IMPERFECT FLUID FRIEDMANNIAN COSMOLOGY

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Abstract. The bulk viscosity is introduced into the frame of ordinary Friedmannian cosmology (under highly idealized assumption of the constant coefficient of bulk viscosity). Explicit solutions are given for the viscous flat universe filled with the dust-substratum and for the viscous radiative universe. The problem, how does the introduction of viscosity affect the appearance of singularity, is briefly discussed.

1. Introduction

The role of the bulk viscosity in the cosmic evolution – especially at its early states – seems to be significant. From the macroscopic point of view the existence of bulk viscosity is equivalent to the existence of slow processes of restoring equilibrium states (Landau and Lifshitz, 1959). The general criterion for vanishing bulk viscosity was given by Weinberg (1971). It appears that vanishing bulk viscosity for a general imperfect fluid is rather an exception than a rule. As stressed by Anderson (1969) and Israel and Vardalas (1970), the bulk viscosity does not vanish for a simple gas at the temperatures between the extreme relativistic and non-relativistic limits. Weinberg (1971) also points out that bulk viscosity may be of importance when considering a fluid composed of a mixture of highly relativistic and non-relativistic particles.

One of us (Klimek, 1971) has paid attention to a different, let us say theoretical, role of viscosity in cosmology. As it is well known, in many hydrodynamical questions singular points appear with density and pressure of considered fluid tending to infinity. A very effective way of avoiding these kinds of singularities consists in introducing a viscosity term into hydrodynamical equations. A one-dimensional model of such processes was considered by Hopf (1950). One may also introduce the so-called generalized solutions in which singular points are contained (Rożdenstwienski and Yanienko, 1968). Olejnik (1957) has shown that under certain assumptions the generalized solutions may be obtained by the introduction of a viscous term and then by tending with it to zero. In such a case the viscous term itself may be considered as a mathematical trick only. The analogy between hydrodynamics and perfect fluid cosmology is evident. It is interesting to find out how do the cosmological solutions of field equations of General Relativity behave after introducing viscosity into them. Will the initial singularity in cosmology remain when the Friedmann equation is enlarged by a viscosity term?

In the present paper we introduce the viscosity term into the frame of Friedmannian

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cosmology. For the sake of simplicity we assume the coefficient of bulk viscosity, ζ, equal to a constant (the shear viscosity term vanishes automatically because of isotropy). It is of course a great simplification of physical reality. This coefficient is *de facto* a function of cosmic time (through the dependence on temperature and pressure); we may suppose, however, that our actual world is described by a series of different models (introduced by us) with different values of the coefficient ζ. We discuss also the question, how does the introduction of viscosity affect the singularity problem in Friedmannian cosmology.

2. General Formulae

In presenting the general formalism for the dissipative processes within the frame of Relativistic Cosmology we follow the approach used by Weinberg (1971). The energy-momentum tensor for an imperfect fluid has the form

$$T^{\mu\nu} = (c^2\varrho + p)\, u^\mu u^\nu - p g^{\mu\nu} + \eta H^{\mu\varkappa} H^{\nu\lambda} (u_{\varkappa;\lambda} + u_{\lambda;\varkappa} - \tfrac{2}{3} g_{\lambda\varkappa} u^{\sigma}_{;\sigma}) + \\ + \zeta H^{\mu\nu} u^{\lambda}_{;\lambda}, \tag{2.1}$$

where $H^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and η, ζ are, respectively, the coefficients of shear and bulk viscosity, other symbols have their usual meanings.

Applying (2.1) to the Robertson-Walker metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - R^2(t)\,\frac{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2}{1 + kr^2/4} \tag{2.2}$$

we obtain

$$T^0_0 = c^2\varrho\,, \\ T^i_k = \begin{cases} -p + 3\zeta\,\dfrac{\dot R}{R} & \text{for} \quad i = k\,, \\ 0 & \text{for} \quad i \neq k\,, \end{cases} \tag{2.3}$$

i, k run from 1 to 3. The shear viscosity disappears on the strength of isotropy. The only effect of dissipative processes is to add a $3\zeta \dot R/R$-term to the macroscopic pressure p.

The field equations with the above forms of metric and energy-momentum tensor are:

$$\varkappa\varrho c^2 = -\Lambda + 3\,\frac{kc^2 + \dot R^2}{c^2 R^2}, \tag{2.4a}$$

$$\varkappa p = \Lambda - \frac{2R\ddot R + \dot R^2 + kc^2}{c^2 R^2} + 2\alpha\,\frac{\dot R}{R}, \tag{2.4b}$$

where $\alpha = \frac{3}{2}\zeta\varkappa$.

The Equations (2.4), when $\Lambda = p = 0$, are formally identical with those of Hoyle's steady-state cosmology (see Hoyle, 1958). In Hoyle's version, however, the α-term is responsible for the 'creation field' and causes non-vanishing of the divergence of the energy-momentum tensor. In our case, on account of dissipation

$$\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V, \tag{2.5}$$

which gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{6\alpha}{\varkappa} \dot{R}^2 R - p \frac{\mathrm{d}R^3}{\mathrm{d}t} = 0. \tag{2.6}$$

The particles conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} (nR^3) = 0, \tag{2.7}$$

(where n is equal to the value of the particle current N_0 as measured in a co-moving frame of reference, in which the 4-vector of velocity has the same space-time direction as the particle current) remains unaffected by the dissipative terms (see Weinberg, 1971).

3. Viscous Dust Universes with $k=0$

The field Equation (2.4b), with the equation of state for the dust universe: $p=0$, condition of flatness: $k=0$, and system of units: $c=\varkappa=1$, takes the form

$$2R\ddot{R} + \dot{R}^2 - 2\alpha R\dot{R} - \Lambda R^2 = 0. \tag{3.1}$$

This equation constitues a counterpart of the known Friedmann equation for non-viscous universes. After the substitution

$$u = \dot{R}/R$$

(3.1) reduces to simple equation of the first order

$$\dot{u} = -\tfrac{3}{2}u^2 + \alpha u + \frac{\Lambda}{2}. \tag{3.2}$$

Denoting by u_1 and u_2 the solutions of $\dot{u}=0$ we get

$$u_1 = \tfrac{1}{3}(\alpha + \beta), \quad u_2 = \tfrac{1}{3}(\alpha - \beta),$$

where $\beta = \sqrt{\alpha^2 + 3\Lambda}$.

The singular points of the Equation (3.1) are determined by the relations

$$\frac{\dot{R}}{R} = u_1, \quad \frac{\dot{R}}{R} = u_2. \tag{3.3}$$

The general solution of the Equation (3.1), according to the values of u_1 and u_2, is of the form

(A) $\Lambda>0$ $(u_1>0,\ u_2<0)$:

$$R = e^{\alpha t/3}(c_1 e^{\beta t/2} + c_2 e^{-\beta t/2})^{2/3} = e^{\alpha t/3} \begin{cases} e^{-\beta t/3} & \text{(I)} \\ (\mathrm{ch}(\beta t/2))^{2/3} & \text{(II)} \\ e^{\beta t/3} & \text{(III)} \\ (\mathrm{sh}(\beta t/2))^{2/3} & \text{(IV)} \end{cases}$$

with $\beta > \alpha$.

(B) $\Lambda = 0 \quad (u_1 > u_2 = 0)$:

as in case (A) but $\beta = \alpha$.

All non-static solutions of this class (just like those of class (A)), when t goes to infinity, change asymptotically into the de Sitter's stationary universe. In view of this, Hoyle (1958) has interpreted solutions of class (B) (with $\Lambda = 0$) as describing a stationary expanding universe with a constant rate of continuous creation of matter. To the steady-state cosmology, however, the α-term was introduced *a priori* in order to satisfy certain philosophical assumptions. In our context the introduction of the α-term has a clear physical meaning.

(C) $\Lambda < 0$; there are three sub-cases:

(1) $\quad \Lambda > \Lambda_\alpha = -\frac{1}{3}\alpha^2 \quad (u_1 > u_2 > 0)$:

as in case (A) but $\beta < \alpha$.

(2) $\quad \Lambda = \Lambda_\alpha \quad (u_1 = u_2 > 0)$:

$$R = e^{\alpha t/3} (c_1 + c_2 t)^{2/3} = e^{\alpha t/3} \begin{cases} t^{2/3} & \text{(I)} \\ 1 & \text{(II)} \end{cases}$$

(3) $\quad \Lambda < \Lambda_\alpha \quad (u_1 = u_2^* \text{ complex})$:

$$R = c e^{\alpha(t - t_0)/3} \left[\sin \frac{|\beta|}{2} (t - t_0) \right]^{2/3}.$$

As is well known, all ordinary Friedmannian world models are iso-entropic. There is no heat flow in these models from one portion of the substratum to another. The only mechanism by which an entropy increase is possible in a homogeneous and isotropic model must be due to a suitable composition of the substratum. Still Tolman (1934) had found that a way of increasing entropy in a series of irreversible expansions and contractions of the universe, leads in consequence to an increasingly greater (without limitation) $R_{\max}$ in the subsequent cycles of oscillation. The above case (C3) may serve as an example of Tolman universes (although Tolman originally considered only closed universes with $\Lambda = 0$).

Graphs (Figure 1) represent the above solutions. Among them some are non-physical solutions, because the condition $\varrho \geqslant 0$ was not imposed on the Equation (2.4a). After imposing this condition, it appeared that only in class (A) solutions were found with negative density (marked on the graphs by a dashed line). The existence of regions with non-physical conditions ($\varrho < 0$) follows from the highly idealized assumption: $\alpha = \text{const}$. It is certain that for $\varrho \to 0$ we should have: $\alpha \to 0$. Examples can easily be given to show that assuming α to be a function with such assymptotic properties, leads to solutions physical everywhere ($\varrho \geqslant 0$).

4. Radiative Viscous Universes

The field Equations (2.4) with the usual equation for the radiation-filled universe:

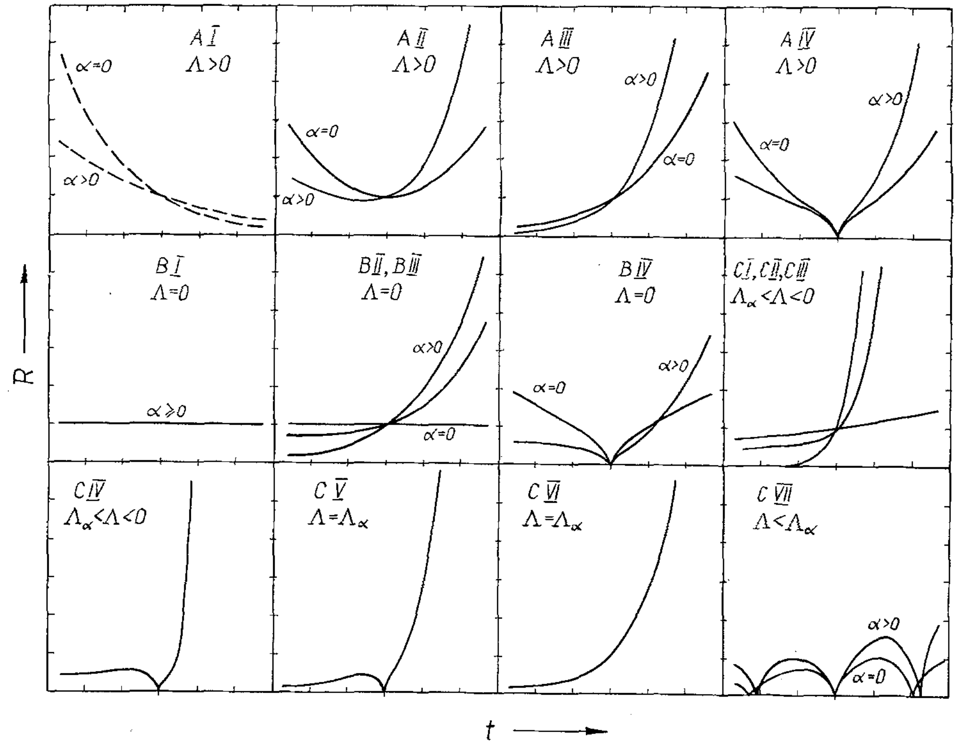

Fig. 1. Viscous dust universes with $k=0$.

$p=\frac{1}{3}\varrho$ take the form

$$R\ddot{R} + \dot{R}^2 - \alpha R\dot{R} - \tfrac{2}{3}\Lambda R^2 + k = 0\,. \tag{4.1}$$

If we substitute

$$X = \tfrac{1}{2}R^2,$$

Equation (4.1) reduces to the simple form

$$\ddot{X} - \alpha\dot{X} - \tfrac{4}{3}\Lambda X + k = 0\,. \tag{4.2}$$

Hence, we obtain the general solution of (4.1) as

$$R^2 = c_1 e^{\omega_1 t} + c_2 e^{\omega_2 t} + \frac{3k}{2\Lambda}, \tag{4.3}$$

where

$$\omega_1 = \tfrac{1}{2}\left[\alpha - \left(\alpha^2 + \frac{16\Lambda}{3}\right)^{1/2}\right],$$

$$\omega_2 = \tfrac{1}{2}\left[\alpha + \left(\alpha^2 + \frac{16\Lambda}{3}\right)^{1/2}\right].$$

The course of the function $R^2(t)$ depends on the values of Λ and on constants c_1 and c_2:

(A) $\Lambda>0$ then: $\omega_2>0>\omega_1$.

(B) $\Lambda=0$ in this case the immediate integration of (4.1) gives

$$R^2(t)=c_1e^{\alpha t}+\frac{2k}{\alpha}t+c_2\,.$$

(C) $\Lambda<0$ three sub-classes appear:

(1) $\Lambda>\Lambda_\alpha=-\frac{3}{16}\alpha^2$:

then $\omega_2>\omega_1>0$.

(2) $\Lambda=\Lambda_\alpha$:

$$R^2(t)=e^{\alpha t/2}(c_1+c_2t)-\frac{8k}{\alpha^2}$$

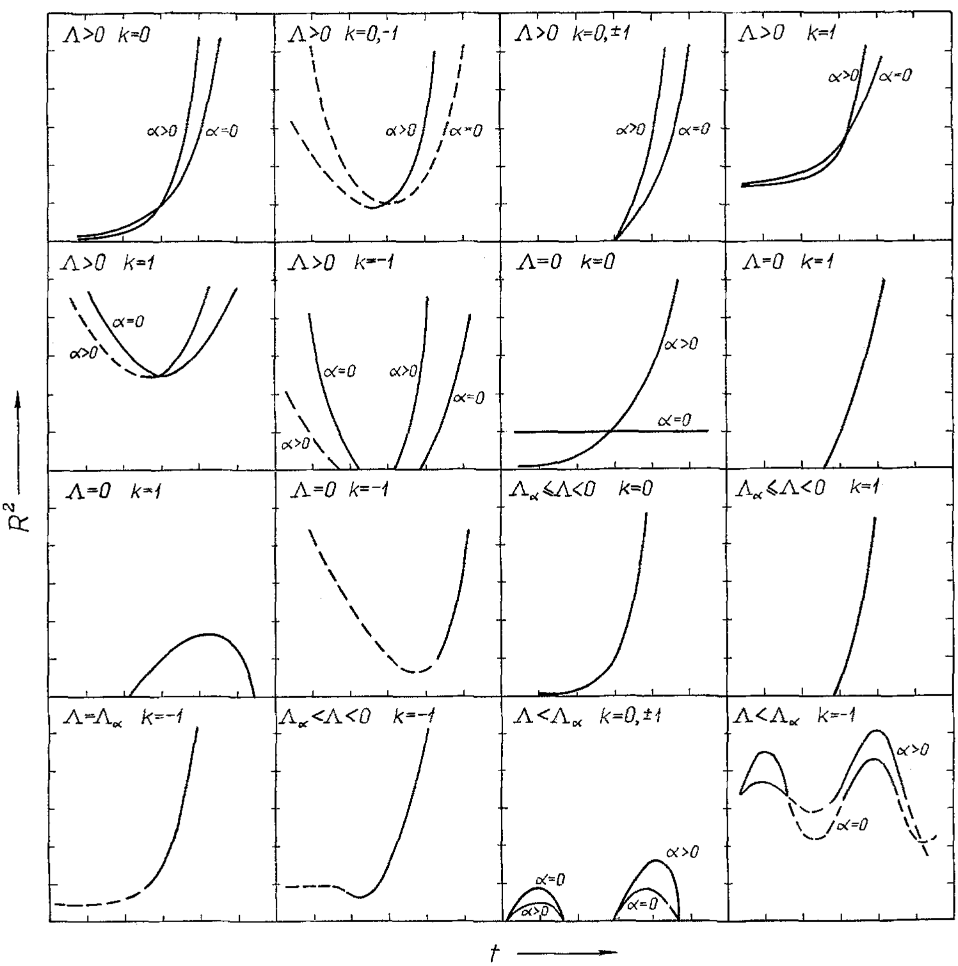

Fig. 2. Radiative viscous universes.

(3) $\Lambda < \Lambda_\alpha \quad (\omega_1 = \omega_2^* \text{ complex})$:

$$\omega_1 = \tfrac{1}{2}(\alpha - i\omega), \qquad \omega_2 = \tfrac{1}{2}(\alpha + i\omega)$$

$$\omega = \left(-\alpha^2 - \frac{16\Lambda}{3}\right)^{1/2}$$

$$R^2(t) = e^{\alpha t/2}\left(c_1 \sin\frac{\omega}{2} t + c_2 \cos\frac{\omega}{2} t\right) + \frac{3k}{\Lambda}.$$

The above solutions are represented by graphs (Figure 2). Domains of solutions for which $\varrho < 0$, are marked by dashed lines. The remark concerning non-physical regions of the solutions, made for dust universes, is also valid here.

5. Viscous Universes and the Singularity Problem

The introduction of the viscosity term into the equations of Friedmannian cosmology does not exclude automatically the appearence of singularities. Within the discussed classes of models singular as well as non-singular solutions appear. We may enrich our exemplicative material by the following lemma:

With $k = +1$ and $\Lambda \leqslant 0$, Equation (3.1) has no solution positive everywhere and continuous for $t \in (-\infty, \infty)$.

Indeed, if at a certain point t_0: $\dot{R}(t) = 0$, then from (3.1),

$$\ddot{R}(t_0) = \frac{\Lambda R^2 - 1}{2R} < 0;$$

so that at the point t_0 there is a maximum. The function $R(t)$, however, cannot tend at $\pm\infty$ to a positive constant. In this case,

$$\lim_{t \to \pm\infty} \dot{R} = \lim_{t \to \pm\infty} \ddot{R} = 0;$$

and from (3.1) we have

$$-\Lambda R^2 - 1 = 0.$$

which is a contradiction. If $R(t)$ were a monotonic function, then at $\pm\infty$ it should tend to a constant and the above argument is still valid.

Above issues are in accordance with the known Hawking-Penrose (1970) theorem about singularities. In all viscous non-singular solutions the so-called energy condition of this theorem is broken (except for the static solution, in which there are no conjugate points). Although our results are only provisional ($\alpha = \text{const!}$) they lead to some tentative conclusions. As it is well known, the Hawking-Penrose theorem was proved only for $\Lambda \leqslant 0$. Hawking and Penrose, however, have put forward a hypothesis that the theorem is valid also for positive values of the cosmological constant. Among our viscous models none is found to be opposed to this hypothesis. Moreover, when considering the ordering of our solutions, we note that the greater the value of Λ, the stronger is the tendency not to fulfill the conditions of the Hawking-Penrose theorem and, consequently, to avoid singularities.

Let us note that, on removing the non-physical part of a given solution, one obtains a geodesically incomplete space-time, which in the sense of certain theorems about singularities (see Geroch, 1967), is understood as a singularity. It is not, however, a 'true' singularity.

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