SOME BIANCHI-TYPE COSMOLOGICAL MODELS IN A NEW SCALAR-TENSOR THEORY

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Abstract. Bianchi type I, III, V, VI_0, and Kantowski–Sachs type models have been investigated in a scalar tensor theory developed by Saez and Ballester (1985) and Saez (1985). The dynamical behaviour of the models has also been analyzed.

1. Introduction

The theories of gravitation involving scalar fields have been extensively studied (Brans and Dicke, 1961; Bergmann, 1968; Nordvedt, 1970; Wagoner, 1970). There are two different types of gravitational theories involving a classical scalar field ϕ. For the first category, the scalar field has the dimension of the inverse of the gravitational constant G. (For example, the Brans–Dicke theory of 1961 and the scalar tetradic theories of Saez, 1983.) The theories of the second type involve a dimensionless scalar field. For example, one has the BWN theory and in particular Barker's (1978) theory. Recently, another theory of the second type has been developed by Saez (1985) and Saez and Ballester (1985) which these authors have referred to as the 'ϕ-coupling'.

The first set of theories have been extensively studied by Singh and Rai (1983). The G-variation has been related with the possible existence of an anti-gravity regime (Linde, 1980; Pollock, 1982).

Saez and Ballester (1985) have developed a theory in which the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of the weak fields. In spite of the dimensionless character of the scalar field, an anti-gravity regime appears. This theory suggests a possible way to solve the missing matter problem in non-flat FRW cosmologies. Saez (1985) discussed the initial singularity and inflationary universe in this theory. He has shown that there is an antigravity regime which could act either at the beginning of the inflationary epoch or before. He has also obtained a non-singular FRW model in the case $k = 0$.

In this work we have studied the ϕ-coupling of gravity for Bianchi class of universes of types I, III, V, VI_0, and Kantowski–Sachs universe (Ryan and Shapley, 1975). We have investigated the dynamical behaviour of these models.

2. Field Equations

Saez and Ballester (1985) start with the Lagrangian

$$L = R - \omega\phi^n(\phi_{,i}\phi^{,i}), \tag{2.1}$$

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where R is the scalar curvature; n, an arbitrary exponent; and ω, a dimensionless coupling constant. The independent variation of the metric tensor g_{ij} and scalar field ϕ leads, respectively, to the field equations

$$R_{ij} - \tfrac{1}{2}g_{ij}R - \omega\phi^n[\phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}(\phi_{,k}\phi^{,k})] = \chi T_{ij}\,, \tag{2.2}$$

$$2\phi^n\phi^{,i}{}_{;i} + n\phi^{n-1}(\phi_{,k}\phi^{,k}) = 0\,, \tag{2.3}$$

g_{ij} being the metric c; R_{ij}, the Ricci tensor; $R = g^{ij}R_{ij}$, T_{ij}, the matter energy-momentum tensor.

The equations of motion

$$T^{ij}{}_{;j} = 0\,, \tag{2.4}$$

are consequences of the field equations (2.2) and (2.3).

3. Bianchi Type-I Model

The Bianchi type-I metric is of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R_1^2\,\mathrm{d}x^2 - R_2^2\,\mathrm{d}y^2 - R_3^2\,\mathrm{d}z^2\,, \tag{3.1}$$

where

$$R_i = R_i(t)\,, \quad i = 1, 2, 3\,.$$

The field equations (2.2) and (2.3) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) = \frac{1}{2}\,\chi(\rho - p)\,, \tag{3.2}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_3}{R_3}\right) = \frac{1}{2}\,\chi(\rho - p)\,, \tag{3.3}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) = \frac{1}{2}\,\chi(\rho - p)\,, \tag{3.4}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -\frac{1}{2}\,\chi(\rho + 3p) + \omega\phi^n\dot{\phi}^2\,, \tag{3.5}$$

$$\ddot{\phi} + 2\dot{\phi}\,\frac{\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,, \tag{3.6}$$

Like in general relativity, $T^{ij}{}_{;j} = 0$ leads to

$$\dot{\rho} + \frac{3\dot{V}}{V}\,(\rho + p) = 0\,, \tag{3.7}$$

where

$$V^3 = R_1 R_2 R_3 \tag{3.8}$$

and a dot denotes differentiation with respect to t.

There are five Equations (3.2)–(3.6) in six unknowns R_1, R_2, R_3, ρ, p, and ϕ. Hence, to solve these equations one can always impose an additional conditions. But it is difficult to find a general solution. Therefore, we consider two particular cases – viz., vacuum ($\rho = p = 0$) – and Zeldovich fluid ($\rho = p$).

Case I. Vacuum ($\rho = p = 0$)

In this case Equations (3.2)–(3.6) reduce to the equations:

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) = 0\,, \tag{3.9}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_3}{R_3} + \frac{\dot{R}_1}{R_1}\right) = 0\,, \tag{3.10}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_2} + \frac{\dot{R}_2}{R_2}\right) = 0\,, \tag{3.11}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = \omega\phi^n\dot{\phi}^2\,, \tag{3.12}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,. \tag{3.13}$$

If we add Equations (3.9)–(3.11) we get

$$\sum \frac{\ddot{R}_1}{R_1} + \sum \frac{\dot{R}_1\dot{R}_2}{R_1R_2} = 0\,. \tag{3.14}$$

Differentiating (3.8) twice with respect to t, we get

$$\sum \frac{\ddot{R}_1}{R_1} - \sum\left(\frac{\dot{R}_1}{R_1}\right)^2 = \frac{3\ddot{V}}{V} - 3\left(\frac{\dot{V}}{V}\right)^2. \tag{3.15}$$

From (3.14) and (3.15) one can obtain

$$\ddot{V} + \frac{2}{V}\,\dot{V}^2 = 0\,, \tag{3.16}$$

It has the solution

$$V^3 = ct + d\,, \tag{3.17}$$

c and d being constants.

Now by use of (3.17) in Equations (3.9)–(3.11) one can obtain in a straightforward manner the values of R_1, R_2, and R_3 given by

$$R_1 = n_1(ct + d)^{l_1/c}, \tag{3.18}$$

$$R_2 = n_2(ct + d)^{l_2/c}, \tag{3.19}$$

$$R_3 = n_3(ct + d)^{l_3/c}, \tag{3.20}$$

where n_1, n_2, n_3; l_1, l_2, l_3 are constants.

If we use (3.17) in (3.13) and integrating once we get

$$\phi^{n/2}\dot{\phi} = n_4(ct + d)^{-1}; \tag{3.21}$$

n_4 being a constant of integration. Integration of (3.21) leads to

$$\phi = \left[\frac{n_4(n+2)}{2c} \log\{n_5(ct + d\}\right]^{2/(n+2)}, \tag{3.22}$$

where n_5 is a constant.

If we use R_1, R_2, R_3 from Equations (3.18), (3.20) and ϕ from (3.22) in Equation (3.12) we obtain a relation between the constants viz.

$$\left(\frac{n_1 l_1}{c}\right)\left(\frac{l_1}{c} - 1\right) + \left(\frac{n_2 l_2}{c}\right)\left(\frac{l_2}{c} - 1\right) + + \left(\frac{n_3 l_3}{c}\right)\left(\frac{l_3}{c} - 1\right) - \omega n_4^2 = 0. \tag{3.23}$$

The dynamical parameters of the model are:

$$\sigma^2 = \frac{1}{12}\left[\left\{\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}}\right\}^2 + \left\{\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}}\right\}^2 + \left\{\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}}\right\}^2\right], \tag{3.24}$$

$$\sigma^2 = \tfrac{2}{3}(ct + d)^{-2}[(l_1^2 + l_2^2 + l_3^2) - (l_1 l_2 + l_2 l_3 + l_1 l_3)];$$

– scalar of expansion

$$\theta = 3\,\frac{\dot{V}}{V} = \frac{(l_1 + l_2 + l_3)}{(ct + d)};$$

– Hubble parameter

$$H = \frac{\dot{V}}{V} = \frac{1}{3}\,\theta,$$

$$\frac{\sigma^2}{\theta} = \frac{2}{3(l_1 + l_2 + l_3)}\,(ct + d)^{-1}[(l_1^2 + l_2^2 + l_3^2) - (l_1 l_2 + l_2 l_3 + l_3 l_1)];$$

– deceleration parameter

$$q = \frac{-V\ddot{V}}{\dot{V}^2} = \left[\frac{3c}{(l_1 + l_2 + l_3)}\right] - 1 ,$$

– the integral

$$\int_{t_0}^{t} \frac{dt'}{V(t')} = \left[\frac{(n_1 n_2 n_3)^{-1/3}}{\{3c - (l_1 + l_2 + l_3)\}} (ct + d)^{\{3c - (l_1 + l_2 + l_3)/3c\}}\right]_{t_0}^{t} \tag{3.25}$$

is convergent. Therefore, the model has a horizon.

The Ricci scalar is

$$R = g^{ij}R_{ij} = 2(ct + d)^{-2}[(l_1^2 + l_2^2 + l_3^2) - c(l_1 + l_2 + l_3) + + (l_1 l_2 + l_2 l_3 + l_3 l_1)] . \tag{3.26}$$

Case II. Zeldovich fluid ($p = \rho$)

In this case the field equations (3.2)–(3.7) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) = 0 , \tag{3.27}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_3}{R_3} + \frac{\dot{R}_1}{R_1}\right) = 0 , \tag{3.28}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) = 0 ; \tag{3.29}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -2\chi\rho + \omega\phi^n\dot{\phi}^2 , \tag{3.30}$$

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{V}}{V} + \frac{n}{2\phi}\dot{\phi}^2 = 0 , \tag{3.31}$$

$$\dot{\rho} + 6\rho\frac{\dot{V}}{V} = 0 . \tag{3.32}$$

If we follow the process similar to the vacuum case, from Equations (3.27)–(3.29) and (3.8) we get

$$V^3 = at + b , \tag{3.33}$$

where a and b are constants.

If we use it in Equations (3.27)–(3.29) we can easily obtain

$$R_1 = m_1(at + b)^{k_1/a}, \quad R_2 = m_2(at + b)^{k_2/a}, \quad R_3 = m_3(at + b)^{k_3/a}; \tag{3.34}$$

where m_1, m_2, m_3; k_1, k_2, k_3 are constants.

Equations (3.32) and (3.33) give

$$\rho = p = m_4(at + b)^{-2}, \tag{3.35}$$

where m_4 is a constant.

By use of (3.33) in (3.31) and integrating once we get

$$\phi^{n/2}\dot{\phi} = m_5(at + b)^{-1}, \tag{3.36}$$

where m_5 is a constant.

Integration of (3.36) gives

$$\phi = \left[\frac{(n + 2)m_5}{2a} \log\{m_6(at + b\}\right]^{2/(n+2)}, \tag{3.37}$$

where m_6 = constant.

If we use R_1, R_2, R_3 from (3.34) and ϕ from (3.37) in (3.30) we get a relation between the constants given by

$$\frac{m_1 k_1}{a}\left(\frac{k_1}{a} - 1\right) + \left(\frac{m_2 k_2}{a}\right)\left(\frac{k_2}{a} - 1\right) + \left(\frac{m_3 k_3}{a}\right)\left(\frac{k_3}{a} - 1\right) + + 2\chi m_4 - \omega m_5^2 = 0. \tag{3.38}$$

The dynamical parameters are defined as:

– shear tensor

$$\sigma_{ij} = \tfrac{1}{2}(u_{i;\,j} + u_{j;\,i}) + \tfrac{1}{2}(\dot{u}_i u_j + \dot{u}_j u_i) - \tfrac{1}{3}h_{ij}(u^k{}_{;\,k}), \tag{3.39}$$

where

$$\sigma^2 = \tfrac{1}{2}\sigma_{ij}\sigma^{ij}, \qquad h_{ij} = g_{ij} - u_i u_j.$$

Scalar of expansion $\theta = u^k{}_{;\,k}$:

– rotation tensor

$$w_{ij} = \tfrac{1}{2}(u_{i;\,j} - u_{j;\,i}) - \tfrac{1}{2}(\dot{u}_i u_j - \dot{u}_j u_i);$$

– rotation

$$w^2 = \tfrac{1}{2}w_{ij}w^{ij}.$$

For the model these parameters are

$$\sigma^2 = \tfrac{1}{3}(at + b)^{-2}[(k_1^2 + k_2^2 + k_3^2) - (k_1k_2 + k_2k_3 + k_3k_1)] , \qquad (3.40)$$

$$\theta = \frac{(k_1 + k_2 + k_3)}{(at + b)} , \quad w_{ij} = 0 ;$$

– Hubble parameter

$$H = \tfrac{1}{3}\theta ;$$

– deceleration parameter

$$q = \left[\frac{3a}{(k_1 + k_2 + k_3)}\right]^{-1} .$$

$$\frac{\sigma^2}{\theta} = \tfrac{1}{3}(at + b)^{-1}\left[\frac{(k_1^2 + k_2^2 + k_3^2) - (k_1k_2 + k_2k_3 + k_3k_1)}{(k_1 + k_2 + k_3)}\right],$$

$$\frac{\sigma^2}{\rho} = \frac{1}{3m_4} [(k_1^2 + k_2^2 + k_3^2) - (k_1k_2 + k_2k_3 + k_3k_1)] .$$

The integral

$$\int_{t_0}^{t} \frac{\mathrm{d}t}{V(t)} = \left[\frac{(m_1m_2m_3)^{-1/3}}{\{3a - (k_1 + k_2 + k_3)\}} (at + b)^{(1/3a)\{3a - (k_1 + k_2 + k_3)\}}\right]_{t_0}^{t} \qquad (3.41)$$

is convergent and, therefore, the model has a horizon.

The model is singular at time $t = -b/a$.

4. Bianchi Type-III Universe

The Bianchi type-III metric is of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R_1^2\,\mathrm{d}r^2 - R_2^2\,[\mathrm{d}\theta^2 + \sinh^2\theta\,\mathrm{d}\phi^2] , \qquad (4.1)$$

where R_1 and R_2 are functions of t only. The field equations (2.2) and (2.3) can be written as

$$\frac{\ddot{R}_1}{R_1} + \frac{2\dot{R}_1\dot{R}_2}{R_1R_2} = \frac{1}{2}\,\chi(\rho - p) , \qquad (4.2)$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} - \frac{1}{R_2^2} = \frac{1}{2}\,\chi(\rho - p) , \qquad (4.3)$$

$$\frac{\ddot{R}_1}{R_1} + 2\,\frac{\ddot{R}_2}{R_2} = -\frac{1}{2}\,\chi(\rho + 3p) + \omega\phi^n\dot{\phi}^2 , \qquad (4.4)$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + 2\,\frac{\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,, \tag{4.5}$$

The equation $T^{ij}{}_{;j} = 0$ leads to

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) = 0\,. \tag{4.6}$$

It is difficult to find a general solution of Equations (4.2)–(4.6). Hence, we consider only two physically interesting cases: namely,

(i) Vacuum ($\rho = p = 0$).

(ii) Zeldovich fluid ($\rho = p$).

Case I. Vacuum Universe ($\rho = p = 0$)

In this case Equations (4.2)–(4.6) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{2\dot{R}_1\dot{R}_2}{R_1R_2} = 0\,, \tag{4.7}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} - \frac{1}{R_2^2} = 0\,, \tag{4.8}$$

$$\frac{\ddot{R}_1}{R_1} + 2\,\frac{\ddot{R}_2}{R_2} = \omega\phi^n\dot{\phi}^2\,, \tag{4.9}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,. \tag{4.10}$$

We use a transformation of the time coordinate by

$$\mathrm{d}t = R_2\,\mathrm{d}\eta\,. \tag{4.11}$$

Furthermore, we use a substitution

$$h = R_1R_2\,. \tag{4.12}$$

Then Equations (4.7)–(4.10) become

$$\frac{R_1''}{R_1} + \frac{R_1'R_2'}{R_1R_2} = 0\,, \tag{4.7a}$$

$$\frac{R_2''}{R_2} + \frac{R_1'R_2'}{R_1R_2} - 1 = 0\,, \tag{4.8a}$$

$$\frac{R_1''}{R_1} - \frac{R_1'R_2'}{R_1R_2} + 2\left[\frac{R_2''}{R_2} - \frac{R_2'^2}{R_2^2}\right] = \omega\phi^n\phi'^2\,, \tag{4.9a}$$

$$\phi'' + \left(\frac{\phi' h'}{h}\right) + \frac{n}{2\phi}\,\phi'^2 = 0\,, \tag{4.10a}$$

where a prime denotes differentiation with respect to η.

From (4.12), (4.7a), and (4.8a) we have

$$h'' - h = 0\,. \tag{4.13}$$

The solution is

$$h = R_1 R_2 = m_1 \sinh(\eta + m_2)\,, \tag{4.14}$$

where m_1 and m_2 are constants.

By use of (4.12) and (4.14) in Equation (4.7a) we can easily obtain

$$R_1 = m_4\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3/m_1}, \tag{4.15}$$

where m_3 and m_4 are constants.

From (4.12), (4.14), and (4.15) we have

$$R_2 = \left(\frac{m_1}{m_4}\right)\sinh(\eta + m_2)\left[\coth\left(\frac{\eta + m_2}{2}\right)\right]^{m_3/m_1}. \tag{4.16}$$

The use of (4.14) in (4.10a) and once integration gives

$$\phi^{n/2}\phi' = \left(\frac{m_5}{m_1}\right)\operatorname{cosech}(\eta + m_2)\,. \tag{4.17}$$

Integration of (4.17) gives

$$\phi = \left[\frac{m_5}{m_1}\left(\frac{n+2}{2}\right)\log\left\{m_6\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]\right\}\right]^{2/(n+2)}, \tag{4.18}$$

where m_5 and m_6 are constants.

The use of R_1, R_2, and ϕ in Equation (4.9a) gives a relation between the constants,

$$2(m_3^2 - m_1^2) = \omega m_5^2\,. \tag{4.19}$$

The kinematical parameters are

– shear

$$\sigma^2 = \frac{2}{3}\frac{m_4^2}{m_1^2}\operatorname{cosech}^2(\eta + m_2)\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{2m_3/m_1} +$$

$$+\left[\frac{2m_3}{m_1}\operatorname{cosech}(\eta + m_2) - \coth(\eta + m_2)\right]^2; \tag{4.20}$$

– scalar of expansion

$$\theta = \frac{m_4}{m_1}\,\mathrm{cosech}\,(\eta + m_2)\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3/m_1} +$$

$$+ [2\coth(\eta + m_2) - (m_3/m_1)\,\mathrm{cosech}\,(\eta + m_2)]\,;$$

– Hubble parameter

$$H = \tfrac{1}{3}\theta\,;$$

– deceleration parameter

$$q = -6\left[2\coth(\eta + m_2) - \frac{m_3}{m_1}\,\mathrm{cosech}\,(\eta + m_2)\right]^{-2},$$

$$\frac{\sigma^2}{\theta} = \frac{2}{3}\left(\frac{\mathrm{m}_4}{\mathrm{m}_1}\right)\mathrm{cosech}\,(\eta + m_2)\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3/m_1} \times$$

$$\times \frac{\left[2\,\frac{m_3}{m_1}\,\mathrm{cosech}\,(\eta + m_2) - \coth(\eta + m_2)\right]}{\left[2\coth(\eta + m_2) - \frac{m_3}{m_1}\,\mathrm{cosech}\,(\eta + m_2)\right]}\,.$$

The Ricci scalar is

$$R = \left(\frac{2m_4^2}{m_1^2}\right)\left(\frac{m_3^2}{m_1^2} - 1\right)\mathrm{cosech}^4(\eta + m_2)\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{2m_3/m_1}. \quad (4.21)$$

Case II. Zeldovich Fluid ($p = \rho$)

In this case Equations (4.2)–(4.6) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{2\dot{R}_1\dot{R}_2}{R_1 R_2} = 0\,, \quad (4.22)$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_1\dot{R}_2}{R_1 R_2} - \frac{1}{R_2^2} = 0\,, \quad (4.23)$$

$$\frac{\ddot{R}_1}{R_1} + \frac{2\ddot{R}_2}{R_2} = 2\chi\rho + \omega\phi^n\dot{\phi}^2\,, \quad (4.24)$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,, \quad (4.25)$$

$$\dot{\rho} + 2\rho\left(\frac{\dot{R}_1}{R_1} + 2\,\frac{\dot{R}_2}{R_2}\right) = 0\,. \tag{4.26}$$

Under the change of the time-coordinate by $\mathrm{d}t = R_2\,\mathrm{d}\eta$ Equations (4.22)–(4.26) are transformed into

$$\frac{R_1''}{R_1} + \frac{R_1' R_2'}{R_1 R_1} = 0\,, \tag{4.27}$$

$$\frac{R_2''}{R_2} + \frac{R_1' R_2'}{R_1 R_2} - 1 = 0\,, \tag{4.28}$$

$$\frac{R_1''}{R_1} - \frac{R_1' R_2'}{R_1 R_2} + 2\,\frac{R_2''}{R_2} - \frac{R_2'^2}{R_2} = -2\chi\rho R_2^2 + \omega\phi^n\phi'^2\,, \tag{4.29}$$

$$\phi'' + \phi'\left(\frac{R_1'}{R_1} + \frac{R_2'}{R_2}\right) + \frac{n}{2\phi}\,\phi'^2 = 0\,, \tag{4.30}$$

$$\rho' + 2\rho\left(\frac{R_1'}{R_1} + 2\,\frac{R_2'}{R_2}\right) = 0\,. \tag{4.31}$$

Now we make the substitution

$$h = R_1 R_2\,. \tag{4.32}$$

Then from Equations (4.27) and (4.28) we can obtain

$$h'' - h = 0\,. \tag{4.33}$$

The solution is

$$h = R_1 R_2 = l_1 \sinh(\eta + l_2)\,, \tag{4.34}$$

l_1, l_2 being constants.

Then from (4.32), (4.34), and (4.27) we can obtain

$$R_1 = l_4\left[\tanh\left(\frac{\eta + l_2}{2}\right)\right]^{l_3/l_1} \tag{4.35}$$

l_3, l_4 are constants.

Now (4.32), (4.34), and (4.35) give

$$R_2 = \left(\frac{l_1}{l_4}\right)\sinh(\eta + l_2)\left[\coth\left(\frac{\eta + l_2}{2}\right)\right]^{l_3/l_2}, \tag{4.36}$$

From (4.32), (4.34), and (4.30) we obtain, after integration

$$\phi^{n/2}\phi' = \left(\frac{l_5}{l_1}\right)\operatorname{cosech}(\eta + l_2)\,. \tag{4.37}$$

If we integrate (4.37) once again, we get

$$\phi = \left[\left(\frac{n+2}{2}\right)\frac{l_5}{l_1}\log\left\{l_6\left[\tanh\left(\frac{\eta+l_2}{2}\right)\right]\right\}\right]^{2/(n+2)}, \tag{4.38}$$

where l_5, l_6 are constants.

If we use the values of R_1, R_2, and ϕ in (4.29) we have

$$\rho = p = \left(\frac{l_4^2}{2l_1^4}\right)[(\omega l_5^2 + 2l_1^2 - 2l_3^2)]\,\mathrm{cosech}^2(\eta + l_2) \times$$
$$\times\left[\tanh\left(\frac{\eta+l_2}{2}\right)\right]^{(2l_3/l_1)}. \tag{4.39}$$

The conservation equation (4.31) is identically satisfied for ρ, R_1, R_2 given by (4.39), (4.35), and (4.36).

The kinematical parameters are

– shear

$$\sigma^2 = \left(\frac{l_4^2}{3l_1^2}\right)\left[\tanh\left(\frac{\eta+l_2}{2}\right)\right]^{2l_3/l_1}\left[2\,\frac{l_3}{l_1}\,\mathrm{cosech}(\eta+l_2) -\right.$$
$$\left. - \coth(\eta+l_2)\right]^2 \mathrm{cosech}^2(\eta+l_2)\,; \tag{4.40}$$

– scalar of expansion

$$\theta = \left(\frac{l_4}{l_1}\right)\mathrm{cosech}(\eta+l_2)\left[\tanh\left(\frac{\eta+l_2}{2}\right)\right]^{l_3/l_1}\left[2\coth(\eta+l_2) -\right.$$
$$\left. - \frac{l_3}{l_1}\,\mathrm{cosech}(\eta+l_2)\right];$$

– Hubble parameter

$$H = \tfrac{1}{3}\theta\,;$$

– deceleration parameter

$$q = -6\left[2\coth(\eta+l_2) - \frac{l_3}{l_1}\,\mathrm{cosech}(\eta+l_2)\right]^{-2},$$

rotation tensor $\omega_{ij} = 0$ identically.

$$\frac{\sigma^2}{\rho} = \frac{2\chi\sinh^2(\eta+l_2)}{3\left[\omega\left(\frac{l_5}{l_1}\right)^2 + 2 - 2\left(\frac{l_3}{l_1}\right)^2\right]}\left[2\,\frac{l_3}{l_1}\,\mathrm{cosech}(\eta+l_2) - \coth(\eta+l_2)\right]^2,$$

$$\frac{\sigma^2}{\theta} = \frac{1}{3}\left(\frac{l_4}{l_1}\right) \operatorname{cosech}(\eta + l_2) \left[\tanh\left(\frac{\eta + l_2}{2}\right)\right]^{l_3/l_1} \times$$

$$\times \frac{\left[2\,\frac{l_3}{l_1}\operatorname{cosech}(\eta + l_2) - \coth(\eta + l_2)\right]^2}{\left[2\coth(\eta + l_2) - \frac{l_3}{l_1}\operatorname{cosech}(\eta + l_2)\right]} .$$

The Ricci scalar is

$$R = \left(\frac{2l_4^2}{l_1^4}\right)(l_3^2 - l_1^2)\operatorname{cosech}^4(\eta + l_2)\,[\tanh(\eta + l_2)]^{2l_3/l_1} . \tag{4.41}$$

The model has singularity at time $\eta = -l_2$.

5. Kantowski–Sachs Universe

The Kantowski–Sachs space-time metric (1966) is of the form

$$ds^2 = dt^2 - R^2\,dt^2 - R_2^2[d\theta^2 + \sin^2\theta\,d\phi^2] , \tag{5.1}$$

where R_1 and R_2 are functions of t only. In this case the field equations (2.2) and (2.3) are

$$\frac{\ddot{R}_1}{R_1} + 2\,\frac{\dot{R}_1\dot{R}_2}{R_1R_2} = \frac{1}{2}\,\chi(\rho - p) , \tag{5.2}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} + \frac{1}{R_2^2} = \frac{1}{2}\,\chi(\rho - p) , \tag{5.3}$$

$$\frac{\ddot{R}_1}{R_1} + 2\,\frac{\ddot{R}_2}{R_2} = \frac{-1}{2}\,\chi(\rho + 3p) + \omega\phi^n\dot{\phi}^2 , \tag{5.4}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0 . \tag{5.5}$$

The conservation equation $T^{ij}{}_{;j} = 0$ is satisfied if

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) = 0 . \tag{5.6}$$

As it is difficult to find a general solution we consider only two particular physically important cases: viz., (i) vacuum and (ii) Zeldovich fluid.

Case I. Vacuum ($\rho = p = 0$)

The field equations (5.2)–(5.6) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{2\dot{R}_1\dot{R}_2}{R_1R_2} = 0\,, \tag{5.7}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2^2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} + \frac{1}{R_2^2} = 0\,, \tag{5.8}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{2\ddot{R}_2}{R_2} = \omega\phi^n\dot{\phi}^2\,, \tag{5.9}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,. \tag{5.10}$$

Under the transformation for time-coordinate $dt = R_2\,d\eta$, Equations (5.7)–(5.10) take the form

$$\frac{R_1''}{R_1} + \frac{R_1'R_2'}{R_1R_2} = 0\,, \tag{5.11}$$

$$\frac{R_2''}{R_2} + \frac{R_1'R_2'}{R_1R_2} + 1 = 0\,, \tag{5.12}$$

$$\frac{R_1''}{R_1} - \frac{R_1'R_2'}{R_1R_2} + 2\left[\frac{R_2''}{R_2} - \left(\frac{R'}{R_2}\right)^2\right] = \omega\phi^n\phi'^2\,, \tag{5.13}$$

$$\phi'' + \phi'\left(\frac{R_1'}{R_1} + \frac{R_2'}{R_2}\right) + \frac{n}{2\phi}\,\phi'^2 = 0\,, \tag{5.14}$$

where a prime denotes differentiation with respect to η. From (5.11) and (5.12) we have

$$h'' + h = 0\,, \tag{5.15}$$

where

$$h = R_1R_2\,. \tag{5.16}$$

The general solution of (5.15) is

$$h = R_1R_2 = c_1\sin(\eta + c_2)\,; \tag{5.17}$$

c_1, c_2 being constants.

From (5.11), (5.16), and (5.17) we can easily obtain

$$R_1 = c_4\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]^{c_3/c_1}\,; \tag{5.18}$$

c_3, c_4 being constants.

From (5.16)–(5.18) we have

$$R_2 = \left(\frac{c_1}{c_4}\right) \sin(\eta + c_2)\left[\cot\left(\frac{\eta + c_2}{2}\right)\right]^{c_3/c_1}. \tag{5.19}$$

If we use (5.18) and (5.19) in (5.14) and integrating once we get

$$\phi^{n/2}\phi' = \left(\frac{c_5}{c_1}\right)\operatorname{cosec}(\eta + c_2), \tag{5.20}$$

where c_5 is a constant.

If we integrate (5.20) once again, we have

$$\phi = \left[\left(\frac{n+2}{2}\right)\left(\frac{c_5}{c_1}\right)\log\left\{c_6\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]\right\}\right]^{2/(n+2)}, \tag{5.21}$$

c_6 is another constant.

Now plugging the values of R_1, R_2, and ϕ in Equation (5.13) we get a relation between the constants.

$$2(c_3^2 - c_1^2) = \omega c_5^2\,. \tag{5.22}$$

Therefore,

(i) when

$$\omega \geq 0\,, \quad |c_3| \geq |c_1|\,,$$

(ii) when

$$\omega < 0\,, \quad |c_3| < |c_1|\,.$$

The dynamical parameters are:

– shear

$$\sigma^2 = \frac{2}{3}\left\{\left(\frac{c_4}{c_1}\right)^2 \operatorname{cosec}^2(\eta + c_2)\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]^{2c_3/c_1} + \right.$$

$$\left. + \left[2\,\frac{c_3}{c_1}\operatorname{cosec}(\eta + c_2) - \cot(\eta + c_2)\right]^2\right\}; \tag{5.23}$$

– scalar of expansion

$$\theta = \left(\frac{c_4}{c_1}\right)\operatorname{cosec}(\eta + c_2)\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]^{c_3/c_1} +$$

$$+ \left[2\cot(\eta + c_2) - \frac{c_3}{c_1}\operatorname{cosec}(\eta + c_2)\right];$$

– Hubble parameter

$$H = \tfrac{1}{3}\theta ;$$

– deceleration parameter

$$q = 6\left[2\cot(\eta + c_2) - \frac{c_3}{c_1}\operatorname{cosec}(\eta + c_2)\right]^{-2},$$

$$\frac{\sigma^2}{\theta} = \frac{2}{3}\left(\frac{c_4}{c_1}\right)\operatorname{cosec}(\eta + c_2)\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]^{c_3/c_1} \times$$

$$\times \frac{\left[2\,\frac{c_3}{c_1}\operatorname{cosec}(\eta + c_2) - \cot(\eta + c_2)\right]^2}{\left[2\cot(\eta + c_2) - \frac{c_3}{c_1}\operatorname{cosec}(\eta + c_2)\right]} .$$

The Ricci scalar is

$$R = \left(\frac{2c_4^2}{c_1^4}\right)(c_3^2 - c_1^2)\operatorname{cosec}^4(\eta + c_2)\left[\tan\left(\frac{\eta + c_2}{2}\right)\right]^{2c_3/c_1} . \qquad (5.24)$$

Case II. Zeldovich Fluid ($p = \rho$)

The field equations reduce, in this case, to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} = 0 , \qquad (5.25)$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2^2}{R_2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} + \frac{1}{R_2^2} = 0 , \qquad (5.26)$$

$$\frac{\ddot{R}_1}{R_1} + \frac{2\ddot{R}_2}{R_2} = -2\chi\rho + \omega\phi^n\dot{\phi}^2 , \qquad (5.27)$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) + \frac{n}{2\phi}\,\dot{\phi}^2 = 0 . \qquad (5.27')$$

The conservation becomes

$$\dot{\rho} + 2\rho\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) = 0 . \qquad (5.28)$$

The transformation $dt = R_2\, d\eta$ changes Equations (5.25)–(5.28) into

$$\frac{R_1''}{R_1} + \frac{R_1' R_2'}{R_1 R_2} = 0\,, \tag{5.29}$$

$$\frac{R_2''}{R_2} + \frac{R_1' R_2'}{R_1 R_2} + 1 = 0\,, \tag{5.30}$$

$$\frac{R_1''}{R_1} - \frac{R_1' R_2'}{R_1 R_2} + 2\left[\frac{R_2''}{R_2} - \left(\frac{R_2'}{R_2}\right)^2\right] = -2\chi\rho R_2^2 + \omega\phi^n \phi'^2\,, \tag{5.31}$$

$$\phi'' + \phi'\left(\frac{R_1'}{R_1} + \frac{R_2'}{R_2}\right) + \frac{n}{2\phi}\,\phi'^2 = 0\,, \tag{5.32}$$

$$\rho' + 2\rho\left(\frac{R_1'}{R_1} + \frac{2R_2'}{R_2}\right) = 0\,, \tag{5.33}$$

where a prime denotes differentiation with respect to η.

From (5.29) and (5.30) we can obtain

$$h'' + h = 0\,, \tag{5.34}$$

where

$$h = R_1 R_2\,. \tag{5.35}$$

The solution of (5.34) is

$$h = R_1 R_2 = k_1 \sin(\eta + k_2)\,, \tag{5.36}$$

where k_1, k_2 are constants.

By use of (5.35) and (5.36) in (5.29) we can easily obtain

$$R_1 = k_4\left[\tan\left(\frac{\eta + k_2}{2}\right)\right]^{k_3/k_1}, \tag{5.37}$$

where k_3, k_4 are constants.

From (5.35)–(5.37) we have

$$R_2 = \left(\frac{k_1}{k_4}\right)\sin(\eta + k_2)\left[\cot\left(\frac{\eta + k_2}{2}\right)\right]^{k_3/k_1}. \tag{5.38}$$

If we use (5.36) in (5.32) and integrating once, we get

$$\phi^{n/2}\phi' = \left(\frac{k_5}{k_1}\right)\operatorname{cosec}(\eta + k_2)\,, \tag{5.39}$$

If we integrate it once again, we have

$$\phi = \left[\left(\frac{n+2}{2}\right)\left(\frac{k_5}{k_1}\right)\log\left\{k_6\left[\tan\left(\frac{\eta+k_2}{2}\right)\right]\right\}\right]^{2/(n+2)}, \tag{5.40}$$

where k_5, k_6 are constants.

If we use the values of R_1, R_2, and ϕ in Equation (5.31) we get

$$p = \rho = \frac{(\omega k_5^2 + 2k_1^2 - 2k_3^2)k_4^2}{2k_1^4}\operatorname{cosec}^4(\eta + k_2) \times$$
$$\times\left[\tan\left(\frac{\eta+k_2}{2}\right)\right]^{2k_3/k_1}. \tag{5.41}$$

Equation (5.33) is identically satisfied by the values of R_1, R_2, and ρ.

The dynamical parameters are:

- Equation (5.42) shear

$$\sigma^2 = \frac{1}{3}\left(\frac{k_4^2}{k_1^2}\right)\operatorname{cosec}^2(\eta + k_2)\left[\tan\left(\frac{\eta+k_2}{2}\right)\right]^{2k_3/k_1} \times$$
$$\times\left[2\,\frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2) - \cot(\eta + k_2)\right]^2; \tag{5.42}$$

- the scalar of expansion

$$\theta = \left(\frac{k_4}{k_1}\right)\operatorname{cosec}(\eta + k_2)\left[\tan\left(\frac{\eta+k_2}{2}\right)\right]^{k_3/k_1} \times$$
$$\times\left[2\cot(\eta + k_2) - \frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2)\right];$$

- Hubble parameter

$$H = \tfrac{1}{3}\theta.$$

- deceleration parameter

$$q = 2\left[2\cot(\eta + k_2) - \frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2)\right]^2,$$

$$\frac{\sigma^2}{\rho} = \frac{2\chi\sin^2(\eta + k_2)}{3\left[\omega\left(\frac{k_5}{k_1}\right)^2 + 2 - 2\left(\frac{k_3}{k_1}\right)^2\right]}\left[2\,\frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2) - \cot(\eta + k_2)\right]^2,$$

$$\frac{\sigma^2}{\theta} = \frac{2}{3}\left(\frac{k_4}{k_1}\right) \operatorname{cosec}(\eta + k_2)\left[\tan\left(\frac{\eta + k_2}{2}\right)\right]^{k_3/k_1} \times$$

$$\times \frac{\left[2\,\frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2) - \cot(\eta + k_2)\right]^2}{\left[2\cot(\eta + k_2) - \frac{k_3}{k_1}\operatorname{cosec}(\eta + k_2)\right]} .$$

The Ricci scalar is

$$R = \left[\frac{2k_4^2(k_3^2 - k_1^2)}{k_1^4}\right]\operatorname{cosec}^4(\eta + k_2)\left[\tan\left(\frac{\eta + k_2}{2}\right)\right]^{2k_3/k_1}, \tag{5.43}$$

6. Bianchi Type-V Universe

The Bianchi type-V metric is of the form

$$ds^2 = dt^2 - R_1^2\,dx^2 - e^{-2a^2x}(R_2^2\,dy^2 + R_3^2\,dz^2), \tag{6.1}$$

where R_1, R_2, R_3 are functions of t only and a = const.

The field equations (2.2) and (2.3) can be written as

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^2}{R_1^2} = \frac{1}{2}\,\chi(\rho - p), \tag{6.2}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^2}{R_1^2} = \frac{1}{2}\,\chi(\rho - p), \tag{6.3}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) - \frac{2a^2}{R_1^2} = \frac{1}{2}\,\chi(\rho - p), \tag{6.4}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -\frac{1}{2}\,\chi(\rho + 3p) + \omega\phi^n\dot{\phi}^2, \tag{6.5}$$

$$\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3} - \frac{2\dot{R}_1}{R_1} = 0, \tag{6.6}$$

$$\ddot{\phi} + 3\,\frac{\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0, \tag{6.7}$$

The conservation equation $T^{ij}{}_{;j} = 0$ leads to

$$\dot{\rho} + (\rho + p)\,\frac{3\dot{V}}{V} = 0, \tag{6.8}$$

where

$$V^3 = R_1 R_2 R_3 . \tag{6.9}$$

Now we consider the two cases corresponding to vacuum and Zeldovich fluid.

Case I. Vacuum Universe ($\rho = p = 0$)

In this case the equations reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^2}{R_1^2} = 0 , \tag{6.10}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^2}{R_1^2} = 0 , \tag{6.11}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) - \frac{2a^2}{R_1^2} = 0 , \tag{6.12}$$

$$\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3} = \frac{2\dot{R}_1}{R_1} , \tag{6.13}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\dot{\phi}^2 = 0 , \tag{6.14}$$

Since it has been possible to get a general solution of Equations (6.10)–(6.14) we make a simple illustrative investigation of these equations.

We assume

$$R_1 = t^{p_1} , \qquad R_2 = t^{p_2} , \qquad R_3 = t^{p_3} , \qquad \phi = \phi_0 t^L , \tag{6.15}$$

where p_1, p_2, p_3, ϕ_0, and L are constants.

Inserting R_1, R_2, R_3, and ϕ in Equations (6.10)–(6.14) we find they are satisfied only when $p_1 = p_2 = p_3 = 1$ and further (i) $L = 0$ or (ii) $n = -2 \cdot L = 0$ leads to $\phi = \text{const.}$, i.e., general relativity. When $n = -2$, the equations become inconsistent.

Hence, there are no solutions of Equations (6.2)–(6.8) in the vacuum case.

Case II. Zeldovich Fluid ($\rho = p$)

The equations are

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^2}{R_1^2} = 0 , \tag{6.16}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_3}{R_3} + \frac{\dot{R}_1}{R_1}\right) - \frac{2a^2}{R_1^2} = 0 , \tag{6.17}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) - \frac{2a^2}{R_1^2} = 0\,, \tag{6.18}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -2\chi\rho + \omega\phi^n\dot{\phi}^2\,, \tag{6.19}$$

$$\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3} = 2\,\frac{\dot{R}_1}{R_1}\,, \tag{6.20}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,. \tag{6.21}$$

The conservation equation is

$$\dot{\rho} + 6\rho\,\frac{\dot{V}}{V} = 0\,, \tag{6.22}$$

Since we have not been able to find a general solution of Equations (6.16)–(6.22), we shall try a simple illustrative solution. We assume that

$$R_1 = t^{p_1}\,, \qquad R_2 = t^{p_2}\,, \qquad R_3 = t^{p_3}\,, \qquad \phi = \phi_0 t^L\,; \tag{6.23}$$

where p_1, p_2, p_3, ϕ_0, and L are constants.

Then Equations (6.16)–(6.21) are satisfied when

$$p_1 = p_2 = p_3 = 1\,, \qquad (n+2)L = -4\,, \tag{6.24}$$

and

$$\rho = \frac{W\phi_0^{n+2}L^2t^{-6}}{2\chi}\,. \tag{6.25}$$

By use of ρ from (6.25) and R_1, R_2, R_3 from (6.23) Equation (6.22) is satisfied identically.

The dynamical parameters are shear $\sigma = 0$, rotation $\omega = 0$, scalar of expansion

$$\theta = t^{-3}\,;$$

– Hubble parameter

$$H = t^{-1}\,,$$

– deceleration parameter

$$q = 0\,.$$

The integral

$$\int_{t_0}^{t} \frac{\mathrm{d}t'}{V(t_0)} = [\tfrac{1}{2}t'^2]_{t'=t_0}^{t'=t}\,,$$

is finite and, therefore, a horizon exists in this model.

The Ricci scalar R is zero. At $t = 0$, the matter density ρ is infinite. Hence, the model has a singular origin at $t = 0$ and for $t > 0$ the expansion slows down continuously.

7. Bianchi Type-VI_0 Universe

The Bianchi type-VI_0 metric is of the form

$$ds^2 = dt^2 - R_1^2\, dx^2 - R_2^2\, e^{-2a^2 x}\, dy^2 - R_3^2\, e^{2a^2 x}\, dz^2\,, \tag{7.1}$$

where R_1, R_2, R_3 are functions of t only and $a = \text{const.}$

The field equations (2.2) and (2.3) lead to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^4}{R_1^2} = \frac{1}{2}\,\chi(\rho - p)\,, \tag{7.2}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_3}{R_3} + \frac{\dot{R}_1}{R_1}\right) = \frac{1}{2}\,\chi(\rho - p)\,, \tag{7.3}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) = \frac{1}{2}\,\chi(\rho - p)\,, \tag{7.4}$$

$$\frac{\dot{R}_2}{R_2} = \frac{\dot{R}_3}{R_3}\,, \tag{7.5}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -\frac{1}{2}\,\chi(\rho + 3p) + \omega\phi^n\dot{\phi}^2\,, \tag{7.6}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,. \tag{7.7}$$

The conservation equation $T^{ij}{}_{;j}$ is satisfied when

$$\dot{\rho} + 3(\rho + p)\,\frac{\dot{V}}{V} = 0\,, \tag{7.8}$$

where $V^3 = R_1 R_2 R_3$.

Since it has not been possible to find a general solution of Equations (7.2)–(7.7) we attempt an illustrative solution in two cases: viz., (i) vacuum and (ii) Zeldovich fluid.

Case I. Vacuum Universe ($\rho = p = 0$)

We assume that

$$R_1 = t^{p_1}\,, \qquad R_2 = t^{p_2}\,, \qquad R_3 = t^{p_3}\,, \qquad \phi = \phi_0 t^L\,, \tag{7.9}$$

where p_1, p_2, p_3, ϕ_0, L are constants.

If we use (7.9) in Equations (7.2)–(7.8) with $p = \rho = 0$, we find that they are satisfied only when $a = 0$ which reduces the metric (7.1) to the Bianchi type-I metric (3.1) already considered. Hence, there are no solutions of Bianchi type-VI_0 in this case.

Case II. Zeldovich Fluid ($\rho = p$)

In this case Equations (7.2)–(7.8) reduce to

$$\frac{\ddot{R}_1}{R_1} + \frac{\dot{R}_1}{R_1}\left(\frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}\right) - \frac{2a^4}{R_1^2} = 0\,, \tag{7.10}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_2}{R_2}\left(\frac{\dot{R}_3}{R_3} + \frac{\dot{R}_1}{R_1}\right) = 0\,, \tag{7.11}$$

$$\frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_3}{R_3}\left(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2}\right) = 0\,, \tag{7.12}$$

$$\frac{\dot{R}_2}{R_2} = \frac{\dot{R}_3}{R_3}\,, \tag{7.13}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = -2\chi\rho + \omega\phi^n\dot{\phi}^2\,, \tag{7.14}$$

$$\ddot{\phi} + 3\,\frac{\dot{\phi}\dot{V}}{V} + \frac{n}{2\phi}\,\dot{\phi}^2 = 0\,, \tag{7.15}$$

$$\dot{\rho} + 6\rho\,\frac{\dot{V}}{V} = 0\,, \tag{7.16}$$

Using R_1, R_2, R_3, and ϕ from (7.9) we find that the Equations (7.10)–(7.15) are satisfied when

$$p_1 = 1\,, \qquad p_2 = 0\,, \qquad p_3 = 0\,, \qquad n = -2\,. \tag{7.17}$$

Then ρ is given by

$$\rho = \frac{WL^2}{t^2}\,. \tag{7.18}$$

Using (7.9), (7.17), and (7.18), the Equation (7.16) is satisfied identically.

The dynamical parameters are

– shear

$$\sigma^2 = \frac{1}{3t^2}\,;$$

– rotation

$$\omega = 0\,;$$

– scalar of expansion

$$\theta = \frac{1}{3t}\,;$$

– deceleration parameter $q = 2$

$$\frac{\sigma^2}{\rho} = \frac{1}{3WL^2}\,, \qquad \frac{\sigma^2}{\theta} = \frac{1}{3t}\,.$$

– Hubble parameter

$$H = \frac{1}{3t} = \frac{1}{3}\,\theta\,. \tag{7.19}$$

The integral

$$\int_{t_0}^{t} \frac{\mathrm{d}t'}{V(t')} = \frac{1}{3}\,[t'^4]_{t'=t_0}^{t} \tag{7.20}$$

is convergent. Therefore, the model has a horizon.

The Ricci scalar is

$$R = \frac{2a^4}{t^2}\,. \tag{7.21}$$

At $t = 0$, $\rho \rightarrow \infty$. Therefore, the model has a singular origin at $t = 0$ and for $t > 0$ the expansion shows down continuously.

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