_j is greater or equal to 4, we say that diffusion has occurred.

5. Variation of the Diffusion Time with the Random Term ε and with the Distance of the Initial Point to the Origin

We intend to estimate the diffusion time as a function of the magnitude of ε and as a function of the initial conditions.

We take the Gambler's ruin model (Feller, 1971) as an approximation of the diffusion process, since the problem is reduced to the study of the jumps of (x_n, y_n) from one elliptic curve to another, up to the ergodic zone, which is considered as an absorbing barrier.

Indeed, the family of invariant curves surrounding the origin can be taken, in a first order approximation, as a continuous elliptic family (cf. Figures 1 and 2).

We consider the family of similar ellipses of equations

$$C = x^2 - a(y^2 + xy), \tag{4}$$

where C is a constant which can be considered as a generalized distance of the points (x, y) to the origin. C takes all values from 0 to $C_{\max}$ corresponding to the largest ellipse, that is to say, the absorbing barrier. Let $N(C)$ be the expected number of iterations which are necessary for reaching the absorbing barrier when starting on the ellipse C, at the generalized distance C from the origin.

We have the fundamental equation

$$N(C) = 1 + \int N(C + \Delta C)\varrho(\Delta C)\,\mathrm{d}(\Delta C), \tag{5}$$

where $\varrho(\Delta C)$ is the probability to jump from an ellipse C to an ellipse $C+\Delta C$.

Expanding to second order (see the Fokker-Planck method), we get

$$N(C) = 1 + N(C) + \frac{\mathrm{d}N(C)}{\mathrm{d}C}\langle\Delta C\rangle + \frac{1}{2}\frac{\mathrm{d}^2N(C)}{\mathrm{d}C^2}\langle\Delta C^2\rangle, \tag{6}$$

we have

$$\Delta C = x^2 - a((y + V)^2 + x(y + V)) - x^2 + a(y^2 + xy), \tag{7}$$

$$\Delta C = -a(x + 2y)V - aV^2, \tag{8}$$

with V being a random variable taking the value 0 and ε with the probability 0.5. Furthermore the points (xy) are distributed on the ellipse C with a density proportional to $ds/\|\text{grad } H\|$, where

$$H(x, y) = x^2 - a(y^2 + xy). \tag{9}$$

In order to compute $\langle \Delta C \rangle$ and $\langle \Delta C^2 \rangle$ we take as new axes the principal axes of the ellipse C. After some algebra we obtain the equations

$$\begin{cases} x = X \cos \delta - Y \sin \delta, \\ y = X \sin \delta + Y \cos \delta, \end{cases} \tag{10}$$

where δ is given by the equations

$$\text{tg } \delta = \frac{1 + a + \sqrt{\Delta}}{a}, \tag{11}$$

$$\Delta = 2a^2 + 2a + 1. \tag{12}$$

Then Equations (4) become

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1, \tag{13}$$

with

$$A^2 = \frac{2C}{1 - a - \sqrt{\Delta}}, \tag{14}$$

$$B^2 = \frac{2C}{1 - a + \sqrt{\Delta}}. \tag{15}$$

Let φ be the eccentric anomaly of the ellipse C, such that

$$X = A \cos \varphi, \qquad Y = B \sin \varphi. \tag{16}$$

Then the density probability is uniform with respect to φ on $[0, 2\Pi]$ – i.e.,

$$P(\varphi) = \frac{1}{2\Pi} d\varphi. \tag{17}$$

From the Equations (8), (10), (17) we get immediately

$$\langle \Delta C \rangle = -\frac{1}{2} a\varepsilon^2. \tag{18}$$

Neglecting terms of order greater than 2 in ε, we have

$$\langle \Delta C^2 \rangle = \frac{1}{2} a^2 \varepsilon^2 \langle (x + 2y)^2 \rangle. \tag{19}$$

From (4) and (17) we get

$$\langle (x + 2y)^2 \rangle = -\frac{4C}{a} + \left(1 + \frac{4}{a}\right) \langle x^2 \rangle; \tag{20}$$

and from (10), (14), (15) and (17)

$$\langle x^2 \rangle = C\left(\frac{1 - a + \sqrt{\Delta}\cos 2\delta}{-a(a + 4)}\right). \tag{21}$$

Using (11) and (12) we obtain after some algebra

$$\langle x^2 \rangle = \frac{2C}{a + 4}; \tag{22}$$

hence, from (20),

$$\langle (x + 2y)^2 \rangle = -\frac{2C}{a}, \tag{23}$$

$$\langle \Delta C^2 \rangle = -a\varepsilon^2 C. \tag{24}$$

The fundamental Equation (6) becomes

$$-\frac{a\varepsilon^2}{2}\frac{\mathrm{d}N(C)}{\mathrm{d}C} - \frac{a\varepsilon^2 C}{2}\frac{\mathrm{d}^2 N(C)}{\mathrm{d}C^2} = -1. \tag{25}$$

Hence, it follows by integration that

$$\varepsilon^2 N(C) = -\frac{2}{a}(C_{\max} - C) + \varepsilon^2 K \log C, \tag{26}$$

where $C_{\max}$ and K are constants.

It is obvious that K is equal to zero to avoid the singularity at $C=0$ which is without any physical meaning.

Finally for small ε we have

$$\varepsilon^2 N(C) \cong -\frac{2}{a}(C_{\max} - C). \tag{27}$$

5.1. Variations with the Strength ε of Random Perturbations

Figure 6 shows the variation of $\varepsilon^2 N(C)$ plotted vs ε in the case $C=0$. In order to eliminate the fluctuations due to the fact that $N(C)$ depends on the sequence of the ε, we take an average of 25 experiments for a given value for ε.

Considering the crude approximations which have been made for obtaining Equation (1), the results are in rather good agreement with the Gambler's ruin model. From the numerical results, we get, taking the mean value of $\varepsilon^2 N(C)$ which is 10.37, the value 6.74 for $C_{\max}$. This value is in rather good agreement with the value we can estimate on Figure 1 taking into account the distortion along the Y axis of the invariant curves near the border of the ergodic zone.

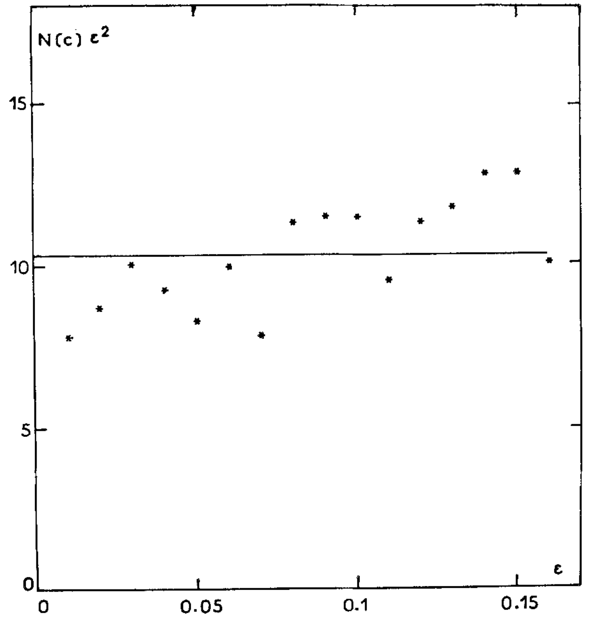

Fig. 6. Verification of the diffusion law $N(C)\varepsilon^2 \simeq$ constant for a given C.

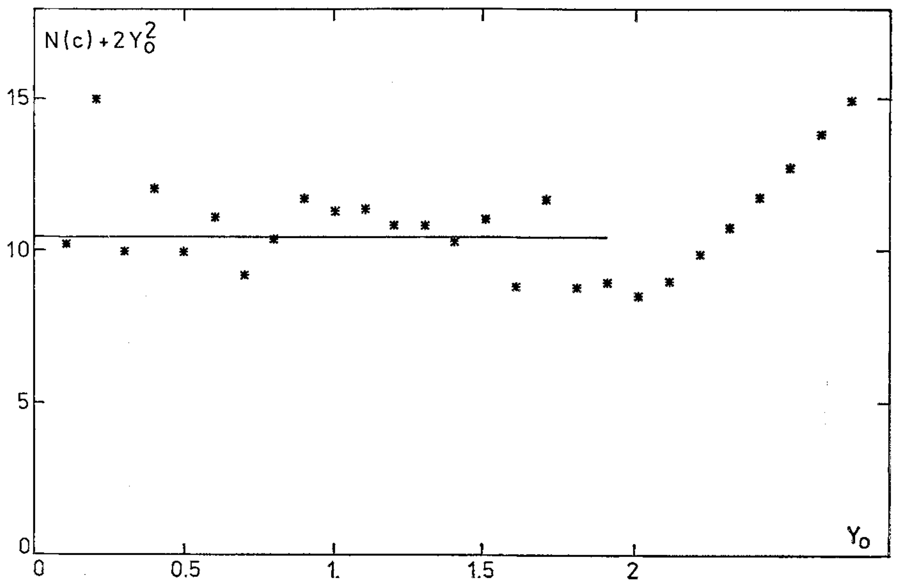

Fig. 7. Verification of the diffusion law $N(C)\varepsilon^2 + 2y_0^2 \simeq$ constant for $\varepsilon = 0.05$, $x_0 = 0$.

5.2. VARIATIONS WITH THE INITIAL CONDITIONS

We take a fixed value of the random term: $\varepsilon=0.05$, and the initial conditions $x_0=0$, y_0 being the parameter. From (27) we get

$$\varepsilon^2 N(C) + 2y_0^2 = -\frac{2}{a} C_{\max}. \tag{28}$$

Thus we plot the values of $\varepsilon^2 N(C)+2y_0^2$ vs y_0 on Figure 7. Each value of $N(C)$ is computed in the same way as previously (average of 25 random sequences).

The straight line represents the average of the values of $\varepsilon^2 N(C)+2y_0^2$ which is found to be equal to 10.44 and, hence, to be very close to 10.37 given previously. This average has been taken for $y_0 \leqslant 2$ because for $y_0 > 2$ the orbit starts immediately in the ergodic zone of T_0 and $\varepsilon^2 N(C)$ is equal to zero.

Also in this case the Gambler's ruin model seems to be a good approximation of the diffusion problem.

6. Stochasticity of the Mapping $\mathscr{T}$

In a previous paper (Froeschlé, 1970), the variation of the largest eigenvalue λ_1^n of the linear tangential mapping T_0^{n*} of the mapping T_0^n has been used as an indicator of stochasticity in the case of a dynamical system with two degrees of freedom. We shall use in the present paper the same indicator not for the mapping $\mathscr{T}$ itself but for a given realization $\mathscr{T}_\omega^n$, which will show us the character of 'C-system' of the mapping $\mathscr{T}$.

To compute the eigenvalues of the Jacobian matrix J which represents the linear tangential mapping $(\mathscr{T}_\omega^i)^*$ of $\mathscr{T}_\omega^i$ at P_0 (P_0 being the initial point) the composite mapping theorem has been used – i.e.,

$$\mathscr{T}_\omega^{i*}(P_0) = T_{\varepsilon_n}^*(\mathscr{T}_\omega^{i-1}(P_0))\, \mathscr{T}_\omega^{i-1*}(P_0).$$

As the elements of this matrix can sometimes exceed the largest number which can be stored by the computer we have used the following device. If $|\lambda_i|$ becomes larger than 10^2 we divide each term of the Jacobian matrix by 10^2 as many times as it is necessary and take

$$|\lambda_i| = 10^{2m}|\lambda_i'|,$$

where m is the number of times the elements are divided and λ_i' is the largest root of the new characteristic equation. We remark that the characteristic equation is reciprocal; therefore the second root is the inverse of the first one.

Figures 8 and 9 display the variations of $\log|\lambda_i|$ plotted vs i. We can see that a sudden change in the slopes of $\log|\lambda_i|$ occurs when the points P_n reach the ergodic zone. However, we note that the values of these slopes are always strictly positive. This means that the orbits have an ergodic behavior and that the dynamical system is close to a C-system, even when the diffusion process is still going on. The values of these slopes, which are characteristics of the orbits (Froeschlé and Scheidecker,

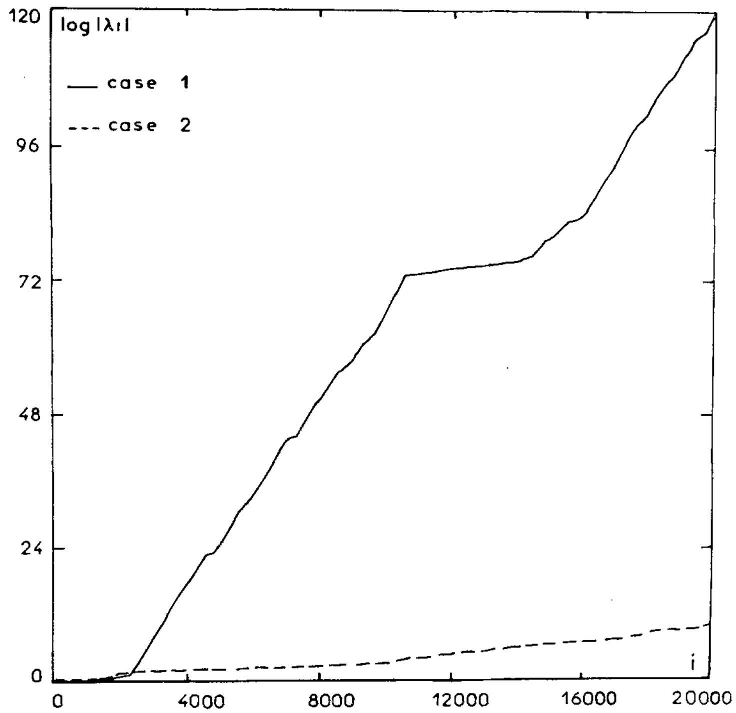

Fig. 8. Variations of $\log_{10}|\lambda_i|$ with i for the same orbits as for Figures 3 and 4.

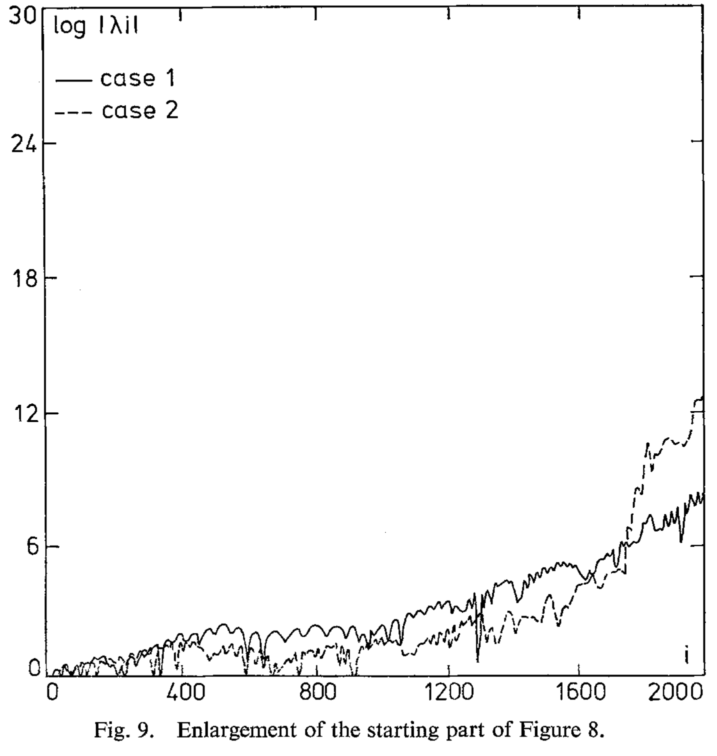

Fig. 9. Enlargement of the starting part of Figure 8.

1973a, b, and c) and which change suddenly, are related to the topological structure of the two-dimensional mappings displayed in Figures 1 and 2.

7. Conclusion

The results obtained show that the classical situation of dynamical systems with two degrees of freedom is completely modified by the introduction of a stochastic element which provides a slow dissolution of the zone of stability.

A characteristic time of dissolution has been defined and it has been shown that the Gambler's ruin problem is a rather good approximation of the diffusion process.

Furthermore the C-system character of the orbits has been shown to appear even during the diffusion process.

Finally, the results given by Froeschlé (1971) and Froeschlé and Schiedecker (1973a, b, and c) concerning dissolutions of isolating integrals for systems with three degrees of freedom but without any stochastic parameters, appear to be very similar to those given in this paper. It confirms the stochastic behavior of orbits during the disappearance of the isolating integrals. However, we note that in opposition to deterministic systems, no well behaved orbits exist for random dynamical systems.

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