# WAVE-TRAINS IN THE SOLAR WIND

## I: *General Theory and Its Application to an Ideal, Isotropic, One-Fluid Plasma*

DIRK J. OLBERS

*Abteilung für Theoretische Geophysik, Universität Hamburg, Germany*

and

ARNE K. RICHTER

*Institut für Reine und Angewandte Kernphysik, Universität Kiel, Germany*

(Received 30 March, 1972)

**Abstract.** The main results of Whitham's averaged Lagrangian method for the treatment of linear wave-trains in a weakly inhomogeneous, moving medium are presented briefly. This method is then applied to an ideal, isotropic, one-fluid plasma which can be taken for the lowest order approximation for the interplanetary solar wind expansion.

## 1. Introduction

In order to obtain the expressions for the parameters of small amplitude, linear, harmonic waves in a given system one usually solves the eigenvalue problem of the characteristic wave equations derived from the equations of motion for this system and a linear, harmonic perturbation ansatz for the corresponding field variables. The disadvantage of such a procedure is, however, that e.g. the propagation equations for the waves' parameter (amplitude, frequency, wave vector, etc.) cannot be stated without difficulties. This is especially true if more realistic problems are considered, e.g., waves propagating in weakly inhomogeneous, moving media.

These difficulties can be omitted if one starts directly from the Lagrangian of the system in question. According to Whitham and others one now makes this perturbation ansatz for the Lagrangian, and obtains thus a Lagrangian for the small amplitude, linear harmonic waves. This Lagrangian is then averaged with respect to all wave phases. The variation of this averaged Lagrangian then yields the characteristic wave equations (which are treated in the usual way), while in addition the propagation equations for the waves' parameters can be derived in a rather uncomplicated and straightforward way from the general Lagrangian theory. Though these propagation equations can also be derived by using a multidimensional W.K.B. approach (Ablowitz and Benney, 1970), the advantage of this averaged Lagrangian method is that there is no change in the formalism if the theory is applied to non-linear systems.

In Section 2 we derive the main results of this averaged Lagrangian method for linear waves in non-moving and moving, weakly inhomogeneous media. This method is then applied to a moving, ideal, isotropic, one-fluid plasma (i.e., zero electrical resistance, viscosity, and thermal conductivity, and a scalar pressure). Such a plasma can be taken for the lowest order approximation for the quietly, large-scale expanding interplanetary solar wind. In Section 3 the Lagrangian density for this plasma, the averaged Lagrangian for the plasma waves, and by variations the characteristic wave equations are obtained. In Section 4 these wave equations are solved generally, and the waves' parameters and their propagation equations are derived.

## 2. Propagation Equations for Wave-Trains in Inhomogeneous, Moving Media

In this chapter we shall briefly explain how Whitham's method of the averaged Lagrangian (Whitham, 1970; Dougherty, 1970; and Bretherton and Garrett, 1969) can be applied to the propagation of wave-trains in a slowly varying background medium.

A wave-train is a superposition of almost sinusoidal, propagating waves. It can be described by a slowly time and space dependent amplitude, a dominant local frequency $\omega$ and wave vector $\mathbf{k}$. Thus the field variables of the wave-train are of the form:

$$\delta\varphi\left(\varepsilon\mathbf{x}, \varepsilon t\right) e^{i\theta(\varepsilon\mathbf{x}, \varepsilon t)/\varepsilon} + \text{c.c.}, \tag{2.1}$$

where $\varepsilon$ is a small parameter that usually characterizes the smallness of the background variations. The phase $\theta(\varepsilon\mathbf{x}, \varepsilon t)/\varepsilon$ is related to the frequency $\omega$ and the wave vector $\mathbf{k}$ by:

$$\omega\left(\varepsilon\mathbf{x}, \varepsilon t\right) := -\frac{\partial}{\partial t}\theta\left(\varepsilon\mathbf{x}, \varepsilon t\right)/\varepsilon, \tag{2.2}$$

$$\mathbf{k}\left(\varepsilon\mathbf{x}, \varepsilon t\right) := \nabla\theta\left(\varepsilon\mathbf{x}, \varepsilon t\right)/\varepsilon. \tag{2.3}$$

It seems to be generally true that equations which admit wave-trains as solutions can be derived from a variational principle:

$$\delta\int \mathrm{d}t \int \mathrm{d}^3x L\left(\varphi_\nu, \frac{\partial\varphi_\nu}{\partial x_i}, \frac{\partial\varphi_\nu}{\partial t}, \mathbf{x}, t\right) = 0, \tag{2.4}$$

were $\nu = 1, \ldots, n$ and $i = 1, 2, 3$. Thereby $L$ is the Lagrangian density for the set of field variables $\varphi_\nu$. The Euler-Lagrange equations obtained from (2.4) are the field equations (or an equivalent set) of the system considered:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial\left(\partial\varphi_\nu/\partial t\right)} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial L}{\partial\left(\partial\varphi_\nu/\partial x_i\right)} - \frac{\partial L}{\partial\varphi_\nu} = 0. \tag{2.5}$$

We now separate each $\varphi_\nu$ into a smoothly varying part $\varphi_{\nu 0}$, and into a small amplitude disturbance (waves) $\varphi_{\nu 1}$ of the form (2.1), i.e.:

$$\varphi_\nu\left(\mathbf{x}, t\right) = \varphi_{\nu 0}\left(\varepsilon\mathbf{x}, \varepsilon t\right) + \varepsilon\varphi_{\nu 1}\left(\mathbf{x}, t\right). \tag{2.6}$$

This yields a formal expansion for the Lagrangian density of the form:

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3 + \cdots. \tag{2.7}$$

Assuming that the $\varphi_{\nu 0}$ satisfy the lowest order of (2.4), then the corresponding Euler-Lagrange equations give the field equations for the background state. As a consequence the variation of $L_1$ with respect to the $\varphi_{\nu 1}$ vanishes identically. Hence, the waves are governed by the variation principle

$$\delta \int \mathrm{d}t \int \mathrm{d}^3 x \left[\varepsilon^2 L_2 + \varepsilon^3 L_3 + \cdots\right] = 0, \tag{2.8}$$

where the variation is performed with respect to the $\varphi_{\nu 1}$. The background state is regarded as given.

In the following we shall neglect all terms of higher order than $L_2$, i.e., we shall only discuss linear waves.

Generally the $\varphi_{\nu 1}$ consist of a number of wave-trains which differ by their wave vectors and/or belong to different modes. We therefore make the following ansatz:

$$\varphi_{\nu 1} = \sum_{\lambda \gtrless 0} \delta\varphi_\nu^\lambda(\varepsilon\mathbf{x}, \varepsilon t)\, e^{i\theta^\lambda(\varepsilon\mathbf{x}, \varepsilon t)/\varepsilon}. \tag{2.9}$$

Thereby $\lambda\,(\lambda > 0)$ counts the different wave-trains, and in order to ensure reality for the $\varphi_{\nu 1}$ it is:

$$\delta\varphi_\nu^{-\lambda} = (\delta\varphi_\nu^{-\lambda})^*, \qquad \theta^{-\lambda} = -\theta^\lambda. \tag{2.10}$$

As a consequence we get $\omega^{-\lambda} = -\omega^\lambda$ and $\mathbf{k}^{-\lambda} = -\mathbf{k}^\lambda$.

The aim is now to derive equations of motion for the parameters $\delta\varphi_\nu^\lambda$, $\omega^\lambda$ and $\mathbf{k}^\lambda$ of the wave-trains. According to Whitham this can be done in the following way: Inserting (2.9) into the Lagrangian $L_2$ of the waves, and averaging then $L_2$ with respect to all phases $\theta^\lambda$ an averaged Lagrangian density

$$\mathscr{L}_2 = \mathscr{L}_2\,(\delta\varphi_\nu^\lambda, \omega^\lambda, \mathbf{k}^\lambda, \varepsilon\mathbf{x}, \varepsilon t) \tag{2.11}$$

is obtained. Bisshopp (1969), Witham (1970) and others proved that the desired equations can now be derived formally by the variation of $\mathscr{L}_2$ with respect to the $\delta\varphi_\nu^{-\lambda}$ and $\theta^\lambda$ (remembering the definitions (2.2) and (2.3)). The variation with respect to the $\delta\varphi_\nu^{-\lambda}$ gives:

$$\frac{\partial \mathscr{L}_2}{\partial \delta\varphi_\nu^{-\lambda}} = 0, \tag{2.12}$$

and thus yields a system of linear equations for the amplitudes $\delta\varphi_\nu^\lambda$. This system can then be treated in the usual way as an eigenvalue problem to give the dispersion relations for the different modes:

$$\omega^\lambda = \Omega^\lambda(\mathbf{k}^\lambda, \varepsilon\mathbf{x}, \varepsilon t), \tag{2.13}$$

and the values for the corresponding amplitudes. From the variation with respect to the $\theta^\lambda$ one obtaines on the other hand:

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}_2}{\partial \omega^\lambda} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\frac{\partial \mathscr{L}_2}{\partial k_i^\lambda} = 0. \tag{2.14}$$

Furthermore, from (2.2) and (2.3) follow the consistency conditions:

$$\frac{\partial}{\partial t}\mathbf{k}^\lambda + \nabla\omega^\lambda = 0 \tag{2.15}$$

$$\nabla \times \mathbf{k}^\lambda = 0. \tag{2.16}$$

The Equations (2.12), (2.14)–(2.16) form a complete set of equations of motion for the wave parameters. They can, however, be cast in a more attractive form by using the energy density of the wave-disturbances. For non-moving media this density is obviously given by:

$$E = \sum_{\nu=1}^{n} \frac{\partial \varphi_{\nu 1}}{\partial t}\frac{\partial L_2}{\partial(\partial\varphi_{\nu 1}/\partial t)} - L_2. \tag{2.17}$$

Using the ansatz (2.9), the Equation (2.2), and averaging then $L_2$ with respect to all phases $\theta^\lambda$, we obtain from (2.17):

$$\langle E\rangle = \sum_\lambda \omega^\lambda \frac{\partial \mathscr{L}_2}{\partial \omega^\lambda} - \mathscr{L}_2 = \sum_\lambda \omega^\lambda \frac{\partial \mathscr{L}_2}{\partial \omega^\lambda} =: \sum_\lambda E^\lambda, \tag{2.18}$$

as, because of (2.12), $\mathscr{L}_2 = 0$. Thus, $\partial\mathscr{L}_2/\partial\omega^\lambda$ can be interpreted as the energy density $E^\lambda$ of the wave $\lambda$ divided by its frequency $\omega^\lambda$. This is just the definition of the wave action density $N^\lambda$:

$$N^\lambda := \frac{\partial \mathscr{L}_2}{\partial \omega^\lambda} = \frac{E^\lambda}{\omega^\lambda}. \tag{2.19}$$

Then equation (2.14) becomes:

$$\frac{\partial}{\partial t}N^\lambda + \nabla\cdot(\mathbf{u}^\lambda N^\lambda) = 0, \tag{2.20}$$

where

$$\mathbf{u}^\lambda := -\frac{\partial\mathscr{L}_2/\partial\mathbf{k}^\lambda}{\partial\mathscr{L}_2/\partial\omega^\lambda} = \frac{\partial\Omega^\lambda}{\partial\mathbf{k}^\lambda} \tag{2.21}$$

is the group velocity of the wave-train $\lambda$. (The second identity follows from $\mathscr{L}_2 = 0$.)

Equation (2.20) shows that $N^\lambda$ is convected along the path $\mathbf{x}^\lambda(t)$ of the wave-train $\lambda$. This path is given by the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^\lambda = \mathbf{u}^\lambda(\varepsilon\mathbf{x}, \varepsilon t) = \frac{\partial\Omega^\lambda}{\partial\mathbf{k}^\lambda}, \tag{2.22}$$

where:

$$\frac{\mathrm{d}}{\mathrm{d}t} := \frac{\partial}{\partial t} + (\mathbf{u}^\lambda\cdot\nabla).$$

The slow modulation of the frequency $\omega^\lambda$ and the wave vector $\mathbf{k}^\lambda$ along this path is given by Equations (2.13), (2.15) and (2.16), i.e., by

$$\frac{\partial\omega^\lambda}{\partial t} = \frac{\partial\Omega^\lambda}{\partial\mathbf{k}^\lambda}\frac{\partial\mathbf{k}^\lambda}{\partial t} + \frac{\partial\Omega^\lambda}{\partial t}$$

and

$$\frac{\partial\mathbf{k}_i^\lambda}{\partial t} = -\sum_{j=1}^{3}\frac{\partial\Omega^\lambda}{\partial k_j^\lambda}\frac{\partial k_j^\lambda}{\partial x_i} - \frac{\partial\Omega^\lambda}{\partial x_i}.$$

These equations can also be written in the form

$$\frac{\mathrm{d}\omega^\lambda}{\mathrm{d}t} = \frac{\partial\Omega^\lambda}{\partial t}, \tag{2.23}$$

and

$$\frac{\mathrm{d}\mathbf{k}^\lambda}{\mathrm{d}t} = -\nabla\Omega^\lambda. \tag{2.24}$$

Thus we obtain the well-known result that the frequency and the wave vector do not change along the wave path if and only if the system is closed (i.e., $L$ does not depend on $\mathbf{x}$ and $t$ explicitly) and if the background is constant. It should be mentioned that the Equations (2.22)–(2.24) are very similar to the Hamiltonian equations, if $\Omega^\lambda$ is regarded as the Hamiltonian.

We want now to generalize the above results, which are valid for a weakly inhomogeneous but not moving background, to the propagation of wave-trains into moving media:

Let $\mathbf{V}_0(\varepsilon\mathbf{x}, \varepsilon t)$ be the velocity field of such a medium, $(\mathbf{x}, t)$ the coordinates of a particle in the fixed frame of reference, and $(\mathbf{x}', t)$ its coordinates in the frame moving with the local velocity $\mathbf{V}_0$, i.e.:

$$\frac{\partial}{\partial t}\mathbf{x}(\mathbf{x}', t) = \mathbf{V}_0(\varepsilon\mathbf{x}, \varepsilon t). \tag{2.25}$$

The relation between $(\omega'^\lambda, \mathbf{k}'^\lambda)$ measured in the moving frame and $(\omega^\lambda, \mathbf{k}^\lambda)$ measured in the fixed frame follows from the transformation of the phase function:

$$\theta'^\lambda(\varepsilon\mathbf{x}', \varepsilon t) = \theta^\lambda(\varepsilon\mathbf{x}(\mathbf{x}', t), \varepsilon t). \tag{2.26}$$

Hence, with (2.2), (2.3) and (2.25) we obtain:

$$\omega'^{\lambda} := -\frac{\partial}{\partial t}\theta'^{\lambda}(\varepsilon\mathbf{x}', \varepsilon t)/\varepsilon = -\frac{\partial}{\partial t}\theta^{\lambda}/\varepsilon - \sum_{i=1}^{3}\frac{\partial\theta^{\lambda}/\varepsilon}{\partial x_i}\frac{\partial x_i}{\partial t}, \tag{2.27}$$

or

$$\omega^{\lambda} = \omega'^{\lambda} + \mathbf{k}^{\lambda}\cdot\mathbf{V}_0 .$$

$$k_j'^{\lambda} := \frac{\partial}{\partial x_j'}\theta'^{\lambda}(\varepsilon\mathbf{x}', \varepsilon t)/\varepsilon = \sum_{i=1}^{3}\frac{\partial\theta^{\lambda}/\varepsilon}{\partial x_i}\frac{\partial x_i}{\partial x_j'} = \sum_{i=1}^{3} k_i^{\lambda}\frac{\partial x_i}{\partial x_j'} \tag{2.28}$$

or

$$k_j^{\lambda} = \sum_{i=1}^{3} k_i'^{\lambda}\frac{\partial x_i'}{\partial x_j} \quad (j = 1, 2, 3).$$

Introducing now the intrinsic dispersion function $\Omega'^{\lambda}(\mathbf{k}'^{\lambda}, \varepsilon\mathbf{x}', \varepsilon t)$ by

$$\Omega^{\lambda}(\mathbf{k}^{\lambda}, \varepsilon\mathbf{x}, \varepsilon t) =: \Omega'^{\lambda}(\mathbf{k}'^{\lambda}, \varepsilon\mathbf{x}', \varepsilon t) + \mathbf{k}^{\lambda}\cdot\mathbf{V}_0 , \tag{2.29}$$

we see that the group velocity $\mathbf{u}^{\lambda}$ becomes

$$\mathbf{u}^{\lambda} = \mathbf{u}'^{\lambda} + \mathbf{V}_0 \quad \text{with} \quad \mathbf{u}'^{\lambda} := \frac{\partial\Omega'^{\lambda}}{\partial\mathbf{k}^{\lambda}} . \tag{2.30}$$

Thus the group velocity $\mathbf{u}^{\lambda}$ is the sum of the bulk velocity $\mathbf{V}_0$ of the background medium plus the intrinsic group velocity $\mathbf{u}'^{\lambda}$.

Considering (2.30) we may therefore say that all the above Equations (2.13)–(2.24) with exception of the Equations (2.17), (2.18) and (2.19) remain valid even for the case of a moving medium. Corresponding to (2.18) we obtain, of course, for the averaged energy density of the disturbances in the moving frame the expression

$$\langle E'\rangle = \sum_{\lambda}\omega'^{\lambda}\frac{\partial\mathscr{L}_2'}{\partial\omega'^{\lambda}} - \mathscr{L}_2' , \tag{2.31}$$

where $\mathscr{L}_2' = \mathscr{L}_2'(\delta\varphi_v'^{\lambda}, \mathbf{k}'^{\lambda}, \omega'^{\lambda}, \varepsilon\mathbf{x}', \varepsilon t)$ is the averaged Lagrangian density in this frame. Thereby $\mathscr{L}_2'$ may be derived from $\mathscr{L}_2$ by transforming the variational principle (2.4) into the moving frame. First we obtain for the Lagrangian densities:

$$L_2'\left(\varphi_v', \frac{\partial\varphi_v'}{\partial x_i'}, \frac{\partial\varphi_v'}{\partial t}, \mathbf{x}', t\right) =$$

$$= \mathscr{J} L_2\left(\varphi_v', \sum_{j=1}^{3}\frac{\partial\varphi_v'}{\partial x_j'}\frac{\partial x_j'}{\partial x_i}, \frac{\partial\varphi_v'}{\partial t} + \sum_{j=1}^{3}\frac{\partial\varphi_v'}{\partial x_j'}\frac{\partial x_j'}{\partial t}, \mathbf{x}(\mathbf{x}', t), t\right), \tag{2.32}$$

where $\mathscr{J}$ is the Jacobian of the transformation $\mathbf{x} = \mathbf{x}(\mathbf{x}', t)$. Then the averaged Lagrangian density becomes using the procedure mentioned before:

$$\mathscr{L}'_2(\delta\varphi'^{\lambda}_{\nu}, \mathbf{k}'^{\lambda}, \omega'^{\lambda}, \varepsilon\mathbf{x}', \varepsilon t) =$$

$$= \mathscr{J}\mathscr{L}_2\left(\delta\varphi'^{\lambda}_{\nu}, \sum_{j=1}^{3} k'^{\lambda}_j \frac{\partial x'_j}{\partial x_i}, \omega'^{\lambda} - \mathbf{k}'^{\lambda}\cdot\frac{\partial \mathbf{x}'}{\partial t}, \varepsilon\mathbf{x}(\mathbf{x}', t), \varepsilon t\right) =$$

$$= \mathscr{J}\mathscr{L}_2(\delta\varphi^{\lambda}_{\nu}, \mathbf{k}^{\lambda}, \omega^{\lambda}, \varepsilon\mathbf{x}, \varepsilon t). \tag{2.33}$$

The last identity follows from (2.27) and (2.28), and the fact that the particle's coordinate $\mathbf{x}'$ is constant along the particle's path, i.e.

$$\frac{\partial}{\partial t}\mathbf{x}' + (\mathbf{V}_0\cdot\nabla)\,\mathbf{x}' = 0. \tag{2.34}$$

Then (2.31) gives together with (2.33) and $\mathscr{L}'_2 = 0$ the relation

$$\langle E'\rangle = \sum_{\lambda} \mathscr{J}\omega'^{\lambda}\frac{\partial\mathscr{L}_2}{\partial\omega^{\lambda}} =: \sum_{\lambda} E'^{\lambda}, \tag{2.35}$$

and thus

$$E^{\lambda} = \mathscr{J}E'^{\lambda} = \omega'^{\lambda}\frac{\partial\mathscr{L}_2}{\partial\omega^{\lambda}} = \omega'^{\lambda}N^{\lambda}. \tag{2.36}$$

Hence, we have shown that in general the action density is the quotient of the perturbation energy density and the *intrinsic* frequency. Furthermore it may be seen from (2.20) and (2.36) that the total wave action is always conserved, while the total wave energy is conserved if and only if the intrinsic frequency is constant along the wave's path.

Finally we want to derive a propagation equation for the amplitudes $\delta\varphi^{\lambda}_{\nu}$ of the wave $\lambda$, strictly speaking for the absolute value of $\delta\varphi^{\lambda}_{\nu}$. Let $\delta\varphi^{\lambda}_{\nu} \equiv \varepsilon^{\lambda}$, for brevity. Then each $\delta\varphi^{\lambda}_{\mu}$ can be expressed by $\varepsilon^{\lambda}$ because of (2.12). Inserting these expressions into $N^{\lambda}$, which is a quadratic form of the $\delta\varphi^{\gamma}_{\nu}$, we obtain an expression of the following general type:

$$N^{\lambda} = \mathrm{sign}(\lambda)\,|\varepsilon^{\lambda}|^2\,\Psi^{\lambda}_0(\mathbf{k}^{\lambda}, \varepsilon\mathbf{x}, \varepsilon t). \tag{2.37}$$

The explicit $(\varepsilon\mathbf{x}, \varepsilon t)$ dependence of $\Psi^{\lambda}_0$ originates from the field variables of the background medium. From (2.20) and (2.37) we then deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}|\varepsilon^{\lambda}| = -\tfrac{1}{2}|\varepsilon^{\lambda}|\left[(\nabla\cdot\mathbf{u}^{\lambda}) + \frac{1}{\Psi^{\lambda}_0}\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{\lambda}_0\right]. \tag{2.38}$$

The last term in the bracket may be cast into the following form by means of (2.22) and (2.24):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{\lambda}_0 = \sum_{i=1}^{3}\left[\frac{\partial\Omega^{\lambda}}{\partial k^{\lambda}_i}\frac{\partial\Psi^{\lambda}_0}{\partial x_i} - \frac{\partial\Omega^{\lambda}}{\partial x_i}\frac{\partial\Psi^{\lambda}_0}{\partial k^{\lambda}_i}\right] + \frac{\partial}{\partial t}\Psi^{\lambda}_0. \tag{2.39}$$

This again shows the close analogy to the Hamiltonian theory with $\Omega^\lambda$ as the Hamiltonian. From (2.38) we may conclude that the waves' amplitudes are constant if and only if the background medium is constant in space and time, which, of course, is a very familiar result.

## 3. Averaged Lagrangian for the Ideal MHD-Plasma

Isotropic, one- or multi-fluid hydromagnetic equations are mainly used in the literature in order to describe the quiet, large-scale solar wind expansion. On the other hand the interplanetary solar wind plasma is treated without exception in the ideal (i.e., zero electrical resistance, viscosity and thermal conductivity), one-fluid approximation for the identification of continuous and discontinuous waves. Thus studying wave-trains in the quietly expanding solar wind by means of the theory presented in the chapter above we shall start with the ideal, isotropic, one-fluid description of the interplanetary medium.

It is well known that in this approximation the equations of motion may be cast into one continuity equation (involving the mean average density $\varrho(\mathbf{x}, t)$ and the average bulk velocity $\mathbf{V}(\mathbf{x}, t)$), into one momentum equation (involving $\mathbf{V}$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$), into one Maxwell's equation (expressing the frozen-in condition of the magnetic field lines), and that the energy equation (using the first law of thermodynamics) may be converted into one propagation-equation for the entropy $s(\mathbf{x}, t)$. Let $e(\varrho, s)$ be the internal energy per unit mass, then the Lagrangian density for the ideal, isotropic, one-fluid hydromagnetic solar wind plasma in the Eulerian description reads (see also Lundgren, 1963; or Seliger and Whitham, 1968):

$$L = \tfrac{1}{2}\varrho V^2 - \varrho e - \frac{1}{8\pi} B^2 + \varphi\, \hat{\mathrm{D}}_t \varrho + \eta\, \hat{\mathrm{D}}_t(\varrho s) + \mathbf{q}\cdot \hat{\mathrm{D}}_t(\varrho \mathbf{r}) - \\ - \mathbf{h}\cdot\left[\frac{\partial}{\partial t}\mathbf{B} - \nabla \times (\mathbf{V}\times\mathbf{B})\right]. \tag{3.1}$$

Thereby $\varphi$, $\eta$, $\mathbf{q}$ and $\mathbf{h}$ are Lagrangian multipliers, and $\mathbf{r}(\mathbf{x}, t)$ the initial coordinates that do not change along the particles' path. Furthermore we used the following abbreviations:

$$\hat{\mathrm{D}}_t := \frac{\partial}{\partial t} + \nabla\cdot(\mathbf{V}), \tag{3.2}$$

$$D_t := \frac{\partial}{\partial t} + (\mathbf{V}\cdot\nabla), \tag{3.3}$$

so that:

$$\hat{\mathrm{D}}_t = D_t + (\nabla\cdot\mathbf{V}). \tag{3.4}$$

The variations with respect to the Lagrangian multipliers and the field variables lead to:

$$\varphi: \quad \hat{\mathrm{D}}_t \varrho = 0 \tag{3.5}$$

$$\eta: \quad D_t s = 0 \tag{3.6}$$

$$\mathbf{q}: \quad D_t \mathbf{r} = 0 \tag{3.7}$$

$$\mathbf{h}: \quad \frac{\partial}{\partial t} \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) \tag{3.8}$$

$$\varrho: \quad \tfrac{1}{2} V^2 - (e + p/\varrho) - D_t \varphi - s D_t \eta - \mathbf{r} \cdot D_t \mathbf{q} = 0 \tag{3.9}$$

$$s: \quad D_t \eta = - T \tag{3.10}$$

$$\mathbf{V}: \quad \mathbf{V} = \nabla \varphi + s \nabla \eta + (\mathbf{r} \cdot \nabla)\, \mathbf{q} - \frac{1}{\varrho} \mathbf{B} \times [\nabla \times \mathbf{h}] \tag{3.11}$$

$$\mathbf{r}: \quad D_t \mathbf{q} = 0 \tag{3.12}$$

$$\mathbf{B}: \quad \frac{\partial}{\partial t} \mathbf{h} = \frac{1}{4\pi} \mathbf{B} + \mathbf{V} \times [\nabla \times \mathbf{h}]\,. \tag{3.13}$$

Thereby $T$ and $p$ denote the mean average temperature and scalar pressure, respectively. In addition we made use of the first law of thermodynamics:

$$T\, \mathrm{d}s = \mathrm{d}e + p\, \mathrm{d}(1/\varrho)\,. \tag{3.14}$$

Thus we obtain by variations of $L$ the known equations of motion for an ideal, isotropic, one-fluid mhd-plasma: For (3.5) is the continuity equation, (3.6) the energy equation, (3.8) the frozen-in magnetic field condition, and (3.7) the conservation equation for the initial coordinates along the particles' path. From (3.9)–(3.13) the momentum equation (the equation for $D_t \mathbf{V}$) may be obtained (Lundgren, 1963) by differentiating Equation (3.11) and then using (3.11), (3.9), (3.10), (3.12), (3.13) and the constraint equations. From (3.11) it follows that the formally introduced Langragian multipliers $\varphi$, $\eta$, $\mathbf{q}$ and $\mathbf{h}$ really are potentials describing the vector field $\mathbf{V}$.

According to the theory and because of (3.11) we make the following perturbation ansatz for the field variables $(\varrho, \mathbf{V}, \mathbf{B}, s, \mathbf{r}) = : \mathbf{A}(\mathbf{x}, t)$ and the potentials $(\varphi, \eta, \mathbf{q}, \mathbf{h}) = \, = : \boldsymbol{\phi}(\mathbf{x}, t)$

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0(\varepsilon \mathbf{x}, \varepsilon t) + \varepsilon \mathbf{A}_1(\mathbf{x}, t) \tag{3.15}$$

$$\boldsymbol{\phi}(\mathbf{x}, t) = \frac{1}{\varepsilon} \boldsymbol{\phi}_0(\varepsilon \mathbf{x}, \varepsilon t) + \varepsilon \boldsymbol{\phi}_1(\mathbf{x}, t)\,. \tag{3.16}$$

For $\mathbf{A}_1$ and $\boldsymbol{\phi}_1$ we now use the ansatz (2.9). We then derive the averaged Lagrangian density (2.11):

$$\mathscr{L}_2 = \sum_{\lambda \gtrless 0} \left\{ \tfrac{1}{2}\varrho_0 |\delta \mathbf{V}^\lambda|^2 - \frac{1}{2}\frac{c_0^2}{\varrho_0} |\delta\varrho^\lambda|^2 - \tfrac{1}{2}\varrho_0 \frac{\partial T_0}{\partial s_0} |\delta s^\lambda|^2 - \frac{1}{\varrho_0}\frac{\partial p_0}{\partial s_0} \delta\varrho^\lambda \, \delta s^{-\lambda} - \right.$$
$$- \frac{1}{8\pi} |\delta \mathbf{B}^\lambda|^2 - \delta \mathbf{h}^{-\lambda} \cdot [- i\omega'^\lambda \, \delta \mathbf{B}^\lambda - i\mathbf{k}^\lambda \times (\delta \mathbf{V}^\lambda \times \mathbf{B}_0)] +$$
$$+ (\delta\varphi^{-\lambda} + s_0 \, \delta\eta^{-\lambda} + \mathbf{r}_0 \cdot \delta\mathbf{q}^{-\lambda}) [- i\omega'^\lambda \, \delta\varrho^\lambda + \varrho_0 i (\mathbf{k}^\lambda \cdot \delta \mathbf{V}^\lambda)] -$$
$$\left. - i\omega'^\lambda \varrho_0 [\delta\eta^{-\lambda} \, \delta s^\lambda + \delta\mathbf{q}^{-\lambda} \cdot \delta \mathbf{r}^\lambda] \right\}. \tag{3.17}$$

Thereby:

$$c_0^2 := \frac{\partial p_0}{\partial \varrho_0}, \tag{3.18}$$

and as denoted before:

$$\omega'^\lambda = \omega^\lambda - (\mathbf{k}^\lambda \cdot \mathbf{V}_0).$$

The variations with respect to the $\delta\boldsymbol{\phi}^{-\lambda}$ and $\delta\mathbf{A}^{-\lambda}$ then yields the following characteristic equations for the different waves:

$$\delta\varphi^{-\lambda}: \quad - i\omega'^\lambda \, \delta\varrho^\lambda + i\varrho_0 (\mathbf{k}^\lambda \cdot \delta\mathbf{V}^\lambda) = 0 \tag{3.19}$$

$$\delta\eta^{-\lambda}: \quad - i\omega'^\lambda \varrho_0 \, \delta s^\lambda = 0 \tag{3.20}$$

$$\delta\mathbf{q}^{-\lambda}: \quad - i\omega'^\lambda \varrho_0 \, \delta\mathbf{r}^\lambda = 0 \tag{3.21}$$

$$\delta\mathbf{h}^{-\lambda}: \quad - i\omega'^\lambda \, \delta\mathbf{B}^\lambda - i\mathbf{k}^\lambda \times (\delta\mathbf{V}^\lambda \times \mathbf{B}_0) = 0 \tag{3.22}$$

$$\delta\varrho^{-\lambda}: \quad - \frac{c_0^2}{\varrho_0} \delta\varrho^\lambda - \frac{1}{\varrho_0}\frac{\partial p_0}{\partial s_0} \delta s^\lambda + i\omega'^\lambda (\delta\varphi^\lambda + s_0 \, \delta\eta^\lambda + \mathbf{r}_0 \cdot \delta\mathbf{q}^\lambda) = 0 \tag{3.23}$$

$$\delta\mathbf{V}^{-\lambda}: \quad \varrho_0 \, \delta\mathbf{V}^\lambda - i \, \delta\mathbf{h}^\lambda (\mathbf{k}^\lambda \cdot \mathbf{B}_0) + i\mathbf{k}^\lambda (\delta\mathbf{h}^\lambda \cdot \mathbf{B}_0) - \\ - i\varrho_0 \mathbf{k}^\lambda (\delta\varphi^\lambda + s_0 \, \delta\eta^\lambda + \mathbf{r}_0 \cdot \delta\mathbf{q}^\lambda) = 0 \tag{3.24}$$

$$\delta\mathbf{B}^{-\lambda}: \quad - \frac{1}{4\pi} \delta\mathbf{B}^\lambda - i\omega'^\lambda \, \delta\mathbf{h}^\lambda = 0 \tag{3.25}$$

$$\delta s^{-\lambda}: \quad - \varrho_0 \frac{\partial T_0}{\partial s_0} \delta s^\lambda - \frac{1}{\varrho_0}\frac{\partial p_0}{\partial s_0} \delta\varrho^\lambda + i\omega'^\lambda \varrho_0 \, \delta\eta^\lambda = 0 \tag{3.26}$$

$$\delta\mathbf{r}^{-\lambda}: \quad i\omega'^\lambda \varrho_0 \, \delta\mathbf{q}^\lambda = 0 . \tag{3.27}$$

As we are in the following only interested in the 'true' wave-trains we require that from now on:

$$\omega'^\lambda \neq 0 . \tag{3.28}$$

Then we obtain from (3.20), (3.21) and (3.27) that:

$$\delta s^{\lambda} = \delta \mathbf{r}^{\lambda} = \delta \mathbf{q}^{\lambda} = 0. \tag{3.29}$$

Thus the characteristic equations reduce by expressing them in terms of $\mathbf{A}_0$ and $\delta\mathbf{A}^{\lambda}$ only to:

$$-\omega'^{\lambda}\,\delta\varrho^{\lambda} + \varrho_0(\mathbf{k}^{\lambda}\cdot\delta\mathbf{V}^{\lambda}) = 0 \tag{3.30}$$

$$-\omega'^{\lambda}\,\delta\mathbf{B}^{\lambda} + \mathbf{B}_0(\mathbf{k}^{\lambda}\cdot\delta\mathbf{V}^{\lambda}) - \delta\mathbf{V}^{\lambda}(\mathbf{k}^{\lambda}\cdot\mathbf{B}_0) = 0 \tag{3.31}$$

$$-\omega'^{\lambda}\,\delta\mathbf{V}^{\lambda} + \frac{c_0^2}{\varrho_0}\,\delta\varrho^{\lambda}\mathbf{k}^{\lambda} + \frac{1}{4\pi\varrho_0}(\delta\mathbf{B}^{\lambda}\cdot\mathbf{B}_0)\,\mathbf{k}^{\lambda} - \frac{1}{4\pi\varrho_0}(\mathbf{k}^{\lambda}\cdot\mathbf{B}_0)\,\delta\mathbf{B}^{\lambda} = 0. \tag{3.32}$$

We now use (3.29) in the expression (3.17) of $\mathscr{L}_2$. Then we form $\partial\mathscr{L}_2/\partial\omega'^{\lambda}$ and $\partial\mathscr{L}_2/$ $/\partial\mathbf{k}^{\lambda}$. By eliminating now the potentials $\delta\boldsymbol{\phi}^{\lambda}$ in terms of $\mathbf{A}_0$ and $\delta\mathbf{A}^{\lambda}$ by means of (3.19)–(3.26) we finally obtain after some vector algebra:

$$\frac{\partial\mathscr{L}_2}{\partial\omega'^{\lambda}} = \frac{1}{\omega'^{\lambda}}\left[\frac{c_0^2}{\varrho_0}\,|\delta\varrho^{\lambda}|^2 + \frac{1}{4\pi}\,|\delta\mathbf{B}^{\lambda}|^2\right], \tag{3.33}$$

$$\frac{\partial\mathscr{L}_2}{\partial\mathbf{k}^{\lambda}} = -\frac{1}{\omega'^{\lambda}}\Bigg[c_0^2\,Re\,\{\delta\varrho^{\lambda}\,\delta\mathbf{V}^{-\lambda}\} + \frac{1}{4\pi}\,Re\,\{(\delta\mathbf{B}^{\lambda}\cdot\mathbf{B}_0)\,\delta\mathbf{V}^{-\lambda} - (\delta\mathbf{B}^{\lambda}\cdot\delta\mathbf{V}^{-\lambda})\,\mathbf{B}_0\}\Bigg] - \mathbf{V}_0\,\frac{\partial\mathscr{L}_2}{\partial\omega'^{\lambda}}. \tag{3.34}$$

We shall now go on in the following way:

i) First we shall treat (3.30)–(3.32) as an eigenvalue problem. Thus we shall obtain the number of the different modes that may exist in the isotropic solar wind plasma with $\omega'^{\lambda}\neq 0$, their dispersion relations $\Omega^{\lambda}(\mathbf{k}^{\lambda}, \mathbf{A}_0)$, and finally the relationship between their characteristic amplitudes $\delta\varrho^{\lambda}$, $\delta\mathbf{V}^{\lambda}$ and $\delta\mathbf{B}^{\lambda}$.

(ii) Inserting these into (3.33) and (3.34) we shall be able to obtain for each different mode separately its group velocity $\mathbf{u}^{\lambda}$ from (2.21), its energy $E^{\lambda}$ and its action density $N^{\lambda}$ from (2.36) and finally the propagation equation for its amplitudes from (2.38).

Thereby we shall treat all necessary equations in the most general way, i.e., we shall not for example introduce special coordinate systems, as it is done normally. Furthermore it should be kept in mind that because of the subsidiary condition $\nabla\cdot\mathbf{B}=0$ the expression

$$\mathbf{k}^{\lambda}\cdot\delta\mathbf{B}^{\lambda} = 0 \tag{3.35}$$

holds throughout. In addition we shall make use of the following abbreviation:

$$\hat{\mathbf{k}}^{\lambda} := \frac{\mathbf{k}^{\lambda}}{|\mathbf{k}^{\lambda}|}.$$

## 4. Solutions of the Characteristic Wave Equations and the Propagation Equations

In this chapter we shall specify the solutions of the characteristic wave Equations (3.30)–(3.32) and the propagation equations for the different modes. By eliminating $\delta\varrho^\lambda$ from (3.30) and $\delta\mathbf{B}^\lambda$ from (3.31) we obtain from (3.32) and the definition of:

$$\mathbf{b}_0 := \frac{\mathbf{B}_0}{\sqrt{4\pi\varrho_0}}$$

an equation for $\delta\mathbf{V}^\lambda$ only:

$$\begin{aligned}\delta\mathbf{V}^\lambda[(\mathbf{k}^\lambda\cdot\mathbf{b}_0)^2 - (\omega'^\lambda)^2] + \mathbf{k}^\lambda[(c_0^2 + b_0^2)(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda) - \\ - (\mathbf{b}_0\cdot\delta\mathbf{V}^\lambda)(\mathbf{k}^\lambda\cdot\mathbf{b}_0)] - \mathbf{b}_0[(\mathbf{k}^\lambda\cdot\mathbf{b}_0)(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda)] = 0.\end{aligned} \tag{4.1}$$

Thus:

$$(\mathbf{b}_0\cdot\delta\mathbf{V}^\lambda) = \frac{c_0^2}{(\omega'^\lambda)^2}(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda)(\mathbf{k}^\lambda\cdot\mathbf{b}_0), \tag{4.2}$$

$$(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda)\left[(\mathbf{k}^\lambda)^2(c_0^2 + b_0^2) - (\omega'^\lambda)^2 - \frac{(\mathbf{k}^\lambda)^2 c_0^2}{(\omega'^\lambda)^2}(\mathbf{k}^\lambda\cdot\mathbf{b}_0)^2\right] = 0. \tag{4.3}$$

We now want to distinguish between the following two cases: Case I with $(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda) = 0$, and Case II with $(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda) \neq 0$.

CASE I: $(\mathbf{k}^\lambda\cdot\delta\mathbf{V}^\lambda) = 0$ (TRANSVERSE WAVES)

Then we obtain from (3.32)

$$\delta\varrho^\lambda = 0. \tag{4.4}$$

Assuming $(\mathbf{k}^\lambda\cdot\mathbf{B}_0) = 0$, then $\delta\mathbf{B}^\lambda = 0$ from (3.31), and thus $\delta\mathbf{V}^\lambda = 0$ from (3.32). To exclude this trivial case, let

$$(\mathbf{k}^\lambda\cdot\mathbf{B}_0) \neq 0. \tag{4.5}$$

Thus if $\delta\mathbf{V}^\lambda = 0$ or $\delta\mathbf{B}^\lambda = 0$, then $\delta\mathbf{B}^\lambda = 0$ or $\delta\mathbf{V}^\lambda = 0$ from (3.32), respectively. We may therefore assume in addition to (4.5):

$$(\delta\mathbf{V}^\lambda, \delta\mathbf{B}^\lambda) \neq 0. \tag{4.6}$$

Then it follows from (4.2) and (3.31) that:

$$(\delta\mathbf{V}^\lambda, \delta\mathbf{B}^\lambda) \perp (\mathbf{k}^\lambda, \mathbf{B}_0). \tag{4.7}$$

Because of (4.7) and (4.6) it follows from (4.1) that:

$$(\omega'^\lambda)^2 = (\mathbf{k}^\lambda\cdot\mathbf{b}_0)^2. \tag{4.8}$$

The mode satisfying (4.4)–(4.8) is called the transverse mode ($\lambda \equiv t$). Its dispersion relation reads:

$$\Omega^t(\mathbf{k}^t, \mathbf{A}_0) = \operatorname{sign}(t)\, |\mathbf{k}^t \cdot \mathbf{b}_0| + \mathbf{k}^t \cdot \mathbf{V}_0\,, \tag{4.9}$$

so that its intrinsic phase velocity becomes:

$$C^t = b_0\, |\cos\alpha|\,, \tag{4.10}$$

where $\sphericalangle\alpha := \sphericalangle(\mathbf{k}^t, \mathbf{b}_0)$.

Let $\mathbf{t}$ be an arbitrary unit vector perpendicular to $\mathbf{k}^t$ and $\mathbf{B}_0$. Then we make the following ansatz because of (4.6):

$$\frac{\delta\mathbf{B}^t}{\sqrt{4\pi\varrho_0}} = (\varepsilon^t b_0)\, \mathbf{t}\,, \tag{4.11}$$

where $|\varepsilon^t|$ denotes the relative change in $\mathbf{B}_0$ originating from this transverse wave.

Thus we obtain from (3.31) together with (4.8):

$$\delta\mathbf{V}^t = -\operatorname{sign}(\mathbf{k}^t \cdot \mathbf{B}_0)\, (\varepsilon^t b_0)\, \mathbf{t}\,. \tag{4.12}$$

By using now these expressions for $\delta\varrho^t$, $\delta\mathbf{V}^t$ and $\delta\mathbf{B}^t$ in Equations (3.33) and (3.34) for $\lambda \equiv t$, we obtain from the equations of Section 2 the group velocity $\mathbf{u}^t$, the energy $E^t$, the action density $N^t$:

$$\mathbf{u}^t = \operatorname{sign}(\mathbf{k}^t \cdot \mathbf{B}_0)\, \mathbf{b}_0 + \mathbf{V}_0\,. \tag{4.13}$$

$$E^t = \varrho_0\, |\varepsilon^t|^2\, b_0^2\,. \tag{4.14}$$

$$N^t = \operatorname{sign}(t)\, |\varepsilon^t|^2\, \frac{\varrho_0 b_0^2}{|\mathbf{k}^t \cdot \mathbf{b}_0|} \tag{4.15}$$

and the amplitude's equation for $\varepsilon$ from (2.38) with:

$$\Psi_0^t := \frac{\varrho_0 b_0^2}{|\mathbf{k}^t \cdot \mathbf{b}_0|}\,. \tag{4.16}$$

*Remark:* The total energy density of the transverse wave is, of course, $E^{+t} + E^{-t} = 2E^t$.

CASE II: $(\mathbf{k}^\lambda \cdot \delta\mathbf{V}^\lambda) \neq 0$ (MAGNETOSONIC WAVES)

From (3.30)–(3.32) it follows that normally

$$(\delta\varrho^\lambda, \delta\mathbf{V}^\lambda, \delta\mathbf{B}^\lambda) \neq 0\,. \tag{4.17}$$

We now form the following expressions:

$$[\mathbf{k}^\lambda \times\,, \mathbf{k}^\lambda \cdot (\mathbf{b}_0 \times] \tag{4.1}$$

and

$$[\delta\mathbf{V}^\lambda \times\,, \delta\mathbf{V}^\lambda \cdot, \mathbf{k}^\lambda \cdot (\mathbf{B}_0 \times] \tag{3.31}$$

From the equations thus obtained and from (4.3) we derive the following results:

(a) $\mathbf{k}^\lambda \parallel \mathbf{B}_0$ if and only if

$$(\omega'^\lambda)^2 = (\mathbf{k}^\lambda)^2\, C_0^2 \quad \text{or} \quad (\omega'^\lambda)^2 = (\mathbf{k}^\lambda)^2\, b_0^2$$

and

$$0 \neq \delta\mathbf{V}^\lambda \parallel \mathbf{k}^\lambda \quad \text{and} \quad \delta\mathbf{B}^\lambda \parallel \mathbf{k}^\lambda, \quad \text{i.e.,} \quad \delta\mathbf{B}^\lambda = 0\,. \tag{4.18}$$

The wave satisfying the first equation of (4.18) is called the (degenerated) fast-mode ($\lambda \equiv f$), while the wave belonging to the second Equation of (4.18) is known as the (degenerated) slow-mode ($\lambda \equiv s$). This last mode propagates with the same velocities as the transverse mode for $\mathbf{k}^t \parallel \mathbf{B}_0$. Thus the dispersion relations read:

$$\begin{aligned} \Omega^f &= \operatorname{sign}(f)\, |\mathbf{k}^f|\, c_0 + \mathbf{k}^f \cdot \mathbf{V}_0 \\ \Omega^s &= \operatorname{sign}(s)\, |\mathbf{k}^s|\, b_0 + \mathbf{k}^s \cdot \mathbf{V}_0\,. \end{aligned} \tag{4.19}$$

Because of the sonic character of these modes we make the following ansatz:

$$\delta\varrho^{f,s} = \varepsilon^{f,s}\varrho_0\,. \tag{4.20}$$

Then we obtain from (3.32) because of $\delta\mathbf{B}^{f,s} = 0$:

$$\delta\mathbf{V}^f = \operatorname{sign}(f)\, \varepsilon^f c_0 \mathbf{k}^f = \operatorname{sign}(f)\, \varepsilon^f \frac{c_0}{b_0}\, \mathbf{b}_0\,, \tag{4.21}$$

and

$$\delta\mathbf{V}^s = \operatorname{sign}(s)\, \varepsilon^s \frac{c_0^2}{b_0}\, \hat{\mathbf{k}}^s = \operatorname{sign}(s)\, \varepsilon^s \frac{c_0^2}{b_0^2}\, \mathbf{b}_0\,.$$

Inserting now the expressions (4.20) and (4.21) into Equations (3.33) and (3.34) for $\lambda \equiv f$ and $s$, we obtain, according to the theory, the group velocities $\mathbf{u}^{f,s}$, the energies $E^{f,s}$, the action densities $N^{f,s}$, and the propagation equations for the amplitudes $\varepsilon^{f,s}$:

$$\mathbf{u}^f = \operatorname{sign}(f)\, c_0 \hat{\mathbf{k}}^f + \mathbf{V}_0 = \operatorname{sign}(f)\, \frac{c_0}{b_0}\, \mathbf{b}_0 + \mathbf{V}_0\,, \tag{4.22}$$

and

$$\mathbf{u}^s = \operatorname{sign}(s)\, b_0 \hat{\mathbf{k}}^s + \mathbf{V}_0 = \operatorname{sign}(s)\, \mathbf{b}_0 + \mathbf{V}_0\,.$$

$$E^{f,s} = \varrho_0 c_0^2\, |\varepsilon^{f,s}|^2\,. \tag{4.23}$$

$$N^f = \operatorname{sign}(f)\, |\varepsilon^f|^2 \frac{\varrho_0 c_0}{|\mathbf{k}^f|} =: \operatorname{sign}(f)\, |\varepsilon^f|^2\, \Psi_0^f\,, \tag{4.24}$$

$$N^s = \operatorname{sign}(s)\, |\varepsilon^s|^2 \frac{\varrho_0 c_0^2}{b_0\, |\mathbf{k}^s|} =: \operatorname{sign}(f)\, |\varepsilon^s|^2\, \Psi_0^s\,,$$

and the equations of motion for the amplitudes from (2.38) together with (4.22) and (4.24).

(b) $\mathbf{k}^\lambda \neq \mathbf{B}_0$ and therefore e.g., $\omega'^\lambda \neq \omega'^t$. We then obtain for the fast- and slow-modes:

$$(\omega'^\lambda)^2 = (\mathbf{k}^\lambda)^2 \tfrac{1}{2}[(c_0^2 + b_0^2) \pm \sqrt{(c_0^2 + b_0^2)^2 - 4c_0^2(\hat{\mathbf{k}}^\lambda \cdot \mathbf{b}_0)^2}] \\ = : (\mathbf{k}^\lambda)^2 (C^{f,s})^2, \tag{4.25}$$

where $f$ belongs to the plus-sign and $s$ to the minus-sign in front of the root. In general the vectors $\delta\mathbf{V}^\lambda$ and $\delta\mathbf{B}^\lambda$ lie in the $\mathbf{k}^\lambda - \mathbf{B}_0$ plane with $\delta\mathbf{B}^\lambda \perp \mathbf{k}^\lambda$. If and only if $\mathbf{k}^\lambda \perp \mathbf{B}_0$ it is $\delta\mathbf{V}^\lambda \parallel \mathbf{k}^\lambda$ and $\delta\mathbf{B}^\lambda \perp \mathbf{k}^\lambda$ but $\delta\mathbf{B}^\lambda \parallel \mathbf{B}_0$. For this special case only the fast-mode, now called the magnetosonic mode, can propagate (with the phase velocity $C^f = (c_0^2 + b_0^2)^{1/2}$).

Thus the general dispersion relations for the fast- and slow-modes read:

$$\Omega^f = \operatorname{sign}(f)\, |\mathbf{k}^f|\, C^f + \mathbf{k}^f \cdot \mathbf{V}_0$$

and

$$\Omega^s = \operatorname{sign}(s)\, |\mathbf{k}^s|\, C^s + \mathbf{k}^s \cdot \mathbf{V}_0 . \tag{4.26}$$

Because of the ansatz (4.20) we obtain from (3.30) and (4.1):

$$\delta\mathbf{V}^{f,s} = \operatorname{sign}(f, s)\, \varepsilon^{f,s} \frac{C^{f,s}[\hat{\mathbf{k}}^{f,s}(C^{f,s})^2 - \mathbf{b}_0(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)]}{[(C^{f,s})^2 - (\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]} . \tag{4.27}$$

And therefore it follows from (3.31) that:

$$(4\pi\varrho_0)^{-1/2}\, \delta\mathbf{B}^{f,s} = \varepsilon^{f,s}(C^{f,s})^2 \frac{[\mathbf{b}_0 - \hat{\mathbf{k}}^{f,s}(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)]}{[(C^{f,s})^2 - (\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]} . \tag{4.28}$$

Using now the expressions (4.20), (4.27) and (4.28) in Equation (3.33) and (3.34) for $\lambda \equiv f$ and $s$, we obtain the group velocities, the energies and the action densities for the fast- and slow-waves:

$$\mathbf{u}^{f,s} = \operatorname{sign}(f, s) \frac{C^{f,s}[(C^{f,s})^4\, \hat{\mathbf{k}}^{f,s} - c_0^2(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)\,\mathbf{b}_0]}{[(C^{f,s})^4 - c_0^2(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]} + \mathbf{V}_0 , \tag{4.29}$$

$$E^{f,s} = \frac{\varrho_0\, |\varepsilon^{f,s}|^2\, [(C^{f,s})^4 - c_0^2(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]}{[(C^{f,s})^2 - (\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]} , \tag{4.30}$$

and

$$N^{f,s} = \operatorname{sign}(f, s)\, |\varepsilon^{f,s}|^2 \frac{\varrho_0[(C^{f,s})^4 - c_0^2(\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]}{|\mathbf{k}^{f,s}|\, C^{f,s}[(C^{f,s})^2 - (\hat{\mathbf{k}}^{f,s}\cdot\mathbf{b}_0)^2]} =: \\ =: \operatorname{sign}(f, s)\, |\varepsilon^{f,s}|^2\, \Psi_0^{f,s} , \tag{4.31}$$

while the propagation equations for the amplitudes $\varepsilon^{f,s}$ follow from (2.38) together with the expressions for $\mathbf{u}^{f,s}$ from (4.29) and $\Psi_0^{f,s}$ from (4.31).

*Remark:* From Equation (4.26) we may deduce that $(C^f)^2 \geqslant \operatorname{Max}(c_0^2, b_0^2)$ and $(C^s)^2 \leqslant \operatorname{Min}(c_0^2, b_0^2)$. Thus we have, of course, $E^{f,s} > 0$.

## 5. Discussions

In this paper we have first discussed the main ideas of the Lagrangian method of wave-trains in a general continuum mechanical system, as it has been proposed by Whitham and others. According to this theory a linear, harmonic perturbation ansatz of the Lagrangian is made instead of the corresponding equations of motion. Thus the Lagrangian $L_2$ for all different wave-modes is obtained. By averaging $L_2$ over all phases an averaged Lagrangian $\mathscr{L}_2$ is derived. The variations of $\mathscr{L}_2$ then yields the characteristic wave equations, from which the number of different modes in such a system, their dispersion relations, and the values of their relative amplitudes may be obtained. The main advantage of this method is, however, that in addition the expressions for the group velocities, energies and action densities, and the conservation equations of the action densities along the waves' trajectories may be derived for these different modes in a rather uncomplicated, straightforward way. Besides one thus obtains coupled systems of partial differential equations which describe the changes in the amplitudes, the wave-vector and the frequency of each different mode propagating into a weakly inhomogeneous, moving (or not moving) medium caused by this medium itself.

We then applied this theory to the quietly, large-scale expanding solar wind plasma. Thereby we used, to begin with, the ideal, isotropic, one-fluid approximation for this interplanetary medium. Considering only wave-trains with non-zero phase velocities relative to the moving plasma, we obtained the propagation equations for the different wave modes that may be possible in such a system, namely for the transverse, the fast- and the slow-modes, respectively.

Now, it is well-known that for a more realistic description of the large-scale solar wind expansion its two-fluidity and/or the influence of the various transport phenomena have in addition to be taken into account. From the general theory it then follows, however, that these different descriptions will have a strong influence on the different wave modes and their propagation equations. These problems will therefore be discussed thoroughly in forthcoming papers of this series by using different solar wind models (reviewed recently by Richter, 1971). In addition we shall also discuss the reaction of the various wave-trains on the different sets of equations of motion of the expanding solar wind.

## Acknowledgements

The authors are indebted to Prof. Dr K. Hasselmann and Dr G. Wibberenz, and very thankful to Miss E. Radmann for typewriting the manuscript.

The work of one of us (A.K.R.) has been supported in part by the Deutsche Forschungsgemeinschaft.

## References

Ablowitz, M. J. and Benney, D. J.: 1970, *Stud. Appl. Math.* **49**, 225.
Bisshopp, F.: 1969, *J. Diff. Eqns.* **5**, 592.
Bretherton, F. P. and Garrett, C. J. R.: 1969, *Proc. Roy. Soc. (London)* **A 302**, 529.

Dougherty, J. P.: 1970, *J. Plasma Phys.* **4**, 761.
Lundgren, T. S.: 1963, *Phys. Fluids* **7**, 898.
Richter, A. K.: WRK-Report of the Institut für Reine und Angewandte Kernphysik der Universität Kiel, December 1971.
Seliger, R. L. and Whitham, G. B.: 1968, *Proc. Roy. Soc. (London)* **A 305**, 1.
Whitham, G. B.: 1970, *J. Fluid Mech.* **44**, 373.