ACCELERATION WAVES, SHOCK FORMATION AND STABILITY IN A GRAVITATING ATMOSPHERE

AUGUSTO MURACCHINI and TOMMASO RUGGERI

Dipartimento di Matematica, Università di Bologna, Bologna, Italy

(Received 30 September, 1988)

Abstract. The shock formation in a gravitating atmosphere is studied by following the general nonlinear theory of discontinuity waves. In particular, we perform a discussion on the stability of an isothermal and isoentropic atmosphere and we evaluate, when the shock appears, the critical time and the critical height. Some numerical results for the solar and terrestrial atmospheres are also given.

1. Introduction

The field equations of a compressible perfect fluid in the presence of the gravity are (the $x1$-axis is vertical)

$$\left.\begin{aligned} &\partial_t \rho + \partial_i(\rho v^i) = 0\,,\\ &\partial_t(\rho v^j) + \partial_i(\rho v^i v^j + p\delta^{ij}) = -\rho g \delta^{j1}\,,\\ &\partial_t \varepsilon + \partial_i((\varepsilon + p)v^i) = -\rho \mathbf{g}\cdot\mathbf{v}\,; \end{aligned}\right\} \tag{1.1}$$

where ρ is the density, $\mathbf{v}$ the velocity, p is the pressure, $\varepsilon = \rho(\frac{1}{2}v^2 + e)$ the total energy, e is the internal energy, and $\mathbf{g}$ the gravity acceleration. As well known, the system (1.1), taking into account the Gibbs relation

$$\theta\,\mathrm{d}S = \mathrm{d}e - \frac{p}{\rho^2}\,\mathrm{d}\rho$$

admits of the supplementary entropy law

$$\partial_t(\rho S) + \partial_i(\rho S v^i) = 0\,, \tag{1.2}$$

where θ is the absolute temperature and S the density of the entropy.

Let us consider the one dimensional problem and, as it is possible for classical solutions, interchange the energy equation with the entropy balance. Therefore, we can rewrite the system (1.1) in the form

$$\begin{aligned} &\rho_t + v\rho_x + \rho v_x = 0\,,\\ &(\rho v)_t + (\rho v^2 + p)_x = -\rho g\,,\\ &S_t + vS_x = 0\,. \end{aligned} \tag{1.3}$$

To the system (1.3), we adjoin the equation of state for an ideal gas, i.e.:

$$p = \mathcal{R}\rho\theta = e^{S/c_v}\rho^\gamma\,; \qquad \gamma = c_p/c_v\,, \qquad \mathcal{R} = c_p - c_v\,;$$

where $\mathscr{R}$ is the gas constant for unit molecular weight $\mathscr{M}$, c_p, and c_v are the specific heats at constant pressure and constant volume, respectively.

The system (1.3) is a particular case of the following quasi-linear first order hyperbolic system

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = \mathbf{f}(\mathbf{u}) , \tag{1.4}$$

where $\mathbf{u}(x, t)$ is the R^N unknown column vector of the system, $\mathbf{A}$ is an $N \times N$ matrix and $\mathbf{f}$ is the R^N source vector. In the present case ($N = 3$), choosing as field

$$\mathbf{u} \equiv (\rho, \rho v, S)^T ; \tag{1.5}$$

we have

$$\mathbf{A} \equiv \begin{vmatrix} 0 & 1 & 0 \\ c^2 - v^2 & 2v & p_S \\ 0 & 0 & v \end{vmatrix} , \qquad \mathbf{f} \equiv \begin{vmatrix} 0 \\ -g\rho \\ 0 \end{vmatrix} . \tag{1.6}$$

In (1.5) the superscript T indicates the transpose of a matrix.

The problem of the nonlinear wave propagation in a gravitational atmosphere, which arises frequently in astrophysics (Einaudi, 1970; Ulmschneider *et al.*, 1977; Ferraioli *et al.*, 1978; Yousaf, 1980; Hariharan, 1987), has been studied by many techniques: in particular we wish to quote the method of asymptotic waves (Choquet-Bruhat, 1969) which is applied by Anile *et al.* (1980) to treat the case of an acoustic wave propagating in an isothermal atmosphere.

The aim of this paper is to study the propagation of acceleration waves associated to the system (1.3) and, in particular, to evaluate the related critical time (i.e., the instant in which starts a shock wave). The plan is as follows.

In Section 2 we sketch briefly the mathematical theory of discontinuity waves. In Sections 3 and 4 we treat, in the most general form, the evolution of discontinuity waves and the related stability problem in a gravitational atmosphere. Finally, in the Sections 5 and 6 we calculate explicitly the critical time and the critical height in the cases of an isothermal and an isoentropic atmosphere giving also some numerical results concerning the terrestrial and the solar atmosphere.

2. Discontinuity Waves

First of all we recall the definition of hyperbolicity for a system of the type (1.4):

DEF. 2.1 (HYPERBOLICITY)

The first-order quasilinear system (1.4) *is said to be hyperbolic in the t-direction if the eigenvalue problem* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{d} = 0$ *admits only real eigenvalues* λ (*characteristic velocities*) *and a set of linearly independent right eigenvectors.*

For a generic system (1.4) it is possible to consider a particular class of solutions that characterize the so-called '*weak discontinuity waves*' or, in the language of continuum mechanics, '*acceleration waves*'. There exists a moving surface (wave front) Γ of the Cartesian equation $\varphi(\mathbf{x}, t) = 0$ that separates the space in two subspaces; ahead the wave front we have the known unperturbed field $\mathbf{u}_0(x, t)$ and behind the unknown perturbed field $\mathbf{u}(\mathbf{x}, t)$. Both the fields $\mathbf{u}_0$ and $\mathbf{u}$ are supposed regular solutions of (1.4) and across the surface Γ are continuous, but have discontinuous normal derivative: i.e.,

$$[\![\mathbf{u}]\!] = 0\,, \qquad [\![\mathbf{u}_\varphi]\!] = \mathbf{\Pi} \neq 0\,, \tag{2.1}$$

where the square bracket indicates the jump (for simplicity we indicate briefly with g and g_0 the values of a generic quantity g evaluated on Γ, respectively, for $\varphi \to 0^-$ and $\varphi \to 0^+$)

$$[\![\,\cdot\,]\!] = (\,\cdot\,)_{\varphi=0^-} - (\,\cdot\,)_{\varphi=0^+}$$

and $\mathbf{u}_\varphi = \partial\mathbf{u}/\partial\varphi$.

As well known the following results hold (Boillat, 1965):

(1) The normal velocity $\lambda = -\varphi_t/|\nabla_\varphi|$ is equal to a characteristic velocity evaluated in $\mathbf{u}_0$

$$\lambda = \lambda(\mathbf{u}_0)\,. \tag{2.2}$$

(2) The jump vector $\mathbf{\Pi}$ is proportional to the right eigenvector $\mathbf{d}$ (corresponding to the eigenvalue λ) evaluated in $\mathbf{u}_0$

$$\mathbf{\Pi} = \Pi\mathbf{d}(\mathbf{u}_0)\,. \tag{2.3}$$

(3) The amplitude Π satisfies the Bernoulli equation

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} + a(t)\Pi^2 + b(t)\Pi = 0\,. \tag{2.4}$$

where $\mathrm{d}/\mathrm{d}t$ indicates the time derivative along the bicharacteristic lines and $a(t)$, $b(t)$ are known functions of the time through $\mathbf{u}_0$.

In the case of one-dimensional space $\mathrm{d}x/\mathrm{d}t = \lambda_0$ is the characteristic line and we have (Boillat and Ruggeri, 1979; Ruggeri, 1980, 1989)

$$a(t) = \varphi_x(\nabla\lambda\cdot\mathbf{d})_0\,, \tag{2.5}$$

$$b(t) = \left\{\mathbf{d}((\nabla\mathbf{l})^T - \nabla\mathbf{l})\,\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + (\nabla\lambda\cdot\mathbf{d})\,(\mathbf{l}\cdot\mathbf{u}_x) - \nabla(\mathbf{l}\cdot\mathbf{f})\cdot\mathbf{d}\right\}_0, \tag{2.6}$$

$$\frac{\mathrm{d}\varphi_x}{\mathrm{d}t} + (\nabla\lambda\cdot\mathbf{u}_x)_0\,\varphi_x = 0\,, \tag{2.7}$$

where $\mathbf{l}$ indicates the left eigenvector of $\mathbf{A}$ that, for the hyperbolicity, it is possible to choose such that $\mathbf{l}\cdot\mathbf{d} = 1$; $\nabla = \partial/\partial\mathbf{u}$.

The solution of (2.4) is

$$\Pi(t) = \frac{\Pi(0)\exp\left[-\int_0^t b(\xi)\,\mathrm{d}\xi\right]}{1 + \Pi(0)\int_0^t a(\zeta)\,exp\left[-\int_0^\zeta b(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\zeta} \,. \tag{2.8}$$

We recall now that, in the literature, a system (1.4) satisfies the so-called genuine nonlinearity if for every eigenvalues λ we have

$$\nabla\lambda^{(i)}\cdot\mathbf{d}^{(i)} \neq 0\,, \quad \forall i = 1, 2, \ldots, N\,. \tag{2.9}$$

Vice versa, a wave is called exceptional if*

$$\nabla\lambda\cdot\mathbf{d} \equiv 0\,. \tag{2.10}$$

The exceptional waves are very common in mathematical physics and they play a relevant role in several questions of nonlinear wave propagation (see Boillat and Ruggeri, 1978).

If the wave is exceptional then the Bernoulli equation become linear because the coefficient $a(t)$ vanishes. Therefore, if $b(t) \geq 0$ the amplitude decays exponentially as in the linear case.

Vice versa if the wave in consideration satisfies the genuine nonlinearity, there exists in general a *critical time* (t_{cr}) such that the denominator of (2.8) tends to zero and the discontinuity becomes unbounded. This instant usually corresponds to the creation of a *strong discontinuity*, i.e., a shock wave and the field itself presents discontinuity across the wave front. In particular, if the wave satisfies (2.9) the coefficient $a(t) \neq 0$ and we can consider it always positive (with an appropriate choice of the right eigenvector). Therefore, from (2.8) a positive critical time exists if the initial perturbation $\Pi(0)$ satisfies

$$\Pi(0) < -\Pi_{cr} \quad \text{where} \quad \Pi_{\mathrm{cr}} = 1\Big/\int_0^\infty \mathrm{a}(\xi)\exp\left[-\int_0^\xi b(\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}\xi \geq 0\,. \tag{2.11}$$

This condition says that for the existence of the critical time it is necessary that the initial amplitude be negative and, in absolute value, greater than a critical quantity. A qualitative analysis of the behaviour of the solution of the Bernoulli equation can be read on the papers of Chen (1973), Ruggeri (1989). The existence of t_{cr} is a nonlinear phenomenon, but in three dimension space, as well known, it is possible to have also a critical instant due to geometrical reasons: the caustic case.

* Of course if the eigenvalue λ have multiplicity p then the condition is valid for all the corresponding eigenvectors $\mathbf{d}_I$ ($I = 1, 2, \ldots, p$).

3. Stability of the Unperturbed Field

It is possible to consider the problem of discontinuity waves as a nonlinear stability problem for the unperturbed field $\mathbf{u}_0$ in the sense of the following definition (Ruggeri, 1989):

DEF. 3.1 (STABILITY WITH RESPECT TO A λ-WAVE)

Let $\mathbf{u}_0(\mathbf{x}, t)$ *a regular solution of* (1.4), *and we consider perturbations in the class of discontinuity waves (piecewise classical* C^1 *solutions). We said that* $\mathbf{u}_0(\mathbf{x}, t)$ *is stable with respect to a discontinuity wave of velocity* $\lambda_0 \equiv \lambda(\mathbf{u}_0(x, t))$, *if given a* $P_0 > 0$, *there exists a* $P(P_0) > 0$, *such that if the initial perturbation* $\Pi(0)$ *satisfies the condition*

$$|\Pi(0)| \leq P_0 , \tag{3.1}$$

we have

$$|\Pi(t)| \leq P , \quad \forall t \geq 0, \forall x_0 \in R . \tag{3.2}$$

where t is the time along the characteristic $\mathscr{C}: \mathrm{d}x/\mathrm{d}t = \lambda_0$ *and* x_0 *is the point at* $t = 0$ *where the discontinuity wave starts.*

Moreover, if

$$\lim_{t \to \infty} |\Pi(t)| = 0 , \tag{3.3}$$

then we have asymptotic stability *and if the inequality* (3.1) *depends on the solution* $\mathbf{u}_0(\mathbf{x}, t)$ *we have the so-called* conditional stability.

We observe that this problem corresponds to the stability of the zero solution of the Bernoulli equation (2.4). If we look the solution (2.8) and the condition (2.11), it is possible to prove the following simple theorem (Ruggeri, 1989):

THEOREM 3.1

A regular solution $\mathbf{u}_0(x, t)$ *of the hyperbolic system* (1.4) *is stable with respect to a* λ-*wave if the corresponding coefficients* $a(t)$ *and* $b(t)$ *in the Bernoulli equation, that we suppose continuous functions for all* $t \geq 0$, *are such that*

$$\int_0^\infty |a(\xi)| \exp\left[- \int_0^\xi b(\zeta)\,\mathrm{d}\zeta \right] \mathrm{d}\xi = K(x_0) < +\infty , \quad \forall x_0 \in R ; \tag{3.4}$$

and there exists a constant m *such that the integral of* $b(t)$ *is bounded from below*: i.e.,

$$\int_0^t b(\xi)\,\mathrm{d}\xi > m , \quad \forall t > 0, \forall x_0 \in R . \tag{3.5}$$

Moreover, if we have also

$$\int_0^\infty b(t)\,\mathrm{d}t = +\infty , \quad \forall x_0 \in R , \tag{3.6}$$

then the solution is asymptotically stable. The initial perturbation satisfies the condition

$$|\Pi(0)| < P_0 = \inf_{x_0 \in R} \frac{1}{K(x_0)} \tag{3.7}$$

and, as P_0 is, therefore, a functional of $\mathbf{u}_0(x, t)$ then we have in any case a conditional stability.

Therefore, for the stability or the instability it is necessary to study the integrability conditions (3.4), (3.5), and (3.6).

4. Acceleration Waves in the Atmosphere

We study now the evolution of discontinuity waves associated to the system (1.3) following the previous general approach.

The possible discontinuity waves propagate with the characteristic velocities

$$\lambda_1 = v + c\,, \qquad \lambda_2 = v - c\,, \qquad \lambda_3 = v\,; \tag{4.1}$$

where c is the sound velocity: $c^2 = p_\rho = (\partial p/\partial \rho)_S$.

The *contact wave* λ_3 is exceptional and, therefore, does not present problems.

For the *acoustic waves* λ_1 and λ_2 choosing as field (1.5), the corresponding right and left eigenvectors are found

$$\mathbf{d}_1 \equiv (1, v + c, 0)^T, \qquad \mathbf{l}_1 \equiv \frac{1}{2c}\,(c - v, 1, p_S/c)\,; \tag{4.2}$$

$$\mathbf{d}_2 \equiv (1, v - c, 0)^T, \qquad \mathbf{l}_2 \equiv \frac{1}{2c}\,(c + v, -1, p_S/c)\,; \tag{4.3}$$

where $p_s = (\partial p/\partial S)_\rho$.

With the previous choice of the right eigenvectors, taking into account (2.3), the amplitude Π represents the jump of the normal derivative of the density and this is related with the jump of the acceleration G through the relations

$$\Pi = [\![\rho_\varphi]\!]\,, \qquad G = [\![v_t]\!] = -c^2\Pi/\rho_0\,, \qquad [\![h_\varphi^0]\!] = -S_0\Pi\,. \tag{4.4}$$

Let us evaluate now, in the most general form, the expression of the functions a and b appearing in the Bernoulli equation (2.4).

We perform the calculations for the fastest wave ($\lambda_1 = v + c$) only: in fact, observe that the results which we obtain are applicable to the wave with velocity $\lambda_2 = v - c$ changing c into $-c$.

By assuming the static solution with zero velocity ($v_0 = 0$) as unperturbed state the system (1.3) reduces to the momentum equation

$$p_x = c^2\rho_x + p_S S_x = -g\rho\,, \tag{4.5}$$

with $p = p(\rho, S)$, $c = c(\rho, S)$.

By choosing a particular law for $S(x)$ or for $\rho(x)$ we select a solution of Equation (4.5).

In this section we suppose that the functions ρ and S are whatever and, in this manner, we find the generic expressions of the coefficients of the Bernoulli equation (2.4).

Instead, in the next sections we shall consider the particular cases of an isothermal (θ = constant) and an isentropic (S = constant) atmosphere.

By some simple calculations, we find first of all (henceforth, we omit, in the right side, the suffix 0 referring to the physical quantities in the unperturbed field $\mathbf{u}_0$)

$$\lambda_0 = c\,, \qquad \mathbf{d}_0 \equiv (1, c, 0)^T, \qquad \mathbf{l}_0 = \left(\frac{1}{2}, \frac{1}{2c}, \frac{p_S}{2c^2}\right); \tag{4.6}$$

$$(\mathbf{u}_x)_0 = \begin{pmatrix} \rho_x \\ 0 \\ S_x \end{pmatrix}, \qquad (\nabla\lambda)_0^T = \left(c_\rho, \frac{1}{\rho}, c_S\right); \tag{4.7}$$

so that

$$(\nabla\lambda \cdot \mathbf{u}_x)_0 = -\frac{g}{2c}\left(\frac{c^2\rho_x}{g\rho} + \gamma\right), \tag{4.8}$$

having the following thermodynamic relations

$$c_\rho = \frac{\gamma - 1}{2\rho}\,, \quad c_S = \frac{c}{2c_v}\,, \quad c_v = \frac{\mathscr{R}}{\gamma - 1}\,, \quad p_S = (\gamma - 1)\rho\theta\,, \quad \theta = \frac{c^2}{R\gamma}\,. \tag{4.9}$$

By observing that, in this case, $\mathrm{d}\mathbf{u}_0/\mathrm{d}t = \lambda_0 \mathbf{u}_{0x}$ and subsequently $\nabla\lambda_0 \cdot \mathbf{u}_{0x} = \mathrm{d}\ln|\lambda_0|/\mathrm{d}t$ we find from (2.7) that

$$\varphi_x(t) = \frac{c_0(0)}{c_0(t)}\,, \tag{4.10}$$

where $c(0)$ represents the value of c at time $t = 0$ on the characteristic lines starting at $x = x_0$.

By taking into account (4.6), (4.7), (4.9) we have also

$$(\nabla\lambda \cdot \mathbf{d})_0 = \frac{c(\gamma + 1)}{2\rho}\,, \qquad \left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right)_0 = c(\mathbf{u}_x)_0 = \begin{pmatrix} c\rho_x \\ 0 \\ cS_x \end{pmatrix}, \tag{4.11}$$

$$\mathbf{d}_0\{\nabla\mathbf{l}^T - \nabla\mathbf{l}\}_0 \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_0 = -\left(\frac{g}{2c} + \frac{c\rho_x}{2\rho}\right) - \left(\frac{g\gamma}{4c} + \frac{3c\rho_x}{4\rho}\right), \tag{4.12}$$

$$(\mathbf{l}\cdot\mathbf{u}_x)_0 = -\frac{\rho g}{2c^2}, \qquad \{\nabla(\mathbf{l}\cdot\mathbf{f})\cdot\mathbf{d}\}_0 = \frac{\gamma-3}{4}\frac{g}{c} \tag{4.13}$$

Then, finally, (2.5) and (2.6) yield

$$a = \left(\frac{\gamma+1}{2}\frac{c}{\rho}\right)\frac{c(0)}{c}, \qquad b = -\left(\frac{3g\gamma}{4c} + \frac{5c\rho_x}{4\rho}\right). \tag{4.14}$$

Because the generic coefficients a and b are functions of the variable x it is more advantageous, in the actual context, to utilize the Bernoulli equation in x instead of (2.4). In fact by observing that $\mathrm{d}\Pi/\mathrm{d}t = (\mathrm{d}\Pi/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}t) = (\mathrm{d}\Pi/\mathrm{d}x)c$ we can write

$$\frac{\mathrm{d}\Pi}{\mathrm{d}x} + a^*(x)\Pi^2 + b^*(x)\Pi = 0, \tag{4.15}$$

where $a^* = a/c$, $b^* = b/c$. The solution of (4.15) is

$$\Pi(x) = \frac{\Pi(0)\exp\left[-\int_{x_0}^{x} b^*(\xi)\,\mathrm{d}\xi\right]}{1 + \Pi(0)\int_{x_0}^{x} a^*(\zeta)\exp\left[-\int_{x_0}^{\zeta} b^*(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\zeta} \tag{4.16}$$

with $\Pi(0) = \Pi(x_0)$.

5. Acceleration Waves in an Isothermal Atmosphere

We consider as unperturbed field the well-known static isothermal solution with zero-velocity of (1.3)

$$v = 0, \qquad \rho = \tilde{\rho}\exp\left(-\frac{gx}{\mathcal{R}\theta}\right), \qquad \theta = \tilde{\theta} \tag{5.1}$$

with $\tilde{\rho}$ and $\tilde{\theta}$ positive constants*. Moreover observe that, in the present case $c = c(0) =$ constant. Then by remembering (4.14) we obtain, for the upwards traveling wave with velocity $\mathrm{d}x/\mathrm{d}t = c$

$$a^*(\zeta) = \frac{\gamma+1}{2\tilde{\rho}}e^{\mu\zeta}, \qquad b^*(\zeta) = \frac{\mu}{2} = \text{constant}, \tag{5.2}$$

where $\mu = g/\mathcal{R}\theta = g\gamma/c^2$.

* Henceforth $\sim$ indicates the value of a generic quantity ψ at $x = 0$ and $\psi(0)$ is the value of ψ at time $t = 0$ along the characteristic line starting at $x = x_0$.

It is a simple matter to see that the stability condition (3.4) is never satisfied and, therefore, the isothermal solution (5.1) is unstable with respect to the upwards wave. In this case, in fact, by evaluating (4.16), we obtain soon that for an arbitrarily small initial acceleration compression jump $G(0)$

$$G(0) > 0 \tag{5.3}$$

there exists a critical height $x_{cr} > x_0$ for which

$$x_{cr} = x_0 + \frac{2}{\mu} \ln\left(1 + \frac{g}{G(0)} \frac{\gamma}{\gamma + 1} e^{-\mu x_0}\right). \tag{5.4}$$

The corresponding critical time is immediately obtained by integrating $dx/dt = c$. One finds that

$$t_{cr} = \frac{2}{\mu c} \ln\left(1 + \frac{g}{G(0)} \frac{\gamma}{\gamma + 1} e^{-\mu x_0}\right) \tag{5.5}$$

with the condition (5.3).

Then it is possible to establish

STATEMENT 5.1

There exists always a critical time (5.5) *for a compression* ($G(0) > 0$) *upwards propagating acoustic wave.*

This statement, in terms of stability problem, reads to

STATEMENT 5.2

The isothermal solution (5.1) *is unstable with respect to the upwards acoustic wave.*

In the case of the acoustic wave propagating with the velocity $\lambda = -c$ we have from (5.2) changing $c \to -c$ and $x \to -x$

$$|a(t)| = c\,\frac{\gamma + 1}{2\tilde{\rho}}\, e^{-\mu c t}, \qquad b(t) = -\frac{\mu}{2} c\,. \tag{5.6}$$

The solution (5.1) is still unstable; in fact, it is easily verified that the integral (3.4) is convergent while the condition (3.5) is never satisfied.

It is a simple matter to prove that the existence of a critical height is ensured actually by the condition

$$G(0) < 0\,, \qquad |G(0)| > \frac{g\gamma}{\gamma + 1} e^{-\mu x_0} \geq \frac{g\gamma}{\gamma + 1} > \frac{g}{2} \tag{5.7}$$

having then

$$x_{cr} = x_0 + \frac{2}{\mu} \ln\left(1 - \frac{g}{|G(0)|} \frac{\gamma}{\gamma + 1} e^{-\mu x_0}\right) \tag{5.8}$$

with $x_{cr} < x_0$. The corresponding critical time is actually

$$t_{cr} = -\frac{2}{\mu c} \ln\left(1 - \frac{g}{|G(0)|} \frac{\gamma}{\gamma+1} e^{-\mu x_0}\right). \tag{5.9}$$

Therefore, so the following statement holds:

STATEMENT 5.3

The critical time of the acoustic wave traveling downward exists only for a sufficient large initial compressive amplitude of the acceleration jump $G(0)$. *The critical amplitude and the critical time are given by* (5.7), (5.9), *respectively.*

6. The Isentropic Atmosphere

We have performed the calculations also in the case of an unperturbed solution corresponding to an isentropic atmosphere although this is of lesser physical interest.

The static isentropic solution of (1.3) is

$$v = 0\,, \qquad \rho(x) = \left\{\tilde{\rho}^{\,\gamma-1} - \frac{g(\gamma-1)}{\alpha\gamma}\, x\right\}^{1/\gamma-1}, \qquad S(x) = \tilde{S}\,, \tag{6.1}$$

where $\tilde{S}$ = constant. In the present case the following hold

$$c^2 = \alpha\gamma\rho^{\gamma-1}\,, \qquad c^2(0) = \alpha\gamma\rho(0)^{\gamma-1}\,; \tag{6.2}$$

where

$$\tilde{\rho}^{\,\gamma-1} = \rho(0)^{\gamma-1} + \frac{g(\gamma-1)}{\alpha\gamma}\, x_0\,. \tag{6.3}$$

By taking into account (4.14) we find

$$a^*(\zeta) = \frac{\gamma+1}{2\rho(\zeta)} \frac{c(0)}{c(\zeta)}\,, \qquad b^*(\zeta) = \frac{5-3\gamma}{4} \frac{g}{c^2(\zeta)}\,, \tag{6.4}$$

so that, actually, $\rho_x = -g\rho/c^2$. By proceeding in the same manner of the previous section we find that the critical height for the fastest wave exists if

$$G(0) > 0\,. \tag{6.5}$$

Then

$$x_{cr} = x_0 + \frac{c^2(0)}{g(\gamma-1)} \left\{1 - \left(\frac{2G(0)}{2G(0)+g}\right)^{[4(\gamma-1)/(\gamma+1)]}\right\}. \tag{6.6}$$

The critical time is obtained, as usual, by integrating $\mathrm{d}x/\mathrm{d}t = c$. But, in this case, $c = c(0) - \beta t$ (with $\beta = g(\gamma-1)/2$) so that the characteristic line

TABLE I

The terrestrial atmosphere with $g = |G(0)| = 9.80665$ m s^{-2}, $\tilde{c} = 340.2937$ m s^{-1}, $\gamma = 1.4$, $\mathscr{M} = 28.9644$ g mole^{-1}, $\mathscr{R}/\mathscr{M} = 287.0524$

Terrestrial atmosphere at $\tilde{\theta} = 288.15$ K.
Critical time and critical height for $\lambda = c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
0	22.78	7.75	21.93	6.99
1	20.69	8.04	21.56	7.75
2	18.77	8.39	21.18	8.52
3	16.99	8.78	20.79	9.28
4	15.35	9.22	20.39	10.04
5	13.85	9.71	19.99	10.81
6	12.48	10.25	19.58	11.57
7	11.23	10.82	19.16	12.33
8	10.10	11.44	18.73	13.09
9	9.07	12.08	18.29	13.86
10	8.13	12.77	17.84	14.62

TABLE II

The terrestrial atmosphere by assuming the same values of the Table I. Observe that, in the isentropic case, the values of x_{cr} are bounded accordingly to (6.12). In fact one has $x_L = 29.52$ km. Furthermore, in this case, x_{cr} must be positive ($x = 0$ is the ground level). In the isothermal case the limiting value of x_0 for which $x_{cr} \geq 0$ is $x_0^* = 5.83$ km. In the isentropic case $x_0^* = 10.94$ km.

Terrestrial atmosphere at $\tilde{\theta} = 288.15$ K.
Critical time and critical height for $\lambda = -c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
6	16.73	0.31	–	–
7	14.55	2.05	–	–
8	12.69	3.68	–	–
9	11.10	5.22	–	–
10	9.73	6.69	–	–
11	8.54	8.09	35.72	0.12
12	7.51	9.44	34.74	1.71
13	6.61	10.75	33.74	3.29
14	5.83	12.01	32.70	4.88
15	5.14	13.25	31.63	6.47
16	4.54	14.45	30.52	8.06

TABLE III

The solar atmosphere with $g = |G(0)| = 273.6\ \mathrm{m\ s^{-2}}$, $\tilde{c} = 6529.734\ \mathrm{m\ s^{-1}}$, $\gamma = \frac{5}{3}$, $\mathcal{M} = 1.3\ \mathrm{g\ mole^{-1}}$, $\mathcal{R}/\mathcal{M} = 6395.615$

Solar atmosphere at $\tilde{\theta} = 4000$ K.
Critical time and critical height for $\lambda = c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
0	13.90	90.79	13.14	77.92
10	12.76	93.35	12.85	84.59
20	11.70	96.40	12.56	91.25
30	10.71	99.93	12.27	97.92
40	9.79	103.92	11.96	104.59
50	8.93	108.34	11.65	111.25
60	8.15	113.19	11.33	117.92
70	7.42	118.43	10.99	124.59
80	6.75	124.06	10.66	131.25
90	6.13	130.03	10.30	137.92
100	5.56	136.34	9.94	144.59

TABLE IV

The values of the physical quantities are the same of the Table III. In this case $x_L = 233.76$ km. Furthermore, in the isothermal case $x_0^* = 67.92$ km. In the isentropic case $x_0^* = 117.00$ km.

Solar atmosphere at $\tilde{\theta} = 4000$ K.
Critical time and critical height for $\lambda = -c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
100	6.91	54.85	–	–
110	6.13	69.96	–	–
120	5.45	84.43	20.69	6.24
130	4.84	98.37	19.76	26.24
140	4.31	111.83	18.78	46.24
150	3.85	124.89	17.75	66.24
160	3.43	137.59	16.66	86.24
170	3.06	149.99	15.49	106.24
180	2.74	162.12	14.22	126.24
190	2.45	174.01	12.83	146.24
200	2.19	185.70	11.27	166.24

TABLE V

The solar atmosphere with $g = 273.6 \text{ m s}^{-2}$, $c = 7300.465 \text{ m s}^{-1}$, $\gamma = \frac{5}{3}$, $\mathscr{M} = 1.3 \text{ g mole}^{-1}$, $\mathscr{R}/\mathscr{M} = 6395.615$

Solar atmosphere at $\tilde{\theta} = 5000$ K.
Critical time and critical height for $\lambda = c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
0	15.54	113.49	14.69	97.40
10	14.52	116.00	14.44	104.07
20	13.55	118.90	14.18	110.73
30	12.63	122.19	13.91	117.40
40	11.76	125.86	13.65	124.07
50	10.94	129.90	13.37	130.73
60	10.17	134.28	13.09	137.40
70	9.45	139.00	12.81	144.07
80	8.77	144.05	12.52	150.73
90	8.14	149.41	12.22	157.40
100	7.54	155.07	11.91	164.07

TABLE VI

The values of the physical quantities are the same of the Table V. In this case $x_L = 292.19$ km. Furthermore, in the isothermal case $x_0^* = 84.58$ km. In the isentropic case $x_0^* = 146.25$ km.

Solar atmosphere at $\tilde{\theta} = 5000$ K.
Critical time and critical height for $\lambda = -c$.

Height x_0 (km)	Isothermal		Isentropic	
	t_{cr} (s)	x_{cr}	t_{cr}	x_{cr}
100	9.89	27.82	–	–
110	8.95	44.66	–	–
120	8.11	60.76	–	–
130	7.37	76.22	–	–
140	6.69	91.13	–	–
150	6.09	105.54	23.13	7.80
160	5.54	119.52	22.30	27.80
170	5.05	133.12	21.44	47.80
180	4.61	146.37	20.55	67.80
190	4.20	159.32	19.61	87.80
200	3.84	171.99	18.62	107.80

$x = c(0)t - (1/2)\beta t^2 + x_0$ follows and

$$t_{cr} = \frac{2c(0)}{g(\gamma - 1)} \left\{ 1 - \left(\frac{2G(0)}{2G(0) + g} \right)^{[2(\gamma - 1)/(\gamma + 1)]} \right\} \tag{6.7}$$

or, equivalently, by (6.2), (6.3)

$$t_{cr} = \frac{2\sqrt{\bar{c}^2 - g(\gamma - 1)x_0}}{g(\gamma - 1)} \left\{ 1 - \left(\frac{2G(0)}{2G(0) + g} \right)^{[2(\gamma - 1)/(\gamma + 1)]} \right\}. \tag{6.8}$$

For the wave traveling with the velocity $\lambda = -c$ we obtain the following existence condition for the critical height:

$$G(0) < 0\,, \qquad |G(0)| > \frac{g}{2}\,. \tag{6.9}$$

If (6.9) is satisfied, one has

$$x_{cr} = x_0 + \frac{c^2(0)}{g(\gamma - 1)} \left\{ 1 - \left(\frac{2\,|G(0)|}{2\,|G(0)| - g} \right)^{[4(\gamma - 1)/(\gamma + 1)]} \right\} \tag{6.10}$$

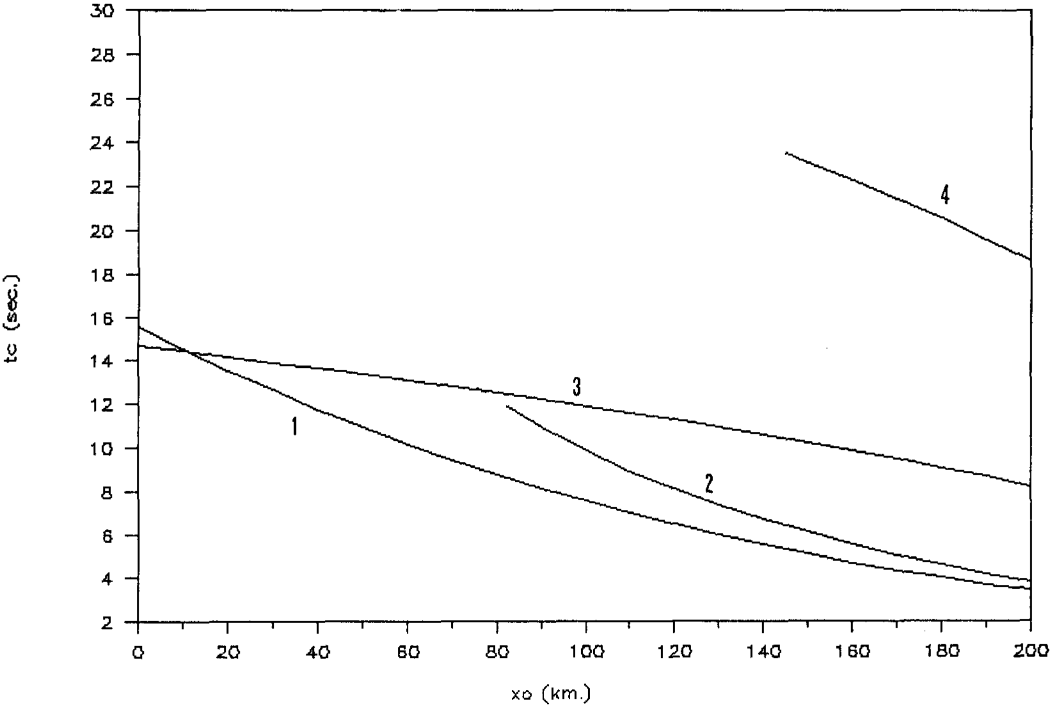

Fig. 1. The critical time as a function of x_0 in the solar atmosphere at 5000 K. (1) Isothermal atmosphere and $\lambda = v + c$. (2) Isothermal atmosphere and $\lambda = v - c$. (3) Isentropic atmosphere and $\lambda = v + c$. (4) Isentropic atmosphere and $\lambda = v - c$.

and

$$t_{cr} = \frac{2c(0)}{g(\gamma - 1)} \left\{ \left(\frac{2\,|G(0)|}{2\,|G(0)| - g} \right)^{[2(\gamma-1)/(\gamma+1)]} - 1 \right\}. \tag{6.11}$$

Note that from (6.1), because $\rho > 0$, it follows that

$$0 \leq x < \frac{\alpha\gamma\tilde{\rho}^{\,\gamma-1}}{g(\gamma-1)} = \frac{\tilde{c}^2}{g(\gamma-1)} \equiv x_L\,; \tag{6.12}$$

and, consequently,

$$0 \leq t < \frac{c(0)}{\beta} \equiv t_L\,. \tag{6.13}$$

The values of x and t are, at present, bounded.

We conclude that in the case of an isentropic atmosphere analogous results to those of the isothermal case hold with the only qualitative difference of the threshold value (6.9) with respect to (5.7).

Let us summarize these results in the following statements:

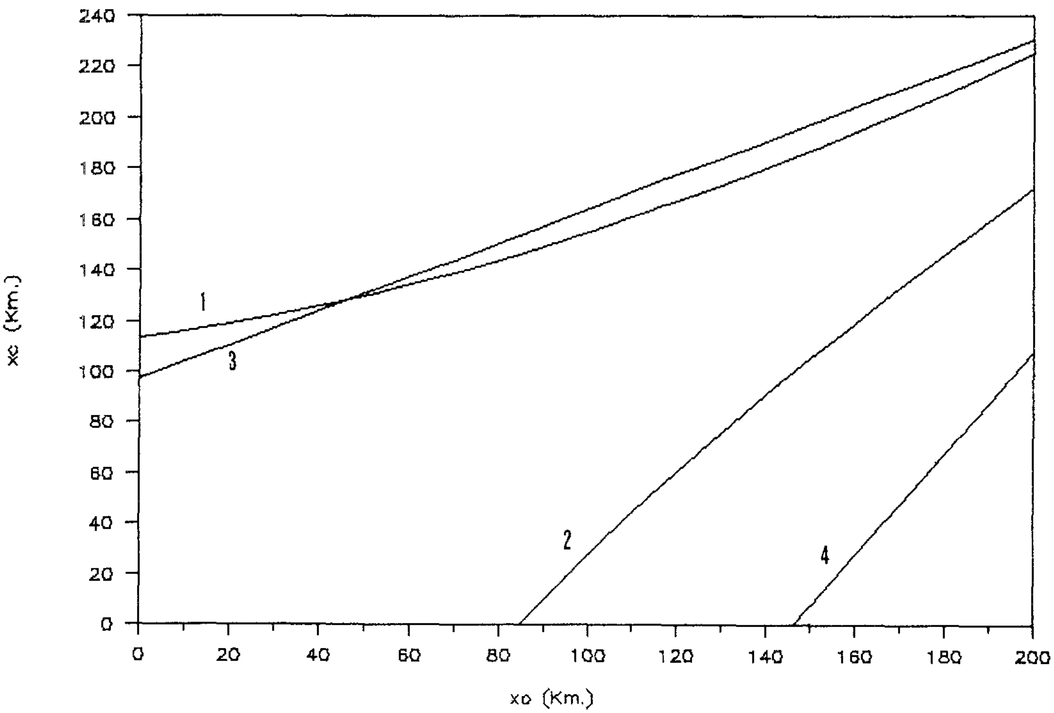

Fig. 2. The critical height as a function of x_0 in the solar atmosphere at 5000 K. (1) Isothermal atmosphere and $\lambda = v + c$. (2) Isothermal atmosphere and $\lambda = v - c$. (3) Isentropic atmosphere and $\lambda = v + c$. (4) Isentropic atmosphere and $\lambda = v - c$.

STATEMENT 6.1

There exists always the critical time (6.7) *for the upwards propagating acoustic wave: it occurs even if one has an arbitrarily small initial acceleration jump* $G(0)$ (>0).

STATEMENT 6.2

The critical time of the wave traveling downward exists only for a sufficiently large initial amplitude of the acceleration jump $G(0)$. *The critical amplitude and the corresponding critical time are given by the conditions* (6.9), (6.11).

Note that, in both cases, we have still nonlinear instability for the reasons analogous to isothermal case.

Finally let us observe that for $g \to 0$ (5.5), (5.9) and (6.7), (6.11) give the same value which represents the critical time in the absence of gravity: i.e.,

$$t_{cr} = \frac{2c}{|G(0)|\,(\gamma + 1)} \tag{6.14}$$

already obtained by Ruggeri (1980). In particular we obtain for the atmosphere of the Earth the value $t_{cr} = 28.92$ s, and for the Sun at 4000 K and 5000 K the values $t_{cr} = 17.90$ s, $t_{cr} = 20.01$ s, respectively, by assuming for the physical quantities entering in (6.14) the values carried in the captions of Tables I–VI.

In Tables I–VI we display the critical heights and the critical times for various values of x_0 in an isothermal and isentropic atmosphere. Tables I and II refer to the case of the Earth. Tables III–VI describe the wave propagation in the solar atmosphere at $\theta = 4000$ K and $\theta = 5000$ K.

Figures 1 and 2 point out the behaviour of t_{cr} and x_{cr} in function of x_0 in the atmosphere of the Sun at 5000 K. Similar graphs are obtained in the case of the terrestrial atmosphere.

References

Anile, A. M., Mulone, G., and Pluchino, S.: 1980, *Wave Motion* **2**, 267.
Boillat, G.: 1965, *La Propagation des Ondes*, Gauthier-Villars, Paris.
Boillat, G. and Ruggeri, T.: 1978, *Boll. U.M.I.* **15A**, 197.
Boillat, G. and Ruggeri, T.: 1979, *Wave Motion* **1**, 149.
Chen, P.: 1973, in S. Flügge (ed.), *Handbuch der Physik* **6A/3**, 303.
Choquet-Bruhat, Y.: 1969, *J. Math. Pures et Appl.* **48**, 117.
Einaudi, F.: 1970, *J. Geophys. Res.* **75**, 193.
Ferraioli, F., Ruggeri, T., and Virgopia, N.: 1978, *Astrophys. Space Sci.* **56**, 303.
Hariharan, S. I.: 1987, *Quart. Appl. Math.* **45**, 735.
Ruggeri, T.: 1980, *Applicable Analysis* **11**, 103.
Ruggeri, T.: 1989, in Longman (ed.), *Contributions on Nonlinear Wave Motion* (to appear).
Ulmschneider, P., Kalkofen, W., Nowak, T., and Bohn, U.: 1977, *Astron. Astrophys.* **54**, 61.
Yousaf, M.: 1980, *Phys. Fluids* **30**, 3669.