SPHERICALLY-SYMMETRIC GRAVITATIONAL SOURCES OF PURELY ELECTROMAGNETIC ORIGIN

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(Received 26 July, 1990)

Abstract. The present paper is a continuation of our earlier work on gravitational sources of purely electromagnetic origin, known in the literature as electromagnetic mass models. Here we have shown that a bounded (regular) interior static spherical-symmetric charged dust, if exists, can only be of electromagnetic origin.

1. Introduction

Sources of purely electromagnetic origin (known as electromagnetic mass models) wherein the gravitational mass is dependent on the electromagnetic field alone have been investigated by many authors (cf. Tiwari *et al.*, 1984; Gautreau, 1985; Grøn, 1985, 1986a, b; Ponce de Leon, 1987, 1988), for the static spherically-symmetric perfect fluid distribution corresponding to the line-element

$$ds^2 = e^{\nu}\,dt^2 - e^{\lambda}\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \qquad (1.1)$$

by imposing the equation of state

$$\rho + p = 0\,, \quad (\rho > 0, p < 0)\,. \qquad (1.2)$$

Equation (1.2) implies that the matter under consideration is in tension. Conceptually, in view of the contemporary developments of the knowledge in physics, in our opinion, such models are not physically unrealistic. However, with a view to know whether there exist sources of purely electromagnetic origin which are not under tension, as a continuation of our previous work, we have recently (cf. Tiwari *et al.*, 1991) investigated static axially-symmetric interior charged dust distribution corresponding to the line element of Levi-Civita and have shown that there do exist some axially-symmetric regular interior charged dust distributions of purely electromagnetic origin which are not under tension. One of the interior solutions obtained in this paper, by applying coordinate transformations, is ultimately expressible in the spherically-symmetric form. This solution corresponds to the source of the well-known Curzon field which, according to Curzon, is the exterior field of an electron. The reason for calling the interior solution to be the source of this Curzon field is two-fold: first, both (the interior and the exterior solutions) are initially static and axisymmetric and then are transformed to spherically-symmetric form. Secondly, both the solutions are of purely electromagnetic origin, i.e., the interior and the exterior, both the space-times become flat when the charge density vanishes.

Astrophysics and Space Science **180**: 143–149, 1991.

The static spherically-symmetric form of the solution obtained above (although initially axisymmetric) obviously suggests that we must investigate in detail the static spherically-symmetric charged dust interiors as well.

In this paper, therefore, we have extended this work to the case of static spherically-symmetric charged dust interiors and have shown that a regular (bounded) interior static spherically-symmetric charged dust, if exists, can only be of purely electromagnetic origin. The result has been verified by examining some of the already static spherically-symmetric available charged dust solutions.

2. Field Equations

The Einstein–Maxwell field equations for the case of charged dust distribution are given by

$$G^i_j = R^i_j - \tfrac{1}{2}g^i_j R = -8\pi[T^{i(m)}_j + T^{i(em)}_j], \tag{2.1}$$

$$(\sqrt{-g}\,F^{ij})_{,j} = 4\pi\sqrt{-g}\,J^i \tag{2.2}$$

and

$$F_{[ij,k]} = 0, \tag{2.3}$$

where the matter and the electromagnetic energy-momentum tensors are given by

$$T^{i(m)}_j = \rho u^i u_j \tag{2.4}$$

and

$$T^{i(em)}_j = \frac{1}{4\pi}[-F_{jk}F^{ik} + \tfrac{1}{4}g^i_j F_{kl}F^{kl}]. \tag{2.5}$$

Here ρ is the proper energy density and u_i is the four-velocity of the matter satisfying the relation

$$u_i u^i = 1. \tag{2.6}$$

The electromagnetic field tensor F_{ij} and the current-four vector J^i are defined as

$$F_{ij} = A_{i,j} - A_{j,i} \tag{2.7}$$

and

$$J^i = \sigma u^i, \tag{2.8}$$

where A_i is the four-potential and σ is the charge density.

In view of the above, the Einstein–Maxwell field equations for static spherically-symmetric charged dust corresponding to the metric (1.1) (cf. Tiwari *et al.*, 1984) are

$$e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi\rho + E^2, \tag{2.9}$$

$$e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = -E^2, \tag{2.10}$$

$$e^{-\lambda}\{(\nu''/2 - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r\} = E^2\,, \tag{2.11}$$

$$[r^2E]' = 4\pi r^2\sigma e^{\lambda/2}\,, \tag{2.12}$$

where $E(r)$, which is analogous to the electric field strength, is defined as

$$E = -e^{-(\nu+\lambda)/2}\varphi' \tag{2.13}$$

and where $F_{01} = -F_{10} = \varphi'$. We have the single component A_0 of the four-potential $A_i = [(\varphi, 0, 0, 0), i = 0, 1, 2, 3]$, as the only surviving component of the field tensor F_{ij}. Here a prime denotes differentiation with respect to r only.

3. Static Spherically-Symmetric Charged Dust Electromagnetic Mass Models

We now assume the charged dust to be confined within a sphere of radius a. Equation (2.12) can also be written in the integral form as

$$E(r) = (1/r^2)\int_0^r 4\pi r^2\sigma(r)e^{\lambda/2}\,\mathrm{d}r = q(r)/r^2\,, \tag{3.1}$$

where $q(r)$ is the total charge within a sphere of radius r.

We also add the equation of continuity $T^i_{j;\,i} = 0$ – viz.,

$$\rho\nu' = (1/4\pi r^4)\,[q^2]' \tag{3.2}$$

to the set of equations for the later convenience (it is also derivable from (2.9) to (2.12)).

From Equation (2.9), we get

$$e^{-\lambda} = 1 - 2M(r)/r\,, \tag{3.3}$$

where

$$M(r) = 4\pi\int_0^r \left(\rho + \frac{E^2}{8\pi}\right)r^2\,\mathrm{d}r\,. \tag{3.4}$$

If we add (2.9) and (2.10), we get

$$e^{-\lambda}(\lambda' + \nu') = 8\pi r\rho\,. \tag{3.5}$$

If we now assume the charge density σ to be zero then from (3.10) we get $E = q = 0$. Thus, from (3.2) we have either (i) $\rho \neq 0,\ \nu' = 0$ or (ii) $\rho = 0,\ \nu' \neq 0$, or (iii) $\rho = \nu' = 0$ (which is quite trivial). In the first case, viz., when $\nu' = 0$ ($\rho \neq 0$), from (2.11) we get λ to be constant which when substituted in (3.5) makes ρ equal to zero and the space-time becomes flat. Similarly, in the second case when $\rho = 0$ ($\nu' \neq 0$), from (3.3) we get λ equal to zero which when used in (3.5) gives ν to be a constant. The space-time becomes flat again. Thus the physical quantities like proper energy density ρ, the effective gravitational mass m, etc., become dependent only on the charge density and vanish when this charge density vanishes. The metric potentials also become constant and the space-time becomes flat. The models survives only when the charge density is non-zero. The static charged dust spherically-symmetric model described by the metric (1.1) is, therefore, of purely electromagnetic origin. We may state the result as follows:

THEOREM

An isolated (bounded) continuous static spherically-symmetric charged dust distribution corresponding to the metric (1.1), if exists, can only be of electromagnetic origin.

In view of this theorem a large number of solutions obtained for bounded static spherically-symmetric charged dust distributions require further investigation. Here we are going to re-examine some of these already available solutions.

(i) *Florides Charged Dust Solution*

More than a decade ago, Florides (1977) obtained a class of solutions for static spherically-symmetric charged dust. The solutions are

$$e^{\nu} = \left[1 - \frac{mr^{n+2}}{a^{n+3}}\right]^{-2/(n+2)}, \tag{3.6}$$

$$e^{-\lambda} = \left[1 - \frac{mr^{n+2}}{a^{n+3}}\right]^{2}, \tag{3.7}$$

$$q = m(r/a)^{n+3}, \tag{3.8}$$

$$\sigma = \pm\rho = \sigma_0 (r/a)^n \left[1 - \frac{mr^{n+2}}{a^{n+3}}\right]. \tag{3.9}$$

By use of Equation (3.4), the total gravitational mass of the charged sphere of radius a is given by

$$m = 4\pi\sigma_0 a^3/(n+3), \quad n \gtrsim 0. \tag{3.10}$$

From (3.6)–(3.10), it is evident that all the physical quantities – viz., the total gravitational mass, etc., vanish (and also the space-time becomes flat) when the charge density vanishes. The solution, therefore, satisfies the criteria of being of purely electromagnetic origin.

(ii) *Charged Dust Solution of Pant and Sah*

The class of solutions obtained by Pant and Sah (1979) for the static spherically-symmetric charged dust is as

$$e^{\nu} = Ar^{2n}, \tag{3.11}$$

$$e^{-\lambda} = (n+1)^{-2}, \tag{3.12}$$

$$q = \left(\frac{n}{n+1}\right) r, \tag{3.13}$$

$$\sigma = \pm\rho = \frac{n}{4\pi r^2(n+1)^2}, \tag{3.14}$$

where

$$A = r^{-2n}(1 - 2m/r + q^2/r^2). \tag{3.14a}$$

The total gravitational mass for this solution is given by

$$m = \left(\frac{n}{n+1}\right)a. \tag{3.15}$$

It may be verified that $\sigma = 0$ implies $n = 0$ which in turn makes the total gravitational mass zero. Hence, the model is of purely electromagnetic origin.

(iii) *Tikekar's Solution*

The spherically-symmetric charged dust solution obtained by Tikekar (1984) is given by

$$e^{\nu} = B^2\left[\frac{\{[1 - k(r^2/R^2)]^{1/2} + \sqrt{k}\,[1 - (r^2/R^2)]^{1/2}\}^{\sqrt{k}}}{[1 - k(r^2/R^2)]^{1/2} + [1 - (r^2/R^2)]^{1/2}}\right]^2, \tag{3.16}$$

$$e^{-\lambda} = \frac{1 - (r^2/R^2)}{1 - k(r^2/R^2)}, \tag{3.17}$$

$$E^2 = q^2/r^4 = (1/r^2)\left[1 - \frac{[1 - r^2/R^2]^{1/2}}{[1 - k(r^2/R^2)]^{1/2}}\right]^2, \tag{3.18}$$

$$\sigma = \pm\rho = (1/4\pi r^2)\left[\frac{[1 - r^2/R^2]^{1/2}}{[1 - k(r^2/R^2)]^{1/2}} - \frac{1 - (r^2/R^2)}{1 - k(r^2/R^2)}\right] + $$
$$+ (1/4\pi R^2)\left(\frac{1-k}{[1 - k(r^2/R^2)]^2}\right), \tag{3.19}$$

where

$$B^2 = \frac{1 - (a^2/R^2)}{1 - k(a^2/R^2)}\left[\frac{[1 - k(a^2/R^2)]^{1/2} + [1 - (a^2/R^2)]^{1/2}}{\{[1 - k(a^2/R^2)]^{1/2} + \sqrt{k}\,[1 - (a^2/R^2)]^{1/2}\}\sqrt{k}}\right]^2.$$

The total gravitational mass for this solution is

$$m = \frac{(1-k)a^3}{2[1 - k(a^2/R^2)]R^2} + \frac{q^2}{2a}, \quad 0 \leq k < 1. \tag{3.20}$$

The above solution is not fully regular in the interior of the distribution (the charge density σ, the proper energy density ρ, etc., are not defined at the origin $r = 0$). We can, however, find a limiting value of σ (or ρ), which from (3.19), is given by $k = 1 - (8\pi\sigma_0 R^2)/3$ (σ_0 is the limiting value of σ as $r \to 0$). Redefining the charge density σ, etc., by taking their limiting value as the actual value, we make the solution regular in the interior of the distribution (including at the origin).

Now from (3.21) it can be easily seen that when σ_0 vanishes, k becomes unity. This, in turn, makes all the physical quantities to vanish and the space-time flat. The model is, therefore, of purely electromagnetic origin.

4. Concluding Remarks

(1) We may note here that a static finite (bounded) spherically-symmetric charged dust exists only when the charge density is non-zero. In the absence of charge the effective gravitational mass becomes zero and the space-time becomes flat (i.e., there do not exist any neutral bounded dust spheres). This, in a way, means that the charge is the primary source which generates the model and the dust (associated with the charge), therefore, does not possess the character of the normal matter. It is in this spirit that this model is called as the source of purely electromagnetic origin.

(2) All the static spherically-symmetric bounded charged dust distributions obtained so far possess the characteristics mentioned above. But that these solutions are of purely electromagnetic origin escaped unnoted by all the workers. The present paper established the fact that all the static (bounded) spherically-symmetric charged dust solutions are of purely electromagnetic origin.

(3) The examples from the set of already known static spherically-symmetric charged dust solutions given in this paper always satisfy the relation $\rho = \pm\sigma$ (wherein the vanishing of σ implies the vanishing of ρ with the underlying space-time becoming flat). This might lead to understand (and, therefore, might lead to conjecture) that the energy density ρ of the spherically-symmetric charged dust, in general, is always a function of charge density (viz., $\rho = \rho(\sigma)$) and that it will always satisfy the criterion mentioned above (viz., when σ vanishes ρ also vanishes). However, as is evident from the theorem proved in this paper, such a relation, satisfying the criterion of vanishing of σ implying the vanishing of ρ, between ρ and σ (although not conclusively established), would hold for bounded charged spheres only. In the general case, for a spherically-symmetric charged dust (with no boundary) there might exist some relation between ρ and σ but might not satisfy the criterion mentioned above. In this connection, we would like to mention the work of Nayak (1980) who has obtained a solution for static spherically-symmetric charged dust with a relation between ρ and σ. But for the solution obtained by Nayak the vanishing of ρ implies the vanishing of σ (which obviously must be true) whereas the vanishing of σ need not imply the vanishing of ρ. In our opinion, however, the solution of Nayak, if considered for a bounded finite sphere and is matched on the boundary of the sphere with the Reissner–Nordström solution, would lead to the same conclusion (viz., the vanishing of σ implying the vanishing of ρ). The solution obtained by Nayak seems, therefore, to be of cosmological interest.

Acknowledgement

We are very grateful to Prof. J. R. Rao for various useful discussions and his valuable comments.

References

Florides, P. S.: 1977, *Nuovo Cimento* **42A**, 343.
Gautreau, R.: 1985, *Phys. Rev.* **D31**, 1860.
Grøn, Ø.: 1985, *Phys. Rev.* **D31**, 2129.
Grøn, Ø.: 1986a, *Am. J. Phys.* **54**, 46.
Grøn, Ø.: 1986b, *Gen. Rel. Grav.* **18**, 591.
Nayak, B. K.: 1980, *Australian J. Phys.* **33**, 765.
Pant, D. N. and Sah, A.: 1979, *J. Maths. Phys.* **20**, 2537.
Ponce de Leon, J.: 1987a, *J. Math. Phys.* **29**, 410.
Ponce de Leon, J.: 1987b, *Gen. Rel. Grav.* **19**, 797.
Ponce de Leon, J.: 1988, *J. Maths. Phys.* **29**.
Tikekar, R.: 1984, *Gen. Rel. Grav.* **21**, 637.
Tiwari, R. N., Rao, J. R., and Kanakamedala, R. R.: 1984, *Phys. Rev.* **D30**, 489.
Tiwari, R. N., Rao, J. R., and Ray, Saibal: 1991, *Astrophys. Space Sci.* **178**, 119.