ON THE DYNAMICAL PROBLEM OF A GENERALIZED THERMOELASTIC GRANULAR INFINITE CYLINDER UNDER INITIAL STRESS

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Abstract. The object of the present paper is to investigate the influence of initial stress on the waves propagation in a generalized thermoelastic granular medium subjected to the boundary conditions that the outer surface is traction free. In addition, it is subjected to temperature boundary conditions. The wave velocity equation for the generalized thermoelastic granular medium Rayleigh wave under the influence of initial stress has been obtained. The classical result has been derived as a limiting case similar to one which was obtained by Ewing *et al.* (1957).

1. Introduction

The dynamical problem in granular medium of a generalized thermoelastic waves has been studied in recent time. This study has been necessitated by its possible application in soil mechanics, geophysical prospecting, mining engineering, etc. The theoretical outline of the development of the subject from the mid-thirties was given by Paria (1960). The frequency equation of Rayleigh waves in a granular over a granular half-space was given by Bhattacharyya (1965). The present paper focusses on the study of the Rayleigh waves with the models which can be used to investigate related research with the Earth. In general, the surface stratum of the Earth is granular and the base is the generalized thermoelastic solid under large initial stresses due to many causes. Such as weight of the substratum, gravity, creep, and inelastic deformation under temperature inside the Earth.

The granular medium under consideration is a discontinuous one and is composed of numerous large or small grains. Unlike a continuous body, each element or grain translates and also rotates about its centre of gravity. This motion is the characteristic of the medium and has an important effect upon the equation of motion to produce internal friction. It was assumed that the medium contains so many grains that they will never be separated from each other during the deformation and each grain has perfect generalized thermoelasticity.

The initial stresses present in the medium also have considerable effect in the propagation of waves (Biot, 1965).

This paper is devoted to the study of the effect of granular body and also of the initial stress in the propagation of Rayleigh waves. Furthermore, a friction coefficient is introduced for the boundary conditions between the granular and generalized thermoelasticity. The frequency equation has been derived in the form of fourth-order determinant. The roots of this equation are in general complex and the imaginary part of an

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appropriate root measures the attenuation of the waves. It is noted that the frequency equation of Rayleigh waves contains the term which involving the initial stress and so the frequency equation of Rayleigh waves changes with respect to this initial stress. When the initial stress vanishes, the derived frequency equation reduces to that one obtained in classical generalized thermoelasticity granular medium.

2. Formulation of the Problem

The dynamical problem of a generalized thermoelastic granular medium in an infinite cylinder of radius R under initial compressive stress P along the r-direction subjected to certain boundary conditions is studied. The state of deformation in the granular medium is described by the displacement vector $\mathbf{U} = (u_r, 0, u_z)$ of the centre of gravity of a grain and the rotation vector $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ of the grain about its centre of gravity.

The dynamic equations of motion in the absence of body forces can be written (cf. Oshima, 1954) as

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + F\nabla \frac{\partial\boldsymbol{\xi}}{\partial t}\, p\nabla \wedge \boldsymbol{\omega} = \gamma\nabla(T + \tau\dot{T}) + \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2}\,, \tag{1}$$

$$-F\,\frac{\partial\boldsymbol{\xi}}{\partial t} + M\nabla^2(\boldsymbol{\xi} + \nabla\wedge\mathbf{u}) = 0\,; \tag{2}$$

and the generalized heat conduction equation is

$$K\nabla^2 T = \rho c_e(\dot{T} + \tau\ddot{T}) + \alpha(3\lambda + 2\mu)T_0\nabla\cdot(\dot{\mathbf{u}} + \tau\ddot{\mathbf{u}})\,, \tag{3}$$

where τ represents the time lag needed to establish steady-state heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element, and will be called the relaxation time. T is the temperature change about the equilibrium temperature T_0; ρ, the density of the medium; λ and μ, Lamé's constants; M, the third elastic constant; F, the coefficient of friction; α, the coefficient volume expansion; c_e, the specific heat per unit mass at constant strain; K, the thermal conductivity $\gamma = \alpha(3\lambda + 2\mu)$; and $\boldsymbol{\omega} = (0, \omega_\theta, 0)$, the rotation vector.

The stress tensors are non-symmetric: i.e.,

$$\tau_{ij} \neq \tau_{ji} \quad \text{and} \quad M_{ij} \neq M_{ji}\,;$$

τ_{ij} can be expressed as the sum of symmetric and anti-symmetric tensors

$$\tau_{ij} = S_{ij} + S'_{ij}\,, \tag{4}$$

where

$$S'_{ij} = \tfrac{1}{2}(\tau_{ij} - \tau_{ji})\,, \tag{5}$$

$$S_{ij} = \tfrac{1}{2}(\tau_{ij} + \tau_{ji})\,, \tag{6}$$

τ_{rr}, τ_{rz}, $\tau_{\theta\theta}$, τ_{zz}, and $\tau_{\theta r}$, ... are components of the resultant force acting on a surface element. Also, M_{rr}, $M_{r\theta}$, etc., are the components of the resultant couple acting on a surface element.

The symmetric tensor $S_{ij} = S_{ji}$ is related to the symmetric strain tensor

$$e_{ij} = e_{ji} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) . \tag{7}$$

The components of stress in a generalized thermoelastic in an infinite cylinder are given by Biot (1965).

$$\begin{aligned}
S_{rr} &= (\lambda + 2\mu + p) \frac{\partial u_r}{\partial r} + (\lambda + p) \frac{\partial u_z}{\partial z} + (\lambda + p) \frac{u_r}{r} - \frac{\gamma}{x_\theta} (T + \tau T) , \\
S_{\theta\theta} &= (\lambda + 2\mu + p) \frac{u_r}{r} + (\lambda + p) \frac{\partial u_r}{\partial r} + (\lambda + p) \frac{\partial u_z}{\partial z} - \frac{\gamma}{x_\theta} (T + \tau T) , \\
S_{zz} &= (\lambda + 2\mu) \frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_r}{\partial r} + \lambda \frac{u_r}{r} - \frac{\gamma}{x_\theta} (T + \tau T) , \\
S_{rz} &= \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) ;
\end{aligned} \tag{8}$$

where x_θ is the isothermal compressibility.

The anti-symmetric stress S_{ij} are given by

$$\begin{aligned}
&S'_{zr} = -F \frac{\partial \eta}{\partial t} , \qquad S'_{r\theta} = -F \frac{\partial \zeta}{\partial t} , \\
&S'_{\theta z} = -F \frac{\partial \xi}{\partial t} \quad \text{and} \quad S'_{rr} = S'_{\theta\theta} = S'_{zz} = 0 ;
\end{aligned} \tag{9}$$

where F is the coefficient of friction between the individual and t is the time.

The stress couple M_{ij} is given by

$$M_{ij} = M \gamma_{ij} . \tag{10}$$

The non-symmetric strain tensor γ_{ij} is defined as

$$\begin{aligned}
&\gamma_{rr} = \frac{\partial \xi}{\partial r} , \qquad \gamma_{zz} = \frac{\partial \zeta}{\partial z} , \qquad \gamma_{\theta\theta} = \frac{\xi}{r} , \\
&\gamma_{z\theta} = \frac{\partial}{\partial} (\omega_\theta + \eta) , \qquad \gamma_{zr} = \frac{\partial \zeta}{\partial z} , \\
&\gamma_{r\theta} = \frac{\partial}{\partial r} (\omega_\theta + \eta) + \frac{1}{r} (\omega_\theta + \eta) , \\
&\gamma_{rz} = \frac{\partial \zeta}{\partial r} , \qquad \gamma_{\theta r} = 0 ,
\end{aligned} \tag{11}$$

Equations (1) and (2) now yield six equations

$$(\lambda + 2\mu + p)\,\frac{\partial \Delta}{\partial r} + 2\left(\mu + \frac{p}{2}\right)\frac{\partial \omega_\theta}{z} - \frac{\gamma}{x_\theta}\,\frac{\partial}{\partial r}\,(T + \tau \dot{T}) - $$

$$- F\,\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial z}\right) = \frac{\partial^2 u_r}{\partial t^2}\,, \tag{12}$$

$$- F\,\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial r}\right) = 0\,, \tag{13}$$

$$(\lambda + 2\mu)\,\frac{\partial \Delta}{\partial z} - \frac{2}{r}\left(\mu - \frac{P}{2}\right)\frac{\partial}{\partial r}\,(r\omega_\theta) - \frac{\gamma}{x_\theta}\,\frac{\partial}{\partial z}\,(T + \tau \dot{T}) +$$

$$+ \frac{F}{r}\,\frac{\partial^2}{\partial r\,\partial t}\,(r\eta) = \rho\,\frac{\partial^2 u_z}{\partial t^2}\,, \tag{14}$$

$$- F\,\frac{\partial \xi}{\partial t} + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\xi = 0\,, \tag{15}$$

$$- F\,\frac{\partial \eta}{\partial t} + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\left(\eta + \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) = 0\,, \tag{16}$$

$$- F\,\frac{\partial \zeta}{\partial t} + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\zeta = 0\,; \tag{17}$$

where

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\,, \qquad \omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right).$$

By use of Helmholtz's theorem (cf. Morse and Feshbach, 1953) and introducing the potential Φ and ψ by the equation

$$\mathbf{U} = \operatorname{grad}\Phi + \operatorname{curl}(0, \psi, 0)\,. \tag{18}$$

From Equations (12), (14), and (18) we get the equations

$$\nabla^2\Phi = \frac{\rho}{(\lambda + 2\mu + P)}\,\frac{\partial^2\Phi}{\partial t^2} + \frac{\gamma(T + \tau T)}{x_\theta(\lambda + 2\mu + P)}\,, \tag{19}$$

$$\nabla^2\Phi = \frac{\rho}{(\lambda + 2\mu)}\,\frac{\partial^2\Phi}{\partial t^2} + \frac{\gamma(T + \tau T)}{x_\theta(\lambda + 2\mu)}\,, \tag{20}$$

$$\nabla^2\psi = \frac{\rho}{\pi + \frac{1}{2}P}\,\frac{\partial^2\psi}{\partial t^2} - \frac{F}{\mu + \frac{1}{2}P}\,\frac{\partial \eta}{\partial t}\,, \tag{21}$$

$$\nabla^2 \psi = \frac{\rho}{\mu - \frac{1}{2}P} \frac{\partial^2 \psi}{\partial t^2} - \frac{F}{\mu - \frac{1}{2}P} \frac{\partial \eta}{\partial t}, \tag{22}$$

respectively.

These equations differ in form and in number from those of the classical theory where there is no initial stress. This is due to the fact that longitudinal and shear wave velocities are not the same in all directions when the medium is subjected to non-hydrostatic initial stress. Since the initial stress has been taken in the direction of r only, the velocity of body waves will be different in r- and z-directions. In the absence of P, Equations (19), (20), (21), and (22) have been reduced to two equations only (19), (22). Now Equations (19) and (20) represent the compressive wave along the r- and z-directions, respectively, and Equations (21) and (22) represent the shear wave along those directions, respectively. Equation (19) represents the longitudinal wave in the direction of r with velocity

$$c_1 = \left(\frac{\lambda + 2\mu + P}{\rho}\right)^{1/2},$$

Equation (22) represents the velocity of the shear wave in the direction of r with velocity

$$c_2 = \left(\frac{\mu - P/2}{\rho}\right)^{1/2}.$$

Equation (20) represents the longitudinal wave in the direction of z with velocity

$$\alpha_1 = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}$$

and Equation (21) represents the shear wave in the direction of z with velocity

$$\alpha_2 = \left(\frac{(\mu + P/2)}{\rho}\right)^{1/2}.$$

In the following discussion compressional and distortional waves along the r-axis are only considered. These waves are represented by Equations (19) and (22), and the generalized heat conduction equation is given by

$$k\nabla^2 T = \rho c_e (T + \tau T) + \alpha(3\lambda + 2\mu) T_0 \nabla^2 \left(\frac{\partial \Phi}{\partial t} + \tau \frac{\partial^2 \Phi}{\partial t^2}\right). \tag{23}$$

Assuming a simple harmonic time-dependent factor $\exp(iwt)$, Equations (15)–(17) and (19)–(23) yield a set of differential equations for ξe^{iwt}, ηe^{iwt}, ζe^{iwt}, Φe^{iwt}, ψe^{iwt}, and Te^{iwt}, i.e.,

$$-iwF\xi + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\xi = 0, \tag{24}$$

$$- iwF\eta + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) \times$$

$$\times \left(\eta + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2}\right) = 0\,, \tag{25}$$

$$- iwF\zeta + M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\zeta = 0\,, \tag{26}$$

$$\nabla^2 \Phi = -\frac{\rho^2}{c_1^2}\,\Phi + \frac{\gamma T(1 + iw\tau)}{x_\theta \rho c_1^2}\,, \tag{27}$$

$$\nabla^2 \psi = -\frac{\rho^2}{c_2^2}\,\psi - \frac{i\mu F\eta}{\rho c_2^2}\,, \tag{28}$$

$$\nabla^2 T = \frac{\rho c_e iw}{k}\,T(1 + iw\tau) + \frac{\gamma T_0}{k}\,\nabla^2\Phi(iw(1 + iw\tau))\,. \tag{29}$$

If we eliminate T between Equations (27) and (29) by substituting Equation (27) into Equation (29) we get

$$\nabla^4\Phi + \left[\frac{\omega^2}{c_1^2} - \frac{\rho c_e iw\tau}{k}\,(1 + \varepsilon\tau)\right]\nabla^2\Phi - \frac{iw^3 \rho c_e \tau\Phi}{kc_1^2} = 0\,, \tag{30}$$

where

$$\varepsilon = \frac{\gamma^2}{\rho^2 c^2 c_e x_\theta}\,, \quad \tau = 1 + iw\tau\,.$$

Also, η can be eliminated by use of Equations (28) and (25). Finally we get the equation of ψ as

$$\left(\nabla^2 - \frac{1}{r^2}\right)^2 \psi - \left[\frac{i\omega\rho c_2^2 F - M\omega^2\rho}{iwFM - \rho c_2^2 M}\right]\left(\nabla^2 - \frac{1}{r^2}\right)\psi - \frac{iw^3\rho F\psi}{iwFM + \rho c_2^2 M} = 0\,. \tag{31}$$

3. Solution of the Problem

General solution of Equations (24), (26), (30), and (31) can be found. Inversion of Hankel transform has been introduced as

$$\Phi(r, z, w) = \int_0^\infty \Phi(\eta, z, w) J_0(\eta r)\eta\, \mathrm{d}\eta\,. \tag{32}$$

If we substitute from Equation (32) into Equation (29) we get

$$\left(\eta^2 - \frac{\partial^2}{\partial z^2}\right)\left(\eta^2 - \frac{\partial^2}{\partial z^2}\right)\Phi - \left[\frac{w^2}{c_1^2} - \frac{\rho c_e iw\tau}{k}(1+\varepsilon\tau)\right] + \\ + \left(\eta^2 - \frac{\partial^2}{\partial z^2}\right)\Phi - \frac{iw^3\rho c_e\tau}{kc_1^2}\Phi = 0 . \quad (33)$$

The indicial equation governing Equation (33) is

$$f^4 = \left[\frac{w^2}{c_1^2} - \frac{\rho c_e iw\tau}{k}(1+\varepsilon\tau)\right] f^2 - \frac{iw^3\rho c_e\tau}{kc_1^2} = 0 . \quad (34)$$

If $\varepsilon = 0$, then the roots of Equation (34) take the form

$$f_1^{*2} = \frac{w^2}{c_1^2} \quad \text{and} \quad f_2^{*2} = \frac{\rho c_e w^2\tau}{k} - \frac{\rho c_e iw}{k} , \quad (35)$$

$$f_1^2 = \frac{w^2}{c_1^2} - \frac{\rho c_e iw^3\tau[w^2k - \rho c_e w^2 c_1^2 k\tau - \rho c_e iwc_1^2 k]}{(w^2k - \rho c_e w^2 c_1^2)^2 + \rho^2 c_1^2 c_2^2 w^2}\varepsilon , \quad (36)$$

$$f_2^2 = -\frac{\rho c_e iw\tau}{k} + \left[\frac{\rho^2 c_e^2 c_1^2 w^2\tau^3[w^2k^2 - \rho c_e w^2 c_1^2 k\tau - \rho c_e iwc_1^2 k]}{(w^2k^2 - \rho c_e w^2 c_1^2 k\tau)^2 - \rho^2 c_e^2 w^2 c_1^2 k^2}\right]\varepsilon . \quad (37)$$

Moreover, let us set

$$\xi_j^2 = \eta^2 - f_j^2 , \qquad R_e(\xi_j) \geq 0 \quad \text{and} \quad j = 1, 2 .$$

Hence, the solution of Equation (33) is of the form

$$\Phi(r, z, w) = \int_0^\infty [A(\eta)\, e^{-\xi_1 z + iwt} + B(\eta)\, e^{-\xi_2 z + iwt}] j_0(\eta r)\eta \,\mathrm{d}\eta . \quad (38)$$

Similarly we can obtain the indicial equation governing Equation (31) which takes the form

$$\lambda^4 + \left[\frac{iw\rho c_2^2 F - Mw^2\rho}{iwFM + \rho c_2^2 M}\right]\lambda^2 + \frac{iw^3\rho F}{iwFM + \rho c_2^2 M} = 0 . \quad (39)$$

For $F = 0$, the roots of Equation (39) are

$$\lambda_1^{*2} = \frac{w^2}{c_2^2} , \qquad \lambda_2^{*2} = 0 . \quad (40)$$

Also, for $F \ll 1$, taking only the first-order terms, then the roots of Equation (39) can be found in the form

$$\lambda_1^2 = \frac{w^2}{c_2^2} - \left[\frac{iw\rho c_2^2 + iw^3\rho M + (iw\rho c_2^2 + iw^3\rho M + 2\rho c_2^2)\rho c_2^2}{2M\rho c_2^2}\right] F,$$

$$\lambda_2^2 = -\left[\frac{iw\rho c_2^2 + iw^3\rho M + (iw\rho c_2^2 + i\rho w^3 M + 2\rho c_2^2 iw)\rho c_2^2}{2M\rho c_2^2}\right] F. \tag{41}$$

Putting $\delta_j^2 = \eta^2 - \lambda_j^2$, $R_e(\delta_j) \geq 0$ and $j = 1, 2$, hence, the solution of Equation (31) is

$$\psi(r, z, t) = \int_0^\infty [c(\eta)\, e^{-\delta_2 z + iwt} + D(\eta)\, e^{-\delta_2 z + iwt}] J(\eta r)\eta\, \mathrm{d}\eta\,. \tag{42}$$

Also, the solution of Equations (24) and (26) can be found as

$$\xi = \int_0^\infty E(\eta)\, e^{-\xi_3 z + iwt} J_1(\eta r)\eta\, \mathrm{d}\eta\,, \tag{43}$$

$$\zeta = \int_0^\infty G(\eta)\, e^{-\xi_3 z + iwt} J_0(\eta r)\eta\, \mathrm{d}\eta\,; \tag{44}$$

where

$$\xi_3^2 = \eta^2 + \frac{iwf}{M}\,.$$

Also, the temperature deviation T, can be obtained by substituting Equation (38) into Equation (27)

$$T(r, z, t) = \frac{\rho x_\theta}{\gamma\tau} \int_0^\infty [A(\eta)\,(w^2 - c_1^2 f_1^2)\, e^{-\xi_1 z + iwt} + $$
$$+ B(\eta)\,(w^2 - c_1^2 f_2^2)\, e^{-\xi_2 z + iwt}] J_0(\eta r)\eta\, \mathrm{d}\eta\,. \tag{45}$$

By substituting Equation (42) into Equation (28) to obtain the value of $\eta(r, z, t)$ which can be written as

$$\eta(r, z, t) = \frac{\rho c_2^2}{iwF} \int_0^\infty \left[c(\eta)\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right) e^{-\delta_1 z + iwt} + \right.$$
$$\left. + D(\eta)\left(\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2}\right) e^{-\delta_2 z + iwt}\right] J_0(\eta r)\eta\, \mathrm{d}\eta\,. \tag{46}$$

The stress τ_{rr}, τ_{rz}, and a secondary stress couple M_{rr}, M_{rz}, $M_{z\theta}$ are given by

$$\tau_{rr} = (\lambda + P)\nabla^2\Phi + 2\mu \frac{\partial^2\psi}{\partial r^2} - 2\mu \frac{\partial^2\psi}{\partial r\,\partial z} - \frac{\gamma}{x_\theta}(T + \tau\dot{T}),$$

$$\tau_{rz} = \mu\left(2\frac{\partial^2\Phi}{\partial r\,\partial z} - \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2}\right) - F\frac{\partial\eta}{\partial t},$$

$$M_{rr} = M\gamma_{rr} = M\frac{\partial\xi}{\partial r},$$

$$M_{rz} = M\gamma_{rz} = M\frac{\partial\zeta}{\partial r},$$

$$M_{z\theta} = M\gamma_{z\theta} = M\frac{\partial}{\partial z}\left[\eta - \frac{1}{2}\left(\frac{\partial^3\psi}{\partial r\,\partial z^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2}\right)\right]. \tag{47}$$

Substituting from Equations (38), (42), (43), (44), (45), and (46) into Equations (21) and (47) the components of displacement and stress can be deduced as

$$U_r = -\int_0^\infty \Big[[A(\eta)\,e^{-\xi_1 z + iwt} + B(\eta)\,e^{-\xi_2 z + iwt}]\,\eta^2 J_1(\eta r) - - [\delta_1 c(\eta)\,e^{-\delta_1 z + iwt} + \delta_2 D(\eta)\,e^{-\delta_2 z + iwt}]\,\eta j_1(\eta r)\Big]\,\mathrm{d}\eta,$$

$$U_z = -\int_0^\infty \Big[[A(\eta)\zeta_1\,e^{-\xi_1 z + iwt} + B(\eta)\xi_2\,e^{-\xi_2 z + iwt}]\,\eta J_0(\eta r) - - [c(\eta)\,e^{-\delta_1 z + iwt} + D(\eta)\,e^{-\delta_2 z + iwt}]\,\eta^2 J_0(\eta r)\Big]\,\mathrm{d}\eta,$$

$$\tau_{rr} = \int_0^\infty \Big[A(\eta)\,(\eta j_0(\eta r)\,(\lambda + P)\,(\xi_1^2 - \eta^2) + + \frac{2\mu\eta^2}{r}J_1(\eta r) - 2\mu\eta^3 J_0(\eta r) - \rho(w^2 - c_1^2 f_1^2)\Big]\,e^{-\xi_1 z + iwt} +$$

$$+ B(\eta)\Big[\eta J_0(\eta r)\,(\lambda + P)\,(\xi_2^2 - \eta^2) + \frac{2\mu\eta^2}{r}j_1(\eta r) - - 2\mu\eta^3 J_0(\eta r) - \rho(w^2 - c_1^2 f_2^2)\Big]\,e^{-\xi_2 z + iwt} +$$

$$+ c(\eta)\left[2\mu\eta\delta_1(\eta J_0(\eta r) - \frac{1}{r} J_1(\eta r)\right] e^{-\delta_1 z + iwt} +$$

$$+ D(\eta)\left[2\mu\eta\delta_2(nJ_0(\eta r) - \frac{1}{r} J_1(\eta r))\right] e^{-\delta_2 z + iwt}] \, d\eta ,$$

$$\tau_{rz} = \int_0^\infty \Big\{ A(\eta)\mu\zeta_1\eta^2 J_1(\eta r)\, e^{-\xi_1 z + iwt} +$$

$$+ B(\eta)\mu\xi_2\eta^2 J_1(\eta r)\, e^{-\xi_2} J_1(\eta r)\, e^{-\xi_2 z + iwt} -$$

$$- c(\eta)\left[\mu(\delta_1^2 + \eta^2) + \rho c_2^2\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right)\right] \times$$

$$\times e^{-\delta_1 z + iwt} J_1(\eta r)\eta - D(\eta)\Big[\mu(\delta_2^2 + \eta^2) +$$

$$+ \rho c_2^2\left(\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2}\right)\Big] e^{-\delta_2 z + iwt} J_1(\eta r)\eta \Big\} \, d\eta ,$$

$$M_{rr} = M \int_0^\infty E(\eta)\left[J_0(\eta r) - \frac{1}{\eta r} J_1(\eta r)\right] e^{-\xi_3 z + iwt}\eta^2 \, d\eta ,$$

$$M_{r\theta} = -M \int_0^\infty G(\eta)\,[\xi_3\, e^{-\xi_3 z + iwt} J_0(\eta r)\eta] \, d\eta ,$$

$$M_{z\theta} = M \int_0^\infty \Big\{ c(\eta)\Big[\frac{1}{2}\left(\delta_1^3\eta^3(J_0(\eta r) - \frac{1}{\eta r} J_1(\eta r)\right) -$$

$$- \delta_1\eta^3 J_1(\eta r) - \frac{\rho c_2^2 \delta_1}{iwF}\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right) J_1(\eta r)\eta\Big] \times$$

$$\times e^{-\delta_1 z + iwt} + D(\eta)\Big[\frac{1}{2}\left(\delta_2^3\eta^2\left(J_0(\eta r) - \frac{1}{\eta r} J_1(\eta r)\right) -\right.$$

$$\left. - \delta_2\eta^3 J_1(\eta r)\right) - \frac{\rho c_2^2\delta_2}{iwF}\left(\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2}\right) J_1(\eta r)\eta\Big] \times$$

$$\times e^{-\delta_2 z + iwt}\Big\} \, d\eta . \tag{48}$$

4. Frequency Equation

In this section frequency equation for the boundary conditions which specify that the outer surface of the cylinder is traction-free is obtained: i.e.,

$$
\begin{aligned}
&\text{(i)} && \tau_{rr} = 0 \\
&\text{(ii)} && \tau_{rz} = 0 \\
&\text{(iii)} && M_{rr} = 0 \\
&\text{(iv)} && M_{rz} = 0 \\
&\text{(v)} && M_{z\theta} = 0 \quad \text{at} \quad r = R\,,
\end{aligned}
\tag{49}
$$

and the temperature boundary condition is

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = R\,. \tag{50}$$

Boundary conditions (iii) and (iv) give, respectively, $G(\eta) = F(\eta) = 0$ and by eliminating constants $A(\eta)$, $B(\eta)$, $c(\eta)$, and $D(\eta)$ by substituting Equations (45) and (48) into the boundary conditions (49) and (50), the frequency equation is given in a form of fourth order determinant as

$$\begin{vmatrix} n_1 & n_2 & n_3 & n_4 \\ n_5 & n_6 & n_7 & n_8 \\ 0 & 0 & n_9 & n_{10} \\ n_{11} & n_{12} & 0 & 0 \end{vmatrix} = 0\,, \tag{51}$$

where

$$n_1 = \left[\eta J_0(\eta R)\,(\lambda + P)\,(\xi_1^2 - \eta^2) + \frac{2\mu\eta^2}{R}\, J_1(\eta R) - \right.$$

$$\left. - 2\mu\eta^3 J_0(\eta R) - \rho(w^2 - c_1^2 f_1^2) \right],$$

$$n_2 = \left[\eta J_0(\eta R)\,(\lambda + P)\,(\xi_2^2 - \eta^2) + \frac{2\mu\eta^2}{R}\, J_1(\eta R) - \right.$$

$$\left. - 2\mu\eta^3 J_0(\eta R) - \rho(w^2 - c_1^2 f_1^2) \right],$$

$$n_3 = 2\mu\eta\delta_1 \left[\eta J_0(\eta R) - \frac{1}{R}\, J_1(\eta R) \right],$$

$$n_4 = 2\mu\eta\delta_2\left[\eta J_0(\eta R) - \frac{1}{R}\, J_1(\eta R)\right],$$

$$n_5 = \xi_1\eta^2 J_1(\eta R),$$

$$n_6 = \xi_2\eta^2 J_1(\eta R),$$

$$n_7 = -\left[\delta_1^2 + \eta^2 + \frac{\rho c_2^2}{\mu}\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right)\right]\eta J_1(\eta R),$$

$$n_8 = -\left[\delta_2^2 + \eta^2 + \frac{\rho c_2^2}{\mu}\left(\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2}\right)\right]\eta J_1(\eta R),$$

$$n_9 = \left[\frac{1}{2}\left(\delta_1^2\eta^2(J_0(\eta R) - \frac{1}{\eta R}\, J_1(\eta R)\right) - \delta_1\eta^3 J_1(\eta R) - \right.$$

$$\left. - \frac{\rho c_2^2}{iwF}\,\delta_1\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right) J_1(\eta R)\right],$$

$$n_{10} = \left[\frac{1}{2}\left(\delta_2^2\eta^2(J_0(\eta R) - \frac{1}{\eta R}\, J_1(\eta R)\right) - \delta_2\eta^3 J_1(\eta R) - \right.$$

$$\left. - \frac{\rho c_2^2}{iwF}\,\delta_2\left(\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2}\right) J_1(\eta R)\right],$$

$$n_{11} = (w^2 - c_1^2 f_1^2), \qquad n_{12} = (w^2 - c_1^2 f_2^2).$$

The transcendental equation (51), in the determinant form, represents the required wave velocity equation of granular generalized thermoelastic medium under initial stress P. It can be seen that Equation (51) has complex roots. The real part gives the velocity and the imaginary part gives the attenuation due to the granular nature of the medium. However, if the coupling factor ε and the coefficient of friction F are assumed to be small, approximate solution to this equation can be found. It is clear from this frequency equation (51) that the phase velocity depends on initial stress in granular generalized thermoelastic medium. Also, Equation (51) is the frequency equation for Rayleigh waves in granular generalized thermoelastic medium under initial stress.

The frequency equation (51) contains the coefficient of friction and initial stress, but in absence of initial stress, the frequency equation of granular generalized thermoelastic medium in infinite cylinder has an expression similar to that which have been obtained by Elnaggar and Abd-Alla (1991). Also, if the granular rotations are ignored, the frequency equation, for Rayleigh waves of a generalized thermoelastic medium has formula which is similar to that which have been obtained by Elnaggar and Abd-Alla (1987). In addition, in absence of initial stress and no coupling between the temperature and the strain fields (i.e., $P = 0$) and ε vanish, one can get the frequency equation of a granular generalized thermoelastic medium similar to that obtained by Oshima (1954) and Bhattacharya (1965). Then Equation (51) take, the form:

$$\begin{vmatrix} n_1 & n_2 & n_3 & n_4 \\ n_5 & n_6 & n_7 & n_8 \\ 0 & 0 & n_9 & n_{10} \\ n_{11} & n_{12} & 0 & 0 \end{vmatrix} = 0\,, \tag{52}$$

where

$$n_1 = \left[\eta J_0(\eta R)\lambda(\xi_1^2 - \eta^2) + \frac{2\mu\eta^2}{R} J_1(\eta R) - \right.$$

$$\left. - 2\mu\eta^3 J_0(\eta R) - \rho(w^2 - c_1^2 f_1^{*2}) \right],$$

$$n_2 = \left[\eta J_0(\eta R)\lambda(\eta_2 - \eta^2) + \frac{2\mu\eta^2}{R} J_1(\eta R) - \right.$$

$$\left. - 2\mu\eta^3 J_0(\eta R) - \rho(w^2 - c_1^2 f_2^{*2}) \right],$$

$$n_3 = 2\mu\delta_1 \left[\eta J_0(\eta R) - \frac{1}{R} J_1(\eta R) \right],$$

$$n_4 = 2\mu\delta \left[\eta J_0(\eta R) - \frac{1}{R} J_1(\eta R) \right],$$

$$n_5 = \xi_1 \eta^2 J_1(\eta R)\,, \qquad n_6 = \xi \eta^2 J_1(\eta R)\,,$$

$$n_7 = - \left[\delta_1^2 + 2\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2} \right] \eta J_1(\eta R)\,,$$

$$n_8 = - \left[\delta_2^2 + 2\eta^2 - \delta_2^2 - \frac{w^2}{c_2^2} \right] \eta J_1(\eta R)\,,$$

$$n_9 = \left[\frac{1}{2} \left(\delta_1^3 \eta^2 \left(J_0(\eta R) - \frac{1}{\eta R} \eta J_1(\eta R) \right) - \delta_1 \eta^3 J_1(\eta R) \right) + \right.$$

$$\left. + \frac{i\mu\delta_1}{wF} \left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2} \right) J_1(\eta R)\eta \right],$$

$$n_{10} = \left[\frac{1}{2}\left(\delta_2^3\eta^2\left(J_0(\eta R) - \frac{1}{\eta R}\,J_1(\eta R)\right) - \delta_2\eta^3 J_1(\eta R)\right) + \right.$$

$$\left. + \frac{i\mu\delta_2}{wF}\left(\eta^2 - \delta_1^2 - \frac{w^2}{c_2^2}\right)J_1(\eta R)\eta\right],$$

$$n_{11} = w^2 - c_1^2 f_1^{*2}, \qquad n_{12} = w^2 - c_1^2 f_2^{*2},$$

$$\xi_1^2 = \eta^2 - f_1^{*2}, \qquad \xi_2^2 = \eta^2 - f_2^{*2},$$

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho},$$

$$\delta_1^2 = \eta^2 - \lambda_1^{*2}, \qquad \delta_2^2 = \eta^2 - \lambda_2^{*2}. \tag{53}$$

It is clear that Equation (53) is the familiar frequency equation of Rayleigh waves of a granular generalized thermoelastic medium. Also, if the initial stress and granular rotations vanish, the frequency equation should reduce to the classical frequency equation as obtained by Ewing (1957).

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