COSMOLOGICAL VACUUM SOLUTIONS IN BRANS AND DICKE'S SCALAR-TENSOR THEORY

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Abstract. The exact cosmological vacuum solutions of Brans and Dicke's scalas-tensor theory are derived when a power law is valid between the gravitational constant κ and the radius of curvature R of the universe. There exist even in the case of the closed 3-dimensional space of positive curvature solutions with increasing R and κ with respect to the age of the universe, whereby the freely available parameter ω of the scalar-tensor theory can take all values greater than $-\frac{3}{2}$. Such solutions are contrary to Dirac's hypothesis as well as to Einstein-Mach's principle.

1. Introduction

In two previous papers we discussed (Dehnen and Obregón, 1971, 1972) the cosmological solutions of Brans and Dicke's scalar-tensor theory of gravitation for the case where a power law is valid between the gravitational constant κ and the radius of curvature R of the universe, and under the natural assumption that the present mass density of the universe does not vanish. In this connection the question arises if possibly non-trivial cosmological *vacuum* solutions exist in the scalar-tensor theory. Evidently such solutions would first of all possess theoretical importance, but they also could be available models for the universe in case that the influence of the matter in the universe upon the structure of the space-time would be negligible in comparison with that of the scalar κ-field.

Recently, this question has been investigated by O'Hanlon and Tupper (1972). These authors have shown that cosmological vacuum solutions exist in the scalar-tensor theory and for the *flat* 3-dimensional space they have given the general solutions. But for the more interesting cases of the non-flat 3-dimensional spaces they have found explicitly only some particular solutions for three special values of the freely available parameter ω of the scalar-tensor theory, namely for $\omega=0$, $-\frac{3}{2}$ and $-\frac{4}{3}$. Therefore it may be justified that we discuss in the following, analogous to our previous papers, the whole class of the cosmological vacuum solutions with a power law between the gravitational constant and the radius of curvature of the universe.

Because the cosmological differential equations used in our previous papers are only valid for non-vanishing mass densities the desired cosmological vacuum solutions cannot be found unambiguously from our previous solutions in the limiting case of vanishing mass density. Thus a new integration of the cosmological vacuum-field equations of the scalar-tensor theory is necessary with the application of a power law

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between the gravitational constant and the radius of curvature of the universe. We find the remarkable result that besides the solutions given by O'Hanlon and Tupper there exist further vacuum solutions for all values of the parameter ω. In case of the non-flat 3-dimensional spaces, $\omega < -\frac{3}{2}$ holds for the space of negative curvature and $\omega > -\frac{3}{2}$ is valid for the closed 3-dimensional space of positive curvature. Just as in our previous solutions, the gravitational constant increases with the expansion of the universe, in contrast to Dirac's hypothesis (1937, 1938), which requires a decrease of the value of the gravitational constant.

The existence of non-trivial vacuum solutions for the closed 3-dimensional space means a direct contradiction to Einstein-Mach's principle (1918), according to which the behaviour of the space-time, and with regard to its interpretation by Brans and Dicke (1961) also the behaviour of the gravitational constant, should be determined solely by the matter in the universe.

2. The Integration of the Cosmological Vacuum-Field Equations

Starting with the Robertson-Walker line-element

$$\mathrm{d}s^2 = -R^2(t)\left\{\frac{\mathrm{d}r^2}{1-\varepsilon r^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\right\} + \mathrm{d}t^2 \tag{1}$$

with $\varepsilon = +1, 0, -1$ for the space of positive, vanishing and negative curvature respectively one finds the following cosmological vacuum differential equations of Brans and Dicke's scalar-tensor theory (see O'Hanlon and Tupper, 1972)

$$\left(\frac{\dot{R}}{R} - \frac{1}{2}\frac{\dot{\kappa}}{\kappa}\right)^2 + \frac{\varepsilon}{R^2} = \tfrac{1}{4}(1 + \tfrac{2}{3}\omega)\left(\frac{\dot{\kappa}}{\kappa}\right)^2, \tag{2}$$

$$\frac{\dot{\kappa}}{\kappa^2} R^3 = -B, \quad B = \text{const}. \tag{3}$$

where κ represents the relativistic gravitational constant, which is inversely proportional to Brans and Dicke's scalar field quantity Φ. At the present time t_0, κ is given by $\kappa_0 = 8\pi G_0/c^4$, wherein G_0 is Newton's gravitational constant.

For solving the two coupled non-linear differential Equations (2) and (3) we assume a power law between the gravitational constant κ and the radius of curvature R of the universe* of the form

$$\kappa R^n = C \quad (C = \text{const}). \tag{4}$$

Elimination of κ and $\dot{\kappa}$ in (3) with the help of (4) yields

$$R^{2+n}\dot{R} = \frac{BC}{n}. \tag{5}$$

* It is easy to show that the solutions following from the application of a power law between κ and t are contained in the solutions discussed below.

With the integration constant D from (5) we obtain

$$R = \left(\frac{n+3}{n} BCt + D\right)^{1/(n+3)}. \tag{6}$$

After the elimination of R, $\dot{R}$, κ and $\dot{\kappa}$ with the use of (4) and (6) the differential Equation (2) results in the algebraic relation

$$\left[\left(\frac{1}{n} + \frac{1}{2}\right)^2 - \frac{1}{4}\left(1 + \frac{2}{3}\omega\right)\right] B^2C^2 + \varepsilon\left(\frac{n+3}{n} BCt + D\right)^{(4+2n)/(3+n)} = 0. \tag{7}$$

To satisfy Equation (7) we distinguish in the following the cases of the flat and the non-flat spaces.

3. Flat Space Solutions

The solutions of this kind are already given in the paper of O'Hanlon and Tupper (1972). But we will discuss briefly their properties especially in view of the application to the universe.

For $\varepsilon=0$ it follows from (7) because of $BC\neq 0$

$$n = \frac{2}{\pm\sqrt{1+\frac{2}{3}\omega} - 1}. \tag{8}$$

Therefore, $\omega \geqslant -\frac{3}{2}$ is necessary.

In the limiting case of $n=-3$ (see Equation (6)), for which according to (8) $\omega = -\frac{4}{3}$, one gets from (6) and (4) with the new constant a:

$$R = ae^{-BCt/3}, \qquad \kappa = Ca^3e^{-BCt}. \tag{9}$$

Because in view of the definition of the Hubble constant H

$$\frac{H}{c} = \left(\frac{\dot{R}}{R}\right)_0 = -\tfrac{1}{3}BC \tag{10}$$

is valid, one obtains from (9)

$$R = ae^{Ht/c}, \qquad \kappa = Ca^3e^{3Ht/c}. \tag{11}$$

According to the solution (11) the age of the universe would be infinite. Furthermore the behaviour of the gravitational constant is contrary to Dirac's hypothesis.

For the remaining values of n and ω the constant D in Equation (6) can be removed by a linear transformation of the time t. In this way we get from (6) and (4) with the use of (8)

$$R = \left(\frac{\pm 3\sqrt{1+\frac{2}{3}\omega} - 1}{2} BCt\right)^{(\pm\sqrt{1+\frac{2}{3}\omega} - 1)/(\pm 3\sqrt{1+\frac{2}{3}\omega} - 1)},$$

$$\kappa = C\left(\frac{\pm 3\sqrt{1+\frac{2}{3}\omega} - 1}{2} BCt\right)^{-2/(\pm 3\sqrt{1+\frac{2}{3}\omega} - 1)}. \tag{12}$$

In view of the definition of the Hubble constant H we find for the age of the universe:

$$t_0 = \frac{c}{H} \frac{\pm\sqrt{1+\frac{2}{3}\omega} - 1}{\pm 3\sqrt{1+\frac{2}{3}\omega} - 1}. \tag{13}$$

Herewith the solution (12) takes the form

$$\begin{aligned} R &= R_0 \left(\frac{t}{t_0}\right)^{(\pm\sqrt{1+\frac{2}{3}\omega}-1)/(\pm 3\sqrt{1+\frac{2}{3}\omega}-1)}, \\ \kappa &= \kappa_0 \left(\frac{t}{t_0}\right)^{-2/(\pm 3\sqrt{1+\frac{2}{3}\omega}-1)}, \end{aligned} \tag{14}$$

wherein

$$\begin{aligned} R_0 &= \left(\frac{\pm\sqrt{1+\frac{2}{3}\omega}-1}{2} BC \frac{c}{H}\right)^{(\pm\sqrt{1+\frac{2}{3}\omega}-1)/(\pm 3\sqrt{1+\frac{2}{3}\omega}-1)}, \\ \kappa_0 &= C\left(\frac{\pm\sqrt{1+\frac{2}{3}\omega}-1}{2} BC \frac{c}{H}\right)^{-2/(\pm 3\sqrt{1+\frac{2}{3}\omega}-1)}. \end{aligned} \tag{15}$$

are the present values of R and κ. From (15) the constants B and C can be determined by the knowledge of R_0 and κ_0. According to (14) the behaviour of the gravitational constant κ is contrary to Dirac's hypothesis for the negative sign of the root and for $\omega < -\frac{4}{3}$. If ω in the range $-\frac{3}{2} \leqslant \omega < -\frac{4}{3}$ approaches to the value $-\frac{4}{3}$ the age of the universe takes according to (13) arbitrary large values.

Finally we note that the solutions (11) and (14) are the general solutions of the differential Equations (2) and (3) for $\varepsilon = 0$, as is easy to show.

4. Non-Flat Space Solutions

In the case of $\varepsilon \neq 0$ the algebraic relation (7) can be satisfied for *all* values of t only if

$$n = -2. \tag{16}$$

Consequently Equation (7) takes the form

$$B^2 C^2 = \frac{4\varepsilon}{1+\frac{2}{3}\omega}. \tag{17}$$

Therefore in case of the closed 3-dimensional space of positive curvature ($\varepsilon = +1$) $\omega > -\frac{3}{2}$ must be chosen, whereas in case of the space of negative curvature ($\varepsilon = -1$) $\omega < -\frac{3}{2}$ is necessary. Herewith it follows from (17) that, in both cases,

$$BC = \pm \frac{2}{\sqrt{|1+\frac{2}{3}\omega|}}. \tag{18}$$

Because the constant D in Equation (6) can be removed by a linear time-transformation the Equations (6) and (4) result with respect to (16) and (18) in

$$R = \mp \frac{t}{\sqrt{|1 + \frac{2}{3}\omega|}}, \qquad \kappa = \frac{Ct^2}{|1 + \frac{2}{3}\omega|}. \tag{19}$$

For the age of the universe we find from (19) by the definition of the Hubble constant H

$$t_0 = \frac{c}{H}. \tag{20}$$

Herewith the solutions (19) become

$$R = R_0 \frac{t}{t_0}, \qquad \kappa = \kappa_0 \left(\frac{t}{t_0}\right)^2, \tag{21}$$

wherein

$$R_0 = \frac{c/H}{\sqrt{|1 + \frac{2}{3}\omega|}}, \qquad \kappa_0 = C \frac{(c/H)^2}{|1 + \frac{2}{3}\omega|}, \tag{22}$$

are the present values of the radius of curvature of the universe and the gravitational constant. It is of interest, that in these solutions it can not be distinguished between the cases $\omega > -\frac{3}{2}$ ($\varepsilon = +1$) and $\omega < -\frac{3}{2}$ ($\varepsilon = -1$) by determination of t_0, R_0 and κ_0 alone.

5. Conclusions

The vacuum solutions for the non-flat spaces are contrary to Dirac's hypothesis as well as the solutions for non-vanishing mass densities given in our previous papers. This fact supports our supposition, that the contradiction to Dirac's hypothesis is an immediate consequence of the assumption of a power law between the gravitational constant and the radius of curvature of the universe.

More remarkable however is the fact that for all values $\omega > -\frac{3}{2}$ cosmological vacuum solutions exist for the closed 3-dimensional space of positive curvature. (There are no analogous cosmological solutions in Einstein-Friedman's cosmology with vanishing cosmological constant.) It was the idea of Brans and Dicke (1961) that as a consequence of Mach's principle the value of the gravitational constant should be determined by the matter in the universe, and they have taken this conception as the reason for generalizing Einstein's theory of general relativity to the scalar-tensor theory of gravitation. Our result shows that the idea of Brans and Dicke is for a large ω-range not completely contained in their theory.

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