ON THE UNDERSTANDING OF THE OBSERVED FLAT OR SLOWLY RISING ROTATION CURVES IN LARGE DISK GALAXIES

MITA GHOSH and B. BASU

Department of Applied Mathematics, University of Calcutta, India

and

TARA BHATTACHARYYA

Department of Mathematics, Jogamaya Devi College, Calcutta, India

(Received 17 August, 1988)

Abstract. Recent observations of the rotation curves of large disk galaxies of all Hubble-types have shown that they possess flat or slowly rising rotation curves up to large distances from the centre. It has been suggested here that such rotation curves are understood under normal fluid dynamical considerations provided that viscous (and/or magnetic) transfer of mass and angular momentum from inner to outer regions of these galaxies is efficient. Flow of gas from halo to the disk in regions close to the axis of rotation is also suggested. The existence of rising rotation curves in some galaxies with varying gradients and flat rotation curves in others suggest that probably these galaxies are not coeval. The formers are probably of more recent origin.

1. Introduction

The attempts to determine the rotation curves of disk galaxies and calculate the masses of these galaxies thereby date back to the late fifties and early sixties. The pioneering works were done by Burbidge *et al.* (1959, 1960, 1962, 1963, 1964) and Rubin *et al.* (1965). These curves were found to rise linearly from the centre, attain peaks at distances of several kiloparsecs from the centre and then fall. At large distances from the centre, these curves were believed to obey the Keplerian law indicating that most of the mass of the Galaxy was concentrated in the central bulge region.

Roberts and Rots (1973) first observed a completely different type of rotation curves for some galaxies (e.g., M31, M101, M81). These curves rise linearly from the centre up to some distances, reach maximum and then remain flat up to large distances (50 kpc or more) from the centre. Similar type of rotation curves were observed by Krumm and Salpeter (1976, 1977). In more recent years Rubin *et al.* (1978a, b), undertaking a massive programme, observed the rotation curves of a large number of more distant and bright galaxies belonging to all Hubble classes. All these curves are seen either to be flat or to rise slowly up to very large distances from the centres of these galaxies.

The flat rotation curve indicates a very widespread mass distribution around the disks of these galaxies thus suggesting the existence of massive, widespread halos around them. The physical state of this nonluminous halo mass is, however, not known. It has been suggested that this mass may be in the form of black holes or dead stars like white dwarfs (black dwarfs, in fact) and faint Population III stars (Rubin *et al.*, 1982).

Astrophysics and Space Science **152** (1989) 131–139.

The present paper deals with a mathematical model which simulates the observed flat or slowly rising rotation curves in disk galaxies. We assume that close to the axis of rotation of the Galaxy, a flow of gas from halo to the disk is present. We have taken the Navier–Stokes equations in cylindrical coordinates and considered the presence of viscosity in the fluid. A plausible density law has also been used. The solution to the equations under the conditions stated above gives flat or slowly rising rotation curves. Whether a galaxy will give a flat or a slowly rising rotation curve depends possibly on its age since its disk was formed. Galaxies with flat rotation curves are probably of earlier origin than those possessing slowly rising rotation curves.

In Section 2, the mathematical formulation of the work has been developed. Section 3 deals with the solution of equations to derive the formula for the rotation curve in the galactic disk. The nature of the rotational motion of the gas in the halo region is derived in Section 4. Section 5 contains the discussion of the transfer of angular momentum from the central to the outer parts of the Galaxy. The summary of the entire work along with the necessary discussions are given in Section 6.

2. Mathematical Formulation of the Problem

We consider here the motion of gas in the Galaxy and try to derive the form of the rotational curve in the presence of viscosity in the gas. Considering the cylindrical coordinates with (r, θ) plane as the plane of the disk of the Galaxy and the Z-axis as its axis of rotational symmetry, the Navier–Stokes equations for viscous compressible fluid are:

$$\frac{\partial u}{\partial t} + \left(u \frac{\partial}{\partial r} + w \frac{\partial}{\partial Z}\right)u - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left[\frac{4}{3}\frac{\partial \chi}{\partial r} + \frac{\partial}{\partial Z}\left(\frac{\partial u}{\partial Z} - \frac{\partial w}{\partial r}\right)\right] + F_r\,, \tag{1}$$

$$\frac{\partial v}{\partial t} + \left(u \frac{\partial}{\partial r} + w \frac{\partial}{\partial Z}\right)v + \frac{uv}{r} = \frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial Z^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right], \tag{2}$$

$$\frac{\partial w}{\partial t} + \left(u \frac{\partial}{\partial r} + w \frac{\partial}{\partial Z}\right)w = -\frac{1}{\rho}\frac{\partial p}{\partial Z} + \frac{\mu}{\rho}\left[\frac{4}{3}\frac{\partial \chi}{\partial Z} + \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial Z}\right) + \frac{1}{r}\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial Z}\right)\right] + F_Z\,; \tag{3}$$

μ being the dynamical coefficient of viscosity, and we have taken $\mathbf{V} \equiv (u, v, w)$, where u and v are the radial and cross-radial velocities parallel to the plane of the disk and

w is the velocity parallel to the axis of rotation. Due to the rotational symmetry, we have taken $\partial/\partial\theta = 0$ for any variable. Also we have used

$$\chi = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial Z} + \frac{u}{r} . \tag{4}$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\rho u) + \frac{\partial}{\partial Z}(r\rho w)\right] = 0 . \tag{5}$$

The density distribution of the gas is given by

$$\rho \sim R^{-n} \quad \text{where} \quad R^2 = r^2 + Z^2 , \quad n > 0 . \tag{6}$$

The equation of state of the gas is

$$p = \frac{R' T \rho}{\mu'} , \tag{7}$$

where p is the gas pressure; T its temperature; R', the universal gas constant; and μ', the mean molecular weight of the gas. F_r and F_Z are the components of gravitational force in r and Z directions, respectively, and are given by

$$F_r = -\frac{GM(R)}{r^2 + Z^2} \frac{r}{\sqrt{r^2 + Z^2}} ,$$

$$F_Z = -\frac{GM(R)}{r^2 + Z^2} \frac{Z}{\sqrt{r^2 + Z^2}} ,$$

where G is the universal gravitational constant and $M(R)$ is the mass of the sphere of radius $R = \sqrt{r^2 + Z^2}$. The density can be written as $\rho = AR^{-n}$, where the constant A will be given by

$$M(R) = \int_0^R 4\pi R^2 \rho \, \mathrm{d}R = \frac{4\pi A R^{3-n}}{3-n} .$$

Since $M(R)$ increases with R, but $M(R)/R^2$ decreases as R increases, we must have $1 < n < 3$ and we now have

$$F_R = -\frac{4\pi A G}{3-n} \frac{r}{(r^2 + Z^2)^{n/2}} , \tag{8}$$

$$F_Z = -\frac{4\pi A G}{3-n} \frac{Z}{(r^2 + Z^2)^{n/2}} . \tag{9}$$

3. Gas Motion in the Disk of the Galaxy

We first solve the equations of motion on the disk of the galaxy where $Z = 0$. In this case, $R = r$ also. Let us assume that $u = B(r)$ at $z = 0$. At any (r, Z), let $w = -Zf(r)$; then $w = 0$, and $\partial w/\partial Z \neq 0$ when $Z = 0$. We assume that the mass moves from the halo to the disk. Then $w \gtrless 0$ according as $Z \lessgtr 0$. Considering the steady motion on the disk and using the density law given by (6), the equation of continuity (5) becomes

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = \frac{n}{r}\, u\,.$$

Substituting the values for u and w, we get

$$\frac{\mathrm{d}B(r)}{\mathrm{d}r} + \frac{1-n}{r}\, B(r) = f(r)\,. \tag{10}$$

The integration yields

$$\frac{B(r)}{r^{n-1}} = \int \frac{f(r)}{r^{n-1}}\, \mathrm{d}r + \text{constant}\,. \tag{11}$$

Since u decreases as r increases and finally tends to zero, $f(r)$ can be assumed to be a slowly decreasing function of r and so, the integration constant in (11) ultimately tends to zero. Let us take $f(r) = Kr^{-m}$, $m > 0$. Then Equation (11) yields

$$u = B(r) = \frac{Kr^{1-m}}{2-m-n}\,, \tag{12}$$

where we must have $m > 1$ and $2 - m - n \neq 0$. Thus we have,

$$\left.\begin{aligned} \omega &\sim Zf(r) \\ u &\sim rf(r) \end{aligned}\right\} \quad \text{where} \quad f(r) \sim r^{-m}\,. \tag{13}$$

In the plane $Z = 0$, Equation (1) becomes

$$u\,\frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{4\mu}{3\rho}\frac{\partial \chi}{\partial r} + F_r\,. \tag{14}$$

Substituting in Equation (14) from (4), (6), (7), and (8), we get

$$v^2 = \frac{K^2(1-m)}{(2-m-n)^2}\, r^{2-2m} + \frac{4\pi AG}{3-n}\, r^{2-n} + \\ + \frac{4\mu}{3A}\,\frac{Kmn}{2-m-n}\, r^{n-m} - \frac{nR'T}{\mu'}\,, \tag{15}$$

where A represents the proportionality constant in the density law (6). Also, using

Equations (2), (6), and (14), we get

$$r^2 \frac{d^2v}{dr^2} + r\frac{dv}{dr} - v - Pr^{-2}\frac{d}{dr}(rv) = 0\,, \tag{16}$$

where

$$L = m + n - 2\,, \qquad P = -\frac{AK}{\mu L}\,. \tag{17}$$

Setting $rv = x$, we find that Equation (16) becomes

$$\frac{d^2x}{dr^2} = \frac{1}{r}\frac{dx}{dr}(1 + Pr^{-1})\,,$$

which yields, on solution

$$x = rv = C\int r\,e^{-(P/L)r^{-L}}\,dr + D\,; \tag{18}$$

where the integration constants $C \equiv C(t)$ and $D \equiv D(t)$ are independent of r, but may be functions of the time t. For large values of r, $r^{-L} \to 0$ when $L > 0$ and $e^{-(P/L)r^{-L}} \to 1$, and $v \to \frac{1}{2}C(t)r$. For very small values of r, $e^{-(P/L)r^{-L}} \to 0$ *if* $P/L > 0$. So there is no singularity at the centre if $P > 0$ that is, if $K < 0$, which implies that the radial flow is outward. Also, when $r \to 0$, $r\,e^{-(P/L)r^{-L}} \to 0$, whence Equation (18) yields $D = 0$.

We now consider the nature of v for large values of r. As $(P/L)r^{-L} \ll 1$, expanding $e^{-(P/L)r^{-L}}$ in series, we get

$$v = \frac{C}{r}\int r\left(1 - \frac{P}{L}r^{-L} + \frac{1}{2}\frac{P^2}{L^2}r^{-2L} - \frac{1}{6}\frac{P^3}{L^3}r^{-3L} + \cdots\right)dr =$$

$$= Cr\left[\frac{1}{2} - \frac{P}{L}\frac{r^{-L}}{2-L} + \frac{1}{2}\frac{P^2}{L^2}\frac{r^{-2L}}{2-2L} + \cdots\right]. \tag{19}$$

The series on the right-hand side of Equation (19) is an alternating series with monotonically decreasing terms and is, therefore, convergent even for large values of r.

Equations (15) and (19) give similar forms of the rotational velocity v at large distances from the galactic centre provided the parameters m and n are suitably chosen under the constraints $1 < n < 3$, $m > 1$ and $m + n - 2 > 0$. These constraints imply that both the non-rotational velocity and the gas density will decrease as the distance from the centre increases, and they ultimately die away.

From Equation (19) we see that for any values of m and n lying in the prescribed range, v tends to $\frac{1}{2}C(t)r$ for large values of r. From Equations (17) and (18) we see that $C(t)$ is of the order of t^{-1}. Hence, for older galaxies (large t) C becomes smaller and smaller. So the increase of v with r may be checked with decreasing C ultimately adjusting with a constant value of v. Physically, this means that the galaxies having flat rotation curves at large distances from the centre may be older than those having slowly rising rotation curves at these distances. This can possibly be verified by observing the oxygen abundance gradient or, in general, the chemical abundance gradient in these two types of galaxies. In this connection it may also be conjectured that spiral arms develop in disk galaxies after they have attained sufficient age.

It is also found from Equation (15) that when $n = m \simeq 2$, v tends to constant at large values or r,and

$$v \sim 4\pi AG + \frac{4\mu}{3A}\,\frac{Kmn}{2 - m - n}\,, \tag{20}$$

the value of the last term in Equation (15) being negligible in comparison with other terms. If we assume that at the periphery of the Galaxy ($r \sim 50$ kpc or more) the density of gas is almost the same as that in the intergalactic medium ($\rho \sim 10^{-2}$ H atoms cm^{-3}) and $u \simeq 10$ km s^{-1} then we get $A \simeq 3.75 \times 10^{20}$ g cm^{-1} and $K/(2 - m - n) \simeq 1.5 \times 10^{29}$ cm^2 s^{-1}. The value of the dynamical coefficient of viscosity μ for non-ionized hydrogen is given by

$$\mu = 5.7 \times 10^{-5}\, T^{1/2} \text{ g cm}^{-1}\text{ s}^{-1}$$

(Lang, 1978). The value $T = 100$ K yields $\mu = 5.7 \times 10^{-4}$ g cm^{-1} s^{-1}. Substitution in (20) yields $v \sim 173$ km s^{-1}. This value is in fairly good agreement with the observed values in galaxies with flat rotation curves at large distances from the centre (e.g., Rubin *et al.*, 1978); 211 km s^{-1} at 34 kpc for NGC 2998, 208 km s^{-1} at 17.6 kpc for NGC 3672, etc.). The second term on the right-hand side of Equation (20) is found to be small compared with the first term. This implies that, at large distances from the galactic centre, the viscosity of gas has little effects on the magnitude of the rotational velocity. But its influence on the form of the rotation curve there is important. The flat or slowly rising rotation curve at large distances from the centre implies that the mass and angular momentum of gas must have been transferred from the central region of the Galaxy to the outer disk by the action of viscous forces. This is discussed in Section 5.

4. The Gas Motion in the Halo

Let us now investigate into the nature of the motion of gas in the halo region. We consider the motion at the point (r, z). The velocity components parallel and perpendicular to the galactic plane are given, respectively, by

$$u_{r,Z} = r\,\frac{v_R}{\sqrt{r^2 + Z^2}}\,, \qquad w_{r,Z} = Z\,\frac{v_R}{\sqrt{r^2 + Z^2}}\,,$$

where v_R is the velocity at a distance R from the centre of the Galaxy along the direction joining the point (r, Z) to the centre. The value of v_R is obtained from Equation (12) by replacing r by $R = \sqrt{r^2 + Z^2}$. Thus,

$$v_R = \frac{K}{2 - m - n} (r^2 + Z^2)^{(1-m)/2} ;$$

so that

$$u_{r,Z} = \frac{K}{2 - m - n} r(r^2 + Z^2)^{-m/2}$$

and

$$w_{r,Z} = \frac{K}{2 - m - n} Z(r^2 + Z^2)^{-m/2} .$$

Substituting these values of $u_{r,Z}$ and $w_{r,Z}$ and also the values of ρ, p, and F_r from Equations (6), (7), and (8), respectively, in Equation (1), we get

$$\begin{aligned} v_{r,Z}^2 = {} & L_1^2(1 - m)r^2(r^2 + Z^2)^{-m} - \frac{nR'T}{\mu'} \times \\ & \times \frac{r^2}{r^2 + Z^2} + \frac{4\mu m L_1}{3A} (3 - m)r^2(r^2 + Z^2)^{[(n-m)/2]-1} + \\ & + \frac{4\pi AG}{3 - n} \frac{r^2}{(r^2 + Z^2)^{n/2}} , \end{aligned} \tag{21}$$

where

$$L_1 = \frac{K}{2 - m - n} .$$

Substitution of $Z = 0$ in Equation (21) yields the value of v^2 on the disk and the result coincides with that already obtianed in Equation (15).

For large values of R, two cases are of particular importance: (i) r is finite and Z tends to infinity, and (ii) Z is finite and r tends to infinity. In the first case the rotational velocity tends to zero and in the second case the rotational velocity is constant if $m = n = 2$. This implies that at sufficiently large distance from the rotation axis, the gas in the inner halo moves in a similar fashion to that in the disk producing a constant rotational velocity there, the values of the rotational smoothly decreasing with increasing values of Z.

5. Transfer of Angular Momentum

The law of conservation of angular momentum gives vr = constant. This means that v is inversely proportional to r, which contradicts the observed flat rotation curve. So the angular momentum I does not remain conserved. The angular momentum must be transported by some agent.

Now, let us consider the fact that the rate of change of angular momentum about the centre is equal to the moment of external forces about it. Since both the gravitational pull and the centrifugal forces pass through the centre, there should exist some tangential force so that the resultant moment about the centre does not vanish. Viscous force and/or azimuthal component of magnetic field may serve this purpose. Here the viscosity of gas in the Galaxy has been taken as an agent for transfer of angular momentum outwards.

Thus the rate of transfer of angular momentum at any point is equal to the tangential component of viscous force at that point multiplied by the distance of its line of action from the galactic centre. This is equal to

$$\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial Z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\, v\right)\right] r = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial Z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\, v\right)\right] r\,. \tag{22}$$

When $v = \text{const.}$ as in Equation (20), this expression reduces to $-\nu/r$:

$$\therefore \frac{\mathrm{d}I}{\mathrm{d}r} = \frac{\mathrm{d}I}{\mathrm{d}t}\bigg/\frac{\mathrm{d}r}{\mathrm{d}t} = \text{constant}\,.$$

Since $\mathrm{d}r/\mathrm{d}t \sim r^{-1}$, that is the rate of radial transport of angular momentum is constant.

Again, substituting the expression for v from Equation (19), we get

$$\frac{\mathrm{d}I}{\mathrm{d}r} = C(t)r\,;$$

i.e., the rate of radial transport of angular momentum increases with r.

As $C \sim t^{-1}$, the transfer rate decreases with time. As the time passes, activity of the nucleus of the galaxy becomes feeble, so the rate of flow of gas from nucleus to the outer region decreases and consequently the rate of transfer of angular momentum decreases.

6. Summary

The work contains a mathematical model showing the nature of rotational velocity of galactic gas at a large distance from the centre. Also the transfer of angular momentum has been discussed here.

We assume the spherical distribution of galactic mass and solve the Navier–Stokes equations for compressible fluid in the cylindrical coordinates. Thus we obtain expressions for rotational and non-rotational velocities of galactic gas in the galactic disk as well as in the halo region. It has been shown that at a large distance from the centre the rotational velocity of gas in the galactic disk either rises very slowly or remains almost constant. Our mathematical model is in agreement with the recently observed flat or slowly rising rotation curves for spiral galaxies. The same type of rotation curves has been derived mathematically for the gas in the galactic halo.

Acknowledgement

One of us (M.G.) expresses her sincere thanks to the Indian Space Research Organisation for facilitating the research work by awarding her a Junior Research Fellowship.

References

Burbidge, E. M., Burbidge, G. R., and Prendergast, K. H.: 1959, *Astrophys. J.* **130**, 739.
Burbidge, E. M., Burbidge, G. R., and Prendergast, K. H.: 1960, *Astrophys. J.* **132**, 640.
Burbidge, E. M., Burbidge, G. R., and Prendergast, K. H.: 1962, *Astrophys. J.* **136**, 339.
Burbidge, E. M., Burbidge, G. R., and Prendergast, K. H.: 1963, *Astrophys. J.* **138**, 375.
Burbidge, E. M., Burbidge, G. R.: Crampin, D. J., Rubin, V. C., and Prendergast, K. H.: 1964, *Astrophys. J.* **139**, 1058.
Krumm, N. and Salpeter, E. E.: 1976, *Astrophys. J.* **208**, L7.
Krumm, N. and Salpeter, E. E.: 1977, *Astron. Astrophys.* **56**, 465.
Lang, K. R.: 1978, *Astrophysical Formulae*, Springer-Verlag, New York, p. 226.
Roberts, M. S. and Rots, A. H.: 1973, *Astron. Astrophys.* **26**, 483.
Rubin, V. C., Burbidge, E. M., Burbidge, G. R., and Prendergast, K. H.: 1965, *Astrophys. J.* **141**(II), 885.
Rubin, V. C., Kent Ford, W., Jr., Strom, K. M., Strom, S. E., and Romanishin, W.: 1978a, *Astrophys. J.* **224**, 782.
Rubin, V. C., Kent Ford, W., Jr., and Thonnard, N.: 1978b, *Astrophys. J.* **225**, L107.
Rubin, V. C., Thonnard, N., Kent Ford, W., Jr.: 1982, *Astron. J.* **87**, 477.