ROBERTSON–WALKER VISCOUS FLUID MODEL

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Abstract. Robertson–Walker cosmological model with bulk viscosity is investigated with equation of state $p = (\gamma - 1)$. The cosmological solution of the model is obtained with the help of the special law of variation for Hubble's deceleration parameter. Some physical consequences of the solution is studied pertaining to two extreme cases of the equation of state.

1. Introduction

The investigation of cosmological problems in Einstein's theory usually deals with the energy-momentum tensor of matter as that due to the perfect fluid. Therefore, Berman (1983) considered a special law of variation for Hubble's parameter in involutionary models with perfect fluid as material source which leads to constant value of the deceleration parameter. Such a realistic matter source necessiates to take into account dissipation process due to viscosity. Murphy (1973) constructed isotropic homogeneous spatially-flat cosmological model with a fluid containing bulk viscosity alone because the shear viscosity cannot exist due to assumption of isotropy. He observed that the Big Bang singularity of finite past may be avoided by introduction of bulk viscosity. The assumption between viscosity coefficient and matter density considered by Murphy (1973) may not be acceptable near high density as studied by Santosh *et al.* (1985). Santosh *et al.* (1985) derived exact solutions for isotropic homogeneous cosmological model with bulk viscous fluid considering the bulk viscous coefficient as power function of mass density.

In the present investigation we extended the work of Murphy (1973) by considering the special law of variation for Hubble's parameter (Berman, 1983) and solved Einstein's field equation in Section 2 when the Universe is filled with bulk-viscous fluid. In Section 3 and 4 the physical behaviour of the solution in connection with the extreme situation of the equation of state is studied.

2. Einstein's Field Equations and Their Solution

Here we considered the space-time described by the isotropic homogeneous R.W. metric

$$ds^2 = dt^2 - Q^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \tag{1}$$

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where K is the curvature index which can take the values $(-1, 0, +1)$ and $Q(t)$ represents the radius of the Universe.

The most general expression for Einstein's relativistic cosmological field equations may be given by

$$G_{ij} \equiv R_{ij} + \Lambda g_{ij} = -kT_{ij}, \tag{2}$$

where T_{ij} is the energy-momentum tensor due to bulk-viscous fluid defined in the form

$$T_{ij} = (\varepsilon + \bar{p})U_i U_j - \bar{p} g_{ij}, \tag{3}$$

where

$$\bar{p} = p - \eta U^i{}_{;i}, \tag{4}$$

$$U^i U_i = 1 \tag{5}$$

and ε is the energy density; p, the pressure; η, the bulk-viscous coefficient; and U_i, the four-velocity vector of the distribution. Hereafter the semi-colon denotes covariant differentiation.

We consider here the special law of variation for Hubble's parameter (Berman, 1983) as

$$H = DQ^{-m}, \tag{6}$$

where H is the Hubble parameter defined by

$$H = \frac{Q_4}{Q}, \tag{7}$$

where D and m are constants and the suffix 4 stands for d/dt.

If we use Equations (6) and (7) we obtain

$$Q = [m(At + B)]^{1/m}, \tag{8}$$

where A and B are taken to be positive constants of integration.

The deceleration parameter is defined by

$$q = -\frac{Q_{44}Q}{Q_4^2}. \tag{9}$$

For special law (6), Equation (9) yields

$$q = (m - 1). \tag{10}$$

For the metric (1), Equation (4) leads to

$$\bar{p} = p - 3\eta H. \tag{11}$$

In a co-moving coordinate system, Equation (5) implies that

$$U_i = \delta_0^i. \tag{12}$$

Now with the aid of Equations (3)–(5) and metric (1) the surviving field equations (2) take the explicit forms

$$\frac{Q_4^2}{Q^2} + \frac{2Q_{44}}{Q} + \frac{K}{Q^2} - \Lambda = -k\bar{p} \tag{13}$$

and

$$-\frac{3K}{Q^2} - \frac{3Q_4^2}{Q^2} + \Lambda = -k\varepsilon\,. \tag{14}$$

The Bianchi identity for the bulk-viscous fluid in the space-time (1) leads to

$$\varepsilon_4 + 3H(\varepsilon + \bar{p}) = 0\,, \tag{15}$$

which, being obtainable from Equations (13) and (14), is redundant.

If we use (8) in Equations (13) and (14), we obtain

$$\bar{p} = \frac{1}{k}\left[\Lambda - \frac{A^2(3-2m)}{\{m(At+B)\}^2} - \frac{K}{\{m(At+B)\}^{2/m}}\right], \tag{16}$$

$$\varepsilon = \frac{1}{k}\left[\frac{3K}{\{m(At+B)\}^{2/m}} + \frac{3A^2}{\{m(At+B)\}^2} - \Lambda\right]. \tag{17}$$

Now restricting the distribution with the barotropic equation of state – i.e.,

$$p = (\gamma - 1)\varepsilon\,, \qquad 0 \leq r \leq 2\,. \tag{18}$$

We obtain the explicit form of the physical quantities p and η as

$$p = (\gamma - 1)\varepsilon = \frac{\gamma - 1}{k}\left[\frac{3K}{\{m(At+B)\}^{2/m}} + \frac{3A^2}{\{m(At+B)\}^2} - \Lambda\right], \tag{19}$$

$$\eta = \frac{1}{3kH}\left[\frac{K(3\gamma-2)}{\{m(At+B)\}^{2/m}} + \frac{A^2(3\gamma-2m)}{\{m(At+B)\}^2} - \Lambda\gamma\right]. \tag{20}$$

The solution obtained – i.e., Equations (8), (17), (19), and (20) – leads to an expanding model of the Universe. As the age of the Universe increases the radius of the Universe increases. At $t = 0$, we have a non-zero radius and ε, p, and η are finite. Thus the model avoids singularity at $t = 0$, which supports the analysis of Murphy (1973) that the introduction of bulk viscous fluid avoids the initial singularity. At the initial epoch the radius of the Universe is finite and, hence, the viscous fluid is confined to a spherical ball with a constant viscosity coefficient.

Moreover, we find that the mass density, pressure, and viscosity of the fluid decrease with the increase of the age of the Universe. Thus it indicates that the early universe being filled with hot dense gaseous matter with large viscosity becomes hotter during evolution.

It may also be verified here that when $\gamma = \frac{2}{3}$ and $m = 1$, we obtain $-p = \frac{1}{3}\varepsilon$. Thus this situation with constant viscosity coefficient is analogous to Einstein's static universe without cosmological constant.

Hawking–Penrose Energy Condition

By use of Einstein's equation, the Hawkiong–Penrose energy condition $R_{ij}U^iU^j \leq 0$ leads to a viscous analogue of the gravitational mass density – i.e., $\sigma = (\varepsilon + 3\bar{p}) \geq 2\Lambda/k$ (McCrea, 1951). The other energy condition used by Murphy (1973) – i.e., $(\bar{p} + \varepsilon) > 0$ – is automatically satisfied for flat and closed elliptical model, i.e., for $K = 0, +1$. For open hyperbolic model ($K = -1$), the second energy condition is satisfied for

$$t = m^{-\{(m+2)/2(m+1)\}} A^{(m-1)^{-1}} - B/A .$$

If the spatial curvature $K = 0$, the model does not reduce to the de Sitter model as in the case studied earlier by us (Mohanty and Pradhan, 1990).

3. False Vacuum Model (i.e., $\gamma = 0$)

If we consider $\gamma = 0$, the distribution reduces to a special case with equation of state $p + \varepsilon = 0$ which is referred to in the literature as 'degenerate vacuum' or 'false vacuum' or 'ρ vacuum'. This problem in non-viscous anisotropic case has already been studied by Mohanty and Pattanaik (1989).

However, in this case the physical quantities take the explicit forms

$$\varepsilon(= -p) = \frac{1}{k}\left[\frac{3K}{\{m(At + B)\}^{2/m}} + \frac{3A^2}{\{m(At + B)\}^2} - \Lambda\right] \tag{21}$$

and

$$\eta = \frac{-2}{3kH}\left[\frac{K}{\{m(At + B)\}^{2/m}} + \frac{mA^2}{\{m(At + B)\}^2}\right]. \tag{22}$$

This model corresponds to a realistic physical situation when $\eta \geq 0$. This is only true when the Hubble parameter $H < 0$.

However, this situation does not yield any solution for a spatially-flat space (i.e., for $K = 0$) as in case of exponential dependence of scale factor on time, studied earlier by Mohanty and Pradhan (1990).

4. Stiff-Fluid Model (i.e., $\gamma = 2$)

If we take $\gamma = 2$, the distribution reduces to a bulk-viscous stiff-fluid model (Barrow, 1978; Zeldovich, 1962) where the density cum pressure and bulk-viscous coefficient take the forms

$$p = \varepsilon = \frac{1}{k}\left[\frac{3K}{\{m(At + B)\}^{2/m}} + \frac{3A^2}{\{m(At + B)\}^2} - \Lambda\right] \tag{23}$$

and

$$\eta = \frac{2}{3kH}\left[\frac{2K}{\{m(At + B)\}^{2/m}} + \frac{A^2(3 - m)}{\{m(At + B)\}^2} - \Lambda\right]. \tag{24}$$

Since $\eta \geq 0$, we obtain $m \leq 3$ for the case $\Lambda = 0 = K$, where the equality corresponds to the non-viscous case already studied by Roy and Verma (1987) for flat space. In this case $q \leq 2$ which is in partial agreement with the experimental value $q_0 = 1.0 + 0.5$ (Adler *et al.*, 1975). This model indicates that the Universe starts expanding from the initial state at $t = 0$ without any Big Bang singularity and at infinite future (i.e., $t \to \infty$) there is a singularity which may correspond to big crunch. It may be mentioned here that the introduction of viscosity avoids the occurrence of 'Big Bang' singularity, which agrees with the work already studied by Murphy (1973).

References

Barrow, J. D.: 1978, *Nature* **272**, 211.
Berman, M. S.: 1983, *Nuovo Cimento* **74B**, 182.
McCrea, W. H. Q.: 1951, *Proc. Roy. Soc. London Ser.* **A206**, 562.
Mohanty, G. and Pattanaik, R. R.: 1989, *Astrophys. Space Sci.* **161**, 253.
Mohanty, G. and Pradhan, B. D.: 1990, *Astrophys. Space Sci.* **165**, 163.
Murphy, G. L.: 1973, *Phys. Rev.* **D8**, 4231.
Santos, N. O., Dias, R. S., and Banerjee, A.: 1985, *J. Math. Phys.* **26**(4).
Zeldovich, Ya. B.: 1962, *Soviet Phys. JETP* **14**, 1143.