THE COLLAPSE OF IRON-OXYGEN STARS: PHYSICAL AND MATHEMATICAL FORMULATION OF THE PROBLEM AND COMPUTATIONAL METHOD

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(Received 14 December, 1976)

Abstract. A statement of the problem of gravitational collapse and a computational method are described. The main feature of the collapse – its extremely high heterogeneity – is taken into account. The structure of a collapsing star is characterized by a dense and hot nucleon core which is opaque with respect to neutrino radiation and is embedded into an extended envelope, almost transparent to neutrinos. The envelope is gradually being accreted onto the core. The enormous amount of energy, radiated in the form of neutrinos and antineutrinos, make us pay particular attention to relatively small absorption of neutrino radiation by extended envelope (so-called energy of deposition). The inclusion of the energy deposition in the calculations is of importance for the problem of transformation of an implosion into an explosion. The deposition is taken into consideration in the approximation of diluted neutrino radiation which escapes from neutrino photosphere and is partially absorbed in the envelope. Both the generation of energy due to deposition and the change of neutron-to-proton ratio are taken into account. The increase of the mass of the core, which is opaque with respect to neutrino radiation, is fully taken into account in the calculations of the gravitational collapse.

1. Introduction

Colgate and White (1966) were the first to take account of the neutrino opacity in the dynamics of the gravitational collapse. As a result, the problem of energy transport from collapsing stellar core to stellar envelope with the aid of neutrinos has been put forward (so-called deposition problem). It was believed that, under favourable conditions, this energy transport could either induce the ejection of an envelope, observed as supernova outburst (pure deposition), or stimulate a flash of unburnt nuclear fuel (oxygen) in stellar envelope (deposition as a triggering mechanism in the Fowler's and Hoyle's (1964) supernova model). The subsequent investigations (Arnett, 1966, 1967; Ivanova *et al.*, 1969) have indicated that the deposition problem is indeed a very complex one, both in the physical and mathematical sense. We have no possibility to go here into details (one can refer to Zel'dovich's and Novikov's book (1971) and to an article by Imshennik and Nadyozhin (1974) presented at Copernicus's Assembly of IAU).

It is worthwhile to note that the main difficulty in the solution of this problem consists in the correct treatment of neutrino interaction with those layers of the collapsing core which are either semitransparent or opaque to neutrino radiation. The straightforward way of investigation of the neutrino deposition problem involves the numerical solution of hydrodynamical equations together with the equation of neutrino transfer (Wilson, 1971). However, a number of inevitable simplifications (very crude physical approximation, inadequate size of difference, mesh, etc.), which arise from the extreme complexity of the problem, bring the calculations of this kind at best to a status of numerical experiment.

A different approach is utilized in the present paper. It deals with the use of two asymptotic solutions of the equation of transfer: one solution being valid inside the collapsing core, which is highly opaque to neutrino radiation, and the other is applicable to neutrino interaction with the nearly transparent stellar envelope. The transfer of energy and lepton charge by means of neutrinos and antineutrinos inside the collapsing stellar core after onset of neutrino opacity is considered here in the neutrino thermal conductivity approximation (Imshennik and Nadyozhin, 1972). The gravitational collapse proves to be highly heterogeneous. Therefore, an intermediate layer between the opaque stellar core and transparent envelope contains a small fraction of stellar mass. This property of the collapse facilitates considerably the fitting of the envelope to the core in the course of computations.

In the present paper we consider the basic equations and the main methodical features of the calculations which were fulfilled for rather massive iron-oxygen stars. The astrophysical side of these calculations will be discussed in the next paper.

2. Initial Models and Differential Equations of the Problem

The initial hydrostatically equilibrium model can be conveniently represented in the form of a gaseous sphere with a polytropic index $n=3$ (Ivanova *et al.*, 1969). It will hereafter be assumed that the interiors of the initial model consists of iron-group nuclei and its envelope is composed of unburnt nuclear fuel (oxygen). The boundary between the iron core and oxygen envelope is chosen as close to the stellar centre as possible in order to meet the requirement of oxygen being unburnt up to the beginning of the collapse. This model gives the maximal efficiency of oxygen burning against a background of the collapse. Indeed, oxygen in this case is in the closest vicinity of the iron core – i.e., under the most favourable conditions for detonation. The stellar structure-like adopted initial model could be formed either as a result of large-scale mixing at the advanced stages of stellar evolution or due to intensive outflow of hydrogen-helium envelope, followed by an extinction of silicon and oxygen burning shells. However, the investigation of the evolution of carbon-oxygen stars (see, for example, Ikeuchi *et al.*, 1971, 1972) within the framework of classical spherically-symmetric theory without mass loss and large-scale mixing (due to meridional circulation, for example) indicates that the forming iron core to be separated from massive envelope by one or two burning shells. The stellar models, obtained in these calculations, have rather a giant-like structure just before the onset of dynamical instability. Therefore, the main storage of unburnt nuclear fuel is settled down at the radii which are considerably greater than in our simplified model.

For a specified mass of initial model, the single arbitrary parameter is the radius of the model. It is fixed by a trial-and-error method at the value which corresponds to the initial model being just at the boundary of dynamical instability, owing to dissociation of iron into alpha-particles and free nucleons. The initial temperature distribution is calculated from known equation of state and known pressure distribution over polytropic gaseous sphere for the initial model to be in hydrostatical equilibrium. With the initial model chosen in this way, the small perturbations, introduced by the difference approximation of differential equations, are quite sufficient to bring the model (after some calculations) into a state of collapse. The collapse of two models of masses 2 $M_\odot$ and 10 $M_\odot$ was computed. The initial characteristics of these models are presented in Table I.

TABLE I

The characteristics of the initial models at the boundary of dynamical stability

The total mass of the star	M_0	2 $M_\odot$	10 $M_\odot$
The mass of the iron core	M_{Fe}	1.82 $M_\odot$	6.48 $M_\odot$
The radius of the star	R_0	7.0×10^{-3} $R_\odot$ 4.9×10^8 cm	3.5×10^{-2} $R_\odot$ 2.45×10^9 cm
The central temperature	T_c	7.42×10^9 K	5.62×10^9 K
The central density	ϱ_c	4.45×10^8 g cm^{-3}	1.78×10^7 g cm^{-3}
The total gravitational energy	E_g	-0.326×10^{52} erg	-1.630×10^{52} erg
The total internal energy	E_{in}	0.272×10^{52} erg	1.460×10^{52} erg
The stores of nuclear energy in oxygen envelope	E_{16}	2.65×10^{50} erg	5.18×10^{51} erg
The energy necessary to dissociate iron core into free nucleons	E_{FN}	3.08×10^{52} erg	1.10×10^{53} erg
The total neutrino luminosity	$L_{\nu 0}$	4.61×10^{47} erg s^{-1}	3.56×10^{48} erg s^{-1}
The contribution of URCA process in total neutrino luminosity	$\frac{L_\nu^{URCA}}{L_{\nu 0}}$	0.072	0.030

Let us write down now the basic differential equations of the problem. At the first stage of the collapse when a star is *transparent* to neutrino radiation the following set of differential equations in Lagrangian form have to be used

$$\frac{\partial r}{\partial t} = u, \tag{1}$$

$$\frac{\partial u}{\partial t} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2}, \tag{2}$$

$$\frac{\partial r^3}{\partial m} = \frac{3}{4\pi\varrho}, \tag{3}$$

$$\frac{\partial E}{\partial t} + P\frac{\partial}{\partial t}\left(\frac{1}{\varrho}\right) = -\varepsilon_\nu + \varepsilon_{16}, \tag{4}$$

$$\frac{\partial X_{16}}{\partial t} = -\varrho X_{16}^2 (T_9/5.3)^{26}, \tag{5}$$

where independent variables are t (time) and m (mass enclosed in a sphere of radius r); ε_ν denotes the volume neutrino energy losses and ε_{16}, which is proportional to the right-hand side of Equation (5), denotes the rate of energy generation due to oxygen burning in reaction $O^{16}+O^{16}$. The right-hand side of Equation (5) approximates Fowler and Hoyle's (1964) data with sufficient accuracy in the relevant range of temperatures, $(2.5\text{–}4)\times 10^9$ K. The same approximation has been used by Ivanova *et al.* (1969). A dependence of pressure P, specific energy E, and ε_ν on temperature T and density ϱ will be discussed in the following sections.

In accordance with the initial model described above, the initial conditions for Equations (1)–(5) may be written down in the form

$$\text{For } t = 0 \begin{cases} r = r_0(m), \qquad \varrho = \varrho_0(m), \qquad T = T_0(m), \\ X_{16}(m) = \begin{cases} 1 & \text{for} \quad M_{\mathrm{Fe}} \leqslant m \leqslant M_0, \\ 0 & \text{for} \quad 0 \leqslant m \leqslant M_{\mathrm{Fe}}, \end{cases} \\ u = 0, \quad \dfrac{\partial u}{\partial t} = 0 \quad \text{for} \quad 0 \leqslant m \leqslant M_0, \end{cases} \tag{6}$$

where the distributions $r_0(m)$, $\varrho_0(m)$, and $T_0(m)$ are determined uniquely by a polytropic structure and a given equation of state, $P=P(\varrho, T)$. The boundary conditions are

$$\begin{aligned} r = 0 \quad \text{and} \quad u &= 0 \quad \text{for} \quad m = 0, \\ P &= 0 \quad \text{for} \quad m = M_0. \end{aligned} \tag{7}$$

The neutrino 'optical' depth τ_ν is increasing slowly as the collapse proceeds. From a certain time, the partial absorption of neutrinos begins to change the temperature distribution within the core in spite of neutrino 'optical' depth at the centre $\tau_{\nu c}$ being much less than 1. This phase of the collapse will henceforth be referred to as a *semitransparent stage*. At the semitransparent stage Equation (4) should be modified to yield

$$\frac{\partial E}{\partial t} + P \frac{\partial}{\partial t}\left(\frac{1}{\varrho}\right) = -\varepsilon_\nu + \varepsilon_{16} + \varepsilon_{\nu d}, \tag{4$'$}$$

where $\varepsilon_{\nu d}$ is the rate of heating due to the partial absorption of neutrinos and antineutrinos.

According to Ivanova *et al.* (1969), the volume energy density of neutrinos with energy e_ν at any radius r is given by

$$U_\nu(e_\nu, r) = \frac{1}{2cr} \int_0^R r' \varrho(r') B_\nu(e_\nu, r') \ln \left| \frac{r + r'}{r - r'} \right| \mathrm{d}r', \tag{8}$$

where R is the radius of the star and $B_\nu \, \mathrm{d}_\nu e$ is the energy radiated by unit mass, at energy interval $\mathrm{d}e_\nu$, per unit time. A quite analogous expression may be written down for antineutrinos. Using (8) and integrating with respect to the energies of neutrinos and antineutrinos, we obtain the expression for $\varepsilon_{\nu d}$ in the form

$$\varepsilon_{\nu d}(r) = \frac{c}{\varrho}\left[\int_0^\infty \frac{U_\nu(e_\nu, r)}{\tilde{l}_\nu(e_\nu, r)}\,\mathrm{d}e_\nu + \int_0^\infty \frac{U_{\tilde{\nu}}(e_{\tilde{\nu}}, r)}{\tilde{l}_{\tilde{\nu}}(e_{\tilde{\nu}}, r)}\,\mathrm{d}e_{\tilde{\nu}}\right], \tag{9}$$

where $\tilde{l}_\nu$ and $\tilde{l}_{\tilde{\nu}}$ are the mean free paths of neutrinos and antineutrinos with allowance for the stimulated absorption (see Equations (43), (44) in Section 4). The collapse could be calculated by use of Equation (4)′ instead of (4) from the beginning. In that case, however, the time of calculation would be highly increased, since the estimation of $\varepsilon_{\nu d}$ involves a double integration with respect to radius and energy for each space mesh point at every time step. Therefore, it is reasonable to make use of Equation (4)′ only for those times when neutrino 'optical' depth exceeds a certain value. We made use of Equation (4)′ in the case of $0.001 \leqslant \tau_{\nu c} \lesssim 1$. In the course of time, a central region of the collapsing star becomes opaque to neutrinos and antineutrinos ($\tau_\nu > 1$). The transport of energy by means of neutrino radiation can be approximated under such conditions by neutrino thermal conductivity equations (Imshennik and Nadyozhin, 1972).

Let us discuss now the equations to be applied to the core which is opaque with respect to the neutrinos and antineutrinos.* The Equations (1) and (3) are valid also in the 'neutrino core'. Equation (2) has the same form as for the transparent stage of the collapse, except that the total pressure P is the sum of material and neutrino-antineutrino pressures. According to Imshennik and Nadyozhin (1972), Equation (4) should be changed radically. One equation of energy has to be replaced now by a couple of differential equations describing diffusion of both energy and lepton charge

$$\frac{\partial E}{\partial t} + P\frac{\partial}{\partial t}\left(\frac{1}{\varrho}\right) = -4\pi\frac{\partial}{\partial m}(r^2 H) - \varepsilon_\mu, \tag{10}$$

$$\frac{\partial A}{\partial t} + 4\pi\frac{\partial}{\partial m}(r^2 F) = 0, \tag{11}$$

where P and E are the total pressure and specific energy with the contribution of neutrino-antineutrino gas being included; A is the mass density of lepton charge

$$A = (n_\nu + n_{e^-} - n_{\tilde{\nu}} - n_{e^+})/\varrho. \tag{12}$$

The temperature within the 'neutrino core' is so high that the heavy nuclei are practically all dissociated into free nucleons. Since, moreover, an inequality $kT \gg m_e c^2$ holds for the 'neutrino core', the transfer coefficients in the expressions for energy flux H and lepton charge flux F can be written out explicitly in terms of the densities of free neutrons n_n and protons n_p and the chemical potentials of neutrinos ψ_ν and

* For simplicity, this core will hereafter be referred to as '*neutrino core*'.

electrons ψ_e (Imshennik and Nadyozhin, 1972): namely,

$$H = -K_1 \frac{(1+\theta)^2}{\theta} T^2 r^2 \left[\frac{A_2}{T}\frac{\partial T}{\partial m} + A_1 \frac{\partial \psi_\nu}{\partial m}\right] \tag{13}$$

$$F = -K_2 \frac{(1+\theta)^2}{\theta} T r^2 \left[\frac{A_1}{T}\frac{\partial T}{\partial m} + \frac{\partial \psi_\nu}{\partial m}\right], \tag{14}$$

where

$$\begin{aligned} A_1 &= \psi_\nu - \frac{\theta - 1}{\theta + 1}, \\ A_2 &= \frac{\pi^2}{3} + \psi_\nu\left(\psi_\nu - 2\frac{\theta - 1}{\theta + 1}\right); \end{aligned} \tag{15}$$

where $\theta = n_n/n_p$ is connected with chemical potentials ψ_ν and ψ_e (in the units of kT) by the condition of statistical equilibrium of beta-processes by

$$\theta = \exp(\psi_e - \psi_\nu). \tag{16}$$

The electroneutrality condition $n_p = n_{e^-} - n_{e^+}$ yields an additional relation between ψ_e and θ:

$$\psi_e^3 + \pi^2 \psi_e = K_3 \varrho / [(1+\theta) T^3]. \tag{17}$$

With the aid of Equation (17) the expression (12) for lepton charge can be reduced to the form

$$\Lambda m_p = \frac{1}{1+\theta} + \frac{T^3}{2K_3 \varrho}(\psi_\nu^3 + \pi^2 \psi_\nu), \tag{18}$$

where m_p is the proton mass and Λm_p is the lepton charge per barion. The above equations for the 'neutrino core' should be supplemented by the equation of state (Section 6), the law of volume energy losses due to muon neutrino (Section 5) and the boundary conditions for the fluxes of energy and lepton charge. When the volume neutrino radiation of the transparent envelope may be neglected in comparison with neutrino emission form the surface of the 'neutrino core' (this is the case for reasonable choice of the 'neutrino core' boundary M_ν, see Section 7), the boundary conditions (Imshennik and Nadyozhin, 1972) are of the form

$$\left.\begin{aligned} \frac{cU}{2} &= H, \\ \frac{cN}{2} &= F. \end{aligned}\right\} \quad \text{for} \quad m = M_0. \tag{19}$$

At the centre of the 'neutrino core' the fluxes of energy and lepton charge have to vanish – i.e.,

$$\left.\begin{aligned} H &= 0 \\ F &= 0 \end{aligned}\right\} \quad \text{for} \quad m = 0. \tag{19$'$}$$

The total energy density of neutrinos and antineutrinos $U = U_\nu + U_{\bar\nu}$ and the difference between the densities of neutrinos and antineutrinos $N = n_\nu - n_{\bar\nu}$ can be expressed in terms of temperature and neutrino chemical potential as

$$U = K_4 T^4 \left(\psi_\nu^4 + 2\pi^2 \psi_\nu + \frac{7\pi^4}{15} \right), \tag{20}$$

$$N = \frac{T^3}{2K_3 m_p} (\psi_\nu^3 + \pi^2 \psi_\nu). \tag{21}$$

The constants K_1, K_2, K_3, and K_4 in Equations (13), (14), (17), (20), and (21) are given by

$$\begin{aligned} K_1 &= \frac{8\pi^2}{3} \frac{k^2}{m_e c} \left(\frac{m_e c}{h} \right)^3 \frac{m_p}{\sigma_0}; \qquad K_2 = K_1/h; \\ K_3 &= \frac{3}{8\pi m_p} \left(\frac{hc}{k} \right)^3; \qquad K_4 = \pi k \left(\frac{k}{hc} \right)^3, \end{aligned} \tag{22}$$

where h is the Planck's constant; k, the Boltzmann's constant; m_e, the mass of the electron; c, the velocity of light; and $\sigma_0 = 1.7 \times 10^{-44}$ cm^2 is the characteristic cross-section of the weak interaction.

After the 'neutrino core' being formed, Equations (1), (2), (3), and (5) remain, nevertheless, valid for the external envelope ($M_\nu < m \leqslant M_0$) which is still transparent to neutrinos and antineutrinos. It is only Equation (4) that should be modified. Though its new form is similar to (4)′, $\varepsilon_{\nu d}$ now has to be calculated from an equation which takes into account the absorption of diluted neutrino-antineutrino radiation escaping from the photosphere of the 'neutrino core' (see Section 4). Besides, the law of the volume neutrino energy losses in the envelope is to be somewhat altered, mainly owing to changes of the neutron-to-proton ratio by means of the interaction of the photospheric neutrinos and antineutrinos with envelope matter.

3. Volume Neutrino Radiation

We shall discuss next the volume energy losses due to neutrino emission which have been used in calculations at various stages of the gravitational collapse.

A. TRANSPARENT STAGE

The universal Fermi interaction (UFI) and URCA processes are the main sources of neutrino and antineutrino emission at this stage. The total rate of energy losses is, therefore, given by

$$\varepsilon_\nu = \varepsilon_{\mathrm{UFI}} + \varepsilon_{\mathrm{URCA}}. \tag{23}$$

For $\varepsilon_{\mathrm{UFI}}$, the interpolation formula of Beaudet *et al.* (1967) was used. It takes into consideration the emission of the neutrino-antineutrino pairs due to the annihilation of the electron-positron pairs, the interaction of photons (γ) with electrons (e^-) and positrons (e^+), and the decay of plasmons (pl) in accordance with the following reactions:

$$\begin{aligned} &e^- + e^+ \rightarrow \nu + \tilde{\nu}, \\ &\gamma + e^\pm \rightarrow e^\mp + \nu + \tilde{\nu}, \\ &pl \rightarrow \nu + \tilde{\nu}. \end{aligned} \tag{24}$$

The energy losses by means of URCA process were calculated with the aid of the following equation, which has been used also by Ivanova *et al.* (1969) and is based on the assumption of beta-processes being in kinetic equilibrium (Imshennik *et al.*, 1966): i.e.,

$$\varepsilon_{\mathrm{URCA}} = 7.8 \times 10^{11} T_9^6 S \phi \text{ erg g}^{-1}\,\text{s}^{-1}, \tag{25}$$

where the dimensionless factors ϕ and S stand for account of degeneracy and dissociation of heavy nuclei into free nucleons, respectively. The factor S is equal to

$$S = X_n + X_p + \frac{(ft)_n}{A_{\mathrm{Fe}}(ft)_{\mathrm{Fe}}} X_{\mathrm{Fe}} \approx X_n + X_p + 0.0017 X_{\mathrm{Fe}}, \tag{26}$$

where X_n, X_p, and X_{Fe} are the mass fractions of the free neutrons, free protons, and iron-group nuclei, respectively; A_{Fe} is the mean atomic weight of iron-group nuclei. The numerical value of the coefficient in front of X_{Fe} in Equation (26) was obtained for the minimal value of $(ft)_{\mathrm{Fe}} \approx 10^4$ (Fowler and Hoyle, 1964). For sufficiently high temperatures and low densities, we have $X_{\mathrm{Fe}} \approx 0$, $X_n + X_p \approx 1$, $\phi \approx 1$ and Equation (26) is reduced to an expression obtained by Imshennik *et al.* (1967) for matter consisting of free nucleons. The factor S may be estimated with allowance for X_n, X_p, X_{Fe}, obtained by Imshennik and Nadyozhin (1965). It depends both on temperature and density. However, in the relevant range of temperatures ($6 \lesssim T_9 \lesssim 20$) in which the dissociation of iron-group nuclei occurs, it depends weakly on density if the density varies in the range of $3 \times 10^8 \lesssim \varrho \lesssim 10^{11}$ g cm^{-3}. These intervals of temperatures and densities just correspond to the physical conditions which stellar matter pass through during the collapse. Therefore, the S was approximated by an interpolation formula with retaining the dependence on temperature only: i.e.,

$$\begin{aligned} S &= 1/600 \quad (T_9 \leqslant 6) \\ \log_{10} S &= 0.38\,(T_9 - 6) - 2.778\,151\,2 \quad (6 < T_9 \leqslant 11) \\ \log_{10} S &= -1.083\,950\,6 \times 10^{-2} T_9^2 + 0.433\,580\,24 T_9 - 4.335\,953\,6 \\ &\quad (11 < T_9 < 20) \\ S &= 1 \quad (20 \leqslant T_9). \end{aligned} \tag{27}$$

The factor ϕ is given by (Ivanova *et al.*, 1969) as

$$\phi = \frac{F_5(\psi_e) + \theta F_5(-\psi_e)}{F_5(0)(1 + \theta)}. \tag{28}$$

where

$$\theta = \frac{n_n}{n_p} = \frac{F_4(\psi_e).}{F_4(-\psi_e)}. \tag{29}$$

In Equations (28) and (29) F_5 and F_4 are the Fermi-Dirac functions of indexes 5 and 4, respectively. They were interpolated by cubic splines, which ensure continuity of the first and the second derivatives in the range of interpolation and make allowance for fitting to the left and right asymptotes with the continuous first derivatives (Ahlberg *et al.*, 1967). If T and ϱ are specified, one can solve Equation (17) along with (29) to obtain the values of ψ_e and θ and calculate afterwards the URCA energy losses with the aid of Equations (28), (27), and (25). In the case of $S=1$, Equation (29) proceeds from the kinetic equilibrium of beta-processes with free nucleons: i.e.,

$$e^- + p \rightarrow n + \nu, \qquad e^+ + n \rightarrow p + \tilde{\nu}. \tag{30}$$

The dependence of ϕ and θ on ϱ/T^3, as given by the solution of Equations (17), (28), and (29), has been tabulated by Ivanova *et al.* (1969). We made, however, use of these equations directly at every step of the calculations.

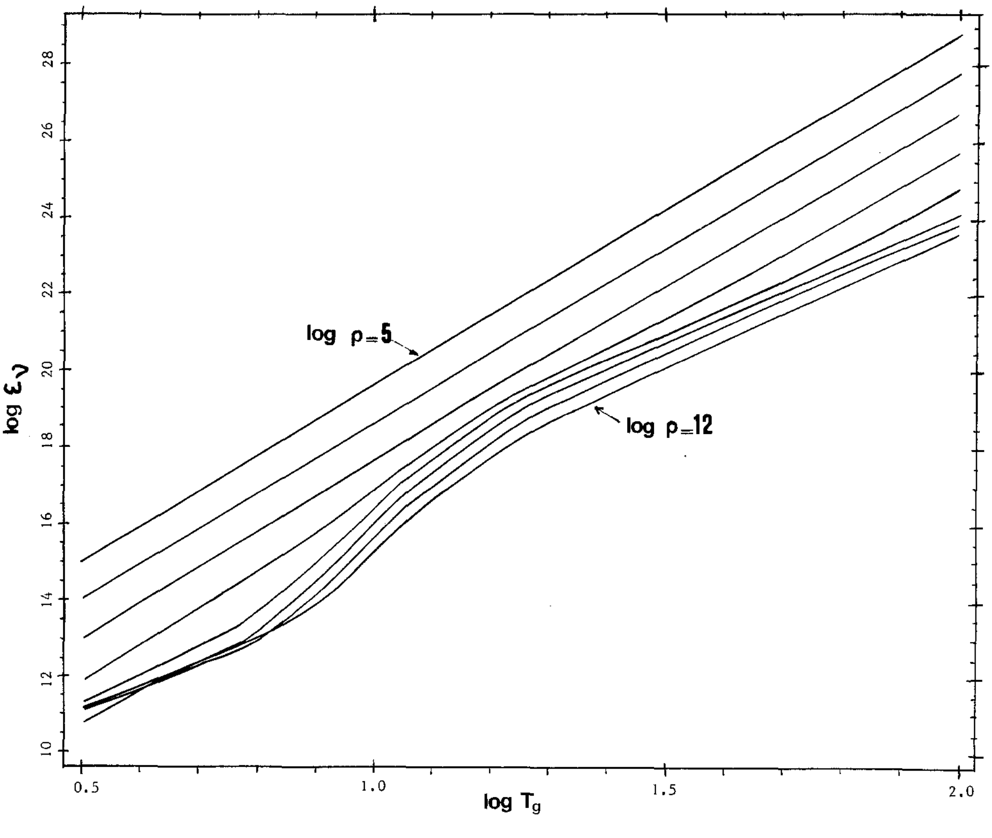

Fig. 1. The overall neutrino energy losses versus temperature for different values of density ϱ from 10^5 g cm^{-3} to 10^{12} g cm^{-3} with the step $\Delta \log_{10} \varrho = 1$.

Figure 1 shows the total rate of energy losses according to Equation (23). The contribution of URCA process to ε_ν is overwhelming at $\varrho > 10^9$ g cm^{-3}.

B. SEMITRANSPARENT STAGE

The law of neutrino emission at this stage was taken in the same form as for preceding transparent stage. It would be desirable to include in the law of neutrino emission the changes of concentrations of free nucleons due to interaction of neutrinos and antineutrinos with stellar matter. However, this would highly complicate the calculations. We took into account only the main effect of energy deposition at this stage (Equations (4)′ and (9); see also Section 4 for further details).

C. OPAQUE STAGE

At this stage the law of neutrino energy losses in the envelope (for $M_\nu < m < M_0$) has to be modified, since the neutron-to-proton ratio θ is affected by the neutrinos and antineutrinos escaping from the surface of the 'neutrino core' (photospheric neutrinos hereafter for simplicity). Let us denote the photospheric radius, temperature, and chemical neutrino potential (in the units of kT) by $R_{\nu p}$, $T_{\nu p}$, and $\psi_{\nu p}$, respectively, and assume the neutrino spectrum to be a Fermi-Dirac distribution with parameters $T_{\nu p}$ and $\psi_{\nu p}$.

The photospheric neutrinos interact with matter of 'optically' thin envelope through the reactions:

$$\begin{aligned} \nu_p + n &\to p + e^-, \\ \tilde{\nu}_p + p &\to n + e^+, \end{aligned} \tag{31}$$

where ν_p and $\tilde{\nu}_p$ are photospheric neutrino and antineutrino. In this case the kinetic equilibrium of the reactions (31) should be considered along with the reactions (30). The resultant expression for θ proves to be rather complex. It generally contains integrals which depend both on photospheric temperature, $T_{\nu p}$, and local one, T. However, T is several times smaller than $T_{\nu p}$ and, besides, $kT_{\nu p}$ is about ten times as large as $m_e c^2$. Therefore, the interaction of photospheric neutrinos with matter of the envelope may be considered with the use of inequalities $T_{\nu p} \gg T$ and $kT_{\nu p} \gg m_e c^2$. Then, the condition of kinetic equilibrium of reactions (30) in common with (31) is greatly simplified and can be reduced to the form

$$\theta = \frac{a_\nu F_4(\psi_e) + F_4(-\psi_{\nu p})(T_{\nu p}/T)^5 W_{\tilde{\nu}}}{a_{\tilde{\nu}} F_4(-\psi_e) + F_4(\psi_{\nu p})(T_{\nu p}/T)^5 W_\nu}. \tag{32}$$

The factors a_ν and $a_{\tilde{\nu}}$ in Equation (32) allow for the Pauli principle in reactions (30) and are given by

$$\begin{aligned} a_\nu &= 1 - \frac{W_\nu}{1 + \exp(-\psi_{\nu p})}, \\ a_{\tilde{\nu}} &= 1 - \frac{W_{\tilde{\nu}}}{1 + \exp(\psi_{\nu p})}, \end{aligned} \tag{33}$$

where W_ν and $W_{\bar{\nu}}$ are the modified dilution factors which are connected with the usual dilution factor

$$W = \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{R_{\nu p}}{r}\right)^2}\right] \tag{34}$$

by the relations

$$W_\nu = \alpha_\nu W, \qquad W_{\bar{\nu}} = \alpha_{\bar{\nu}} W, \tag{35}$$

where factors α_ν and $\alpha_{\bar{\nu}}$ take into account the partial absorption of neutrinos and antineutrinos in the envelope (Section 4).

The function ϕ is also affected by the Pauli principle, and Equation (28) should be replaced by

$$\phi = \frac{a_\nu F_5(\psi_e) + a_{\bar{\nu}}\theta F_5(-\psi_e)}{F_5(0)(1+\theta)}. \tag{36}$$

Moreover, the Pauli principle has an influence on the processes of the Universal Fermi

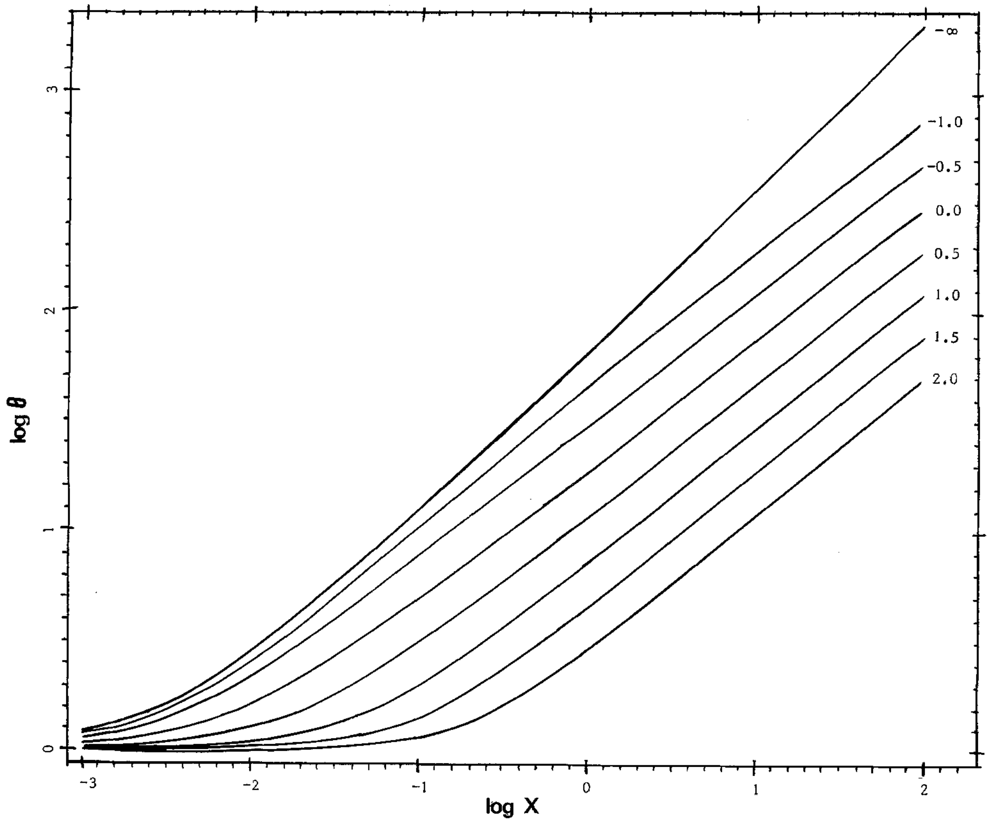

Fig. 2. The dependence of θ on $x = \varrho_7/T_9^3$ for different values of $q = (W_0/2 - W_0)(T_{\nu p}/T)^5$; $\log_{10} q$ is shown at the right ends of the curves.

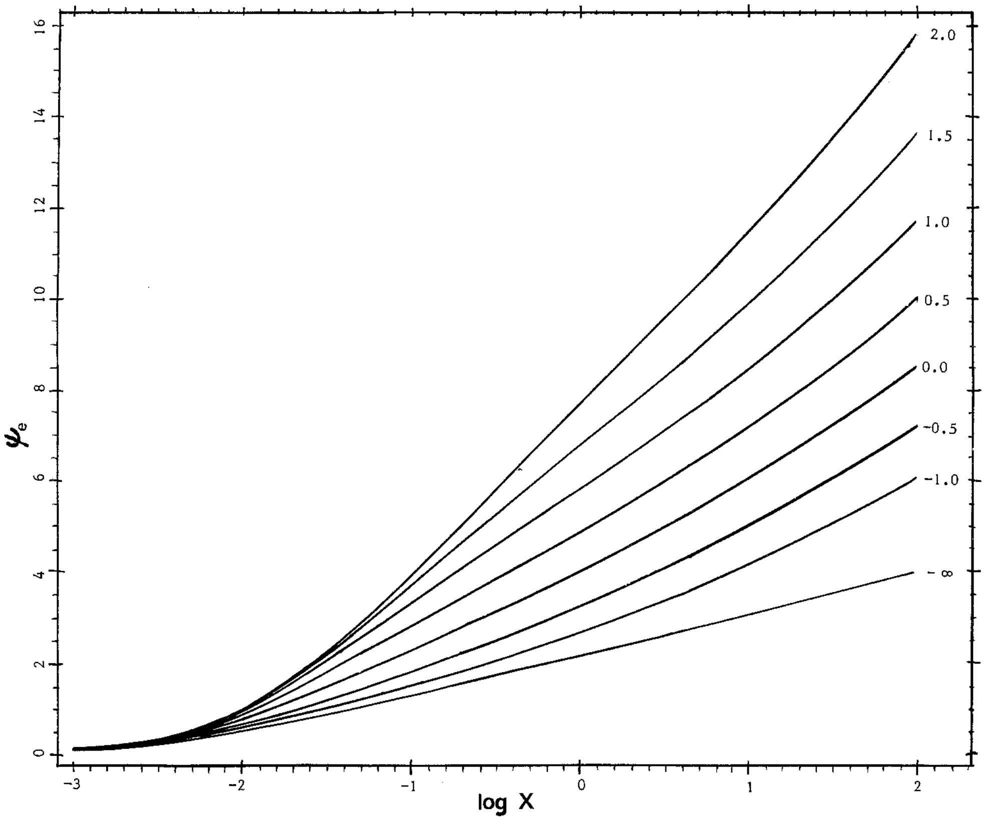

Fig. 3. The dependence of ψ_e on x for the same values of q as in Figure 2.

Interactions. Inasmuch as each of processes (24) produce a $\nu\tilde{\nu}$ pair with ν and $\tilde{\nu}$ energies of the order of kT, a correction for the Pauli principle is reduced in the approximation $T_{\nu p} \gg T$ merely to multiplication of $\varepsilon_{\mathrm{UFI}}$ in Equation (23) by a_ν and $a_{\tilde{\nu}}$: i.e.,

$$\tilde{\varepsilon}_{\mathrm{UFI}} = \varepsilon_{\mathrm{UFI}} a_\nu a_{\tilde{\nu}} . \tag{37}$$

It should be noted here that the corrections, brought in by the Pauli principle, are all unimportant quantitatively owing to $T_{\nu p} \gg T$; but, nevertheless, they are of basic and methodical interest.

Figures 2 and 3 illustrate the effect of photospheric neutrinos on the values of θ and ψ_e in the stellar envelope. A simplified case with $\psi_{\nu p} = -\psi_{\tilde{\nu} p} = 0$ and $W_\nu = W_{\tilde{\nu}} = W_0$ is presented in Figures 2 and 3 for this effect to be clarified. In this case θ and ψ_e, as given by the solution of Equations (17) and (32), depend on two variables: $x = \varrho / T^3$ and $q = (W_0/2 - W_0)(T_{\nu p}/T)^5$. When the photospheric neutrinos are absent ($q = 0$) we obtain the results of Ivanova *et al.* (1969). The photospheric neutrinos lead to noticeable decrease of θ and an increase of ψ_e.

4. Neutrino Energy Deposition

The energy deposition was taken into account both at semitransparent and opaque stages of collapse. In both the cases Equation (9) is a starting point for calculation of $\varepsilon_{\nu d}$.

A. SEMITRANSPARENT STAGE

The total rate of the volume energy losses per unit mass may be obtained by integration of specific volume emissivities B_ν and $B_{\bar{\nu}}$ with respect to neutrino and antineutrino energies

$$\varepsilon_\nu(r) = \int_0^\infty B_\nu(e_\nu, r)\,\mathrm{d}e_\nu + \int_0^\infty B_{\bar{\nu}}(e_{\bar{\nu}}, r)\,\mathrm{d}e_{\bar{\nu}}. \tag{38}$$

Then Equation (9), with allowance for Equations (8) and (38), can be reduced to the form

$$\varepsilon_{\nu d} = \frac{1}{2\varrho r}\int_0^R r'\varrho(r')\varepsilon_\nu(r')\ln\left|\frac{r + r'}{r - r'}\right|\frac{\mathrm{d}r'}{\bar{l}(r', r)}, \tag{39}$$

where $\bar{l}(r', r)$ is the effective neutrino-antineutrino mean free path at radius r, averaged over the spectrum of volume energy losses at radius r', and is given by

$$[\bar{l}(\bar{r}', r)]^{-1} = \frac{1}{\varepsilon_\nu(r')}\left[\int_0^\infty \frac{B_\nu(e_\nu, r')}{\bar{l}(e_\nu, r)}\,\mathrm{d}e_\nu + \int_0^\infty \frac{B_{\bar{\nu}}(e_{\bar{\nu}}, r')}{\bar{l}_{\bar{\nu}}(e_{\bar{\nu}}, r')}\,\mathrm{d}e_{\bar{\nu}}\right]. \tag{40}$$

Consider the case of free nucleons. Then, according to Ivanova *et al.* (1969), we get

$$B_\nu(e_\nu, r') = \frac{4\pi\sigma_0 c}{m_p}\left(\frac{m_e c}{h}\right)^3 \frac{1}{1 + \theta'}\frac{(e_\nu/m_e c^2)^5}{1 + \exp\left(\dfrac{e_\nu}{kT'} - \psi'_e\right)}, \tag{41}$$

$$B_{\bar{\nu}}(e_{\bar{\nu}}, r') = \frac{4\pi\sigma_0 c}{m_p}\left(\frac{m_e c}{h}\right)^3 \frac{\theta'}{1 + \theta'}\frac{(e_{\bar{\nu}}/m_e c^2)^5}{1 + \exp\left(\dfrac{e_{\bar{\nu}}}{kT'} + \psi'_e\right)}, \tag{42}$$

$$\begin{aligned}\bar{l}_\nu(e_\nu, r) = \frac{m_p}{\sigma_0}\frac{1 + \theta}{\varrho\theta}\left(\frac{m_e c^2}{e_\nu}\right)^2\left[1 + \exp\left(-\frac{e_\nu}{kT} + \psi_e\right)\right] \times \\ \times\left[1 + \exp\left(-\frac{e_\nu}{kT} + \psi_\nu\right)\right]^{-1},\end{aligned} \tag{43}$$

$$\begin{aligned}\bar{l}_{\bar{\nu}}(e_{\bar{\nu}}, r) = \frac{m_p}{\sigma_0}\frac{1 + \theta}{\varrho}\left(\frac{m_e c^2}{e_{\bar{\nu}}}\right)^2\left[1 + \exp\left(-\frac{e_{\bar{\nu}}}{kT} - \psi_e\right)\right] \times \\ \times\left[1 + \exp\left(-\frac{e_{\bar{\nu}}}{kT} - \psi_\nu\right)\right]^{-1}.\end{aligned} \tag{44}$$

The primed quantities θ', ψ_e', T' in Equations (41)–(44) depend on r' and unprimed θ, ψ_e, ψ_ν, T on r. The factors in the first square brackets in Equations (43) and (44) proceed from the Pauli principle for electrons and positrons, and in the second square brackets – from the effect of stimulated absorption of neutrinos and antineutrinos. The quantity ψ_ν in Equations (43) and (44) is connected with θ and ψ_e by Equation (16); however, for the determination of θ and ψ_e as functions of T and ϱ the thermodynamically nonequilibrium expression (29) should be used together with Equation (17). The substitution of Equations (41)–(44) in (40) results in the integrals which depend parametrically on T, T', ψ_e, ψ_e', and ψ_ν. They could be integrated for example, numerically with the Gaussian method. However, the account of energy deposition at the semitransparent stage is necessary mainly for the temperature profile to be fitted smoothly to the solution of neutrino thermal conductivity equations at the following opaque stage of the collapse.

The principal property of the function $\varepsilon_{\nu d}(r)$ – the ability to maintain the qualitative correctness of the temperature profile – is kept even with the Pauli principle and stimulated absorption is neglected by omission of the square brackets in Equations (43) and (44). In this case, the integrations with respect to energy may be completed in terms of Fermi-Dirac functions $F_5(x)$ and $F_7(x)$, which are evaluated easily by means of the rapidly convergent expansions in a series (Nadyozhin, 1974). The eventual relations to be calculated have the form

$$\bar{l}(r', r) = \frac{m_p}{\sigma_0}\frac{1+\theta}{\varrho}\left(\frac{m_e c^2}{kT}\right)^2 \frac{1}{S(T)}\frac{F_5(\psi_e') + \theta' F_5(-\psi_e')}{\theta F_7(\psi_e') + \theta' F_7(-\psi_e')}, \tag{45}$$

$$\varepsilon_{\nu d}(r) = \frac{\sigma_0}{m_p}\frac{(kT/m_e c^2)^2}{2r(1+\theta)} S(T) \int\limits_0^R r'\varrho(r') \ln\left|\frac{r+r'}{r-r'}\right| \times$$

$$\times \frac{\theta F_7(\psi_e') + \theta' F_7(-\psi_e')}{F_5(\psi_e') + \theta' F_5(-\psi_e')}\, \varepsilon_\nu(r')\, \mathrm{d}r', \tag{46}$$

where an additional factor $S(T)$ is introduced in Equation (45) in order for the properties of beta-processes with heavy nuclei to be taken into account effectively; for $\varepsilon_\nu(r')$ the Equation (25) should be used since at the semitransparent stage the URCA process is the main contributor to the total neutrino energy losses. The value of $\varepsilon_{\nu d}$ at each mesh point was estimated numerically by the summation of the integrand in Equation (46) over the all difference mesh points. This procedure was repeated anew at every time step of the calculations.

B. OPAQUE STAGE

The neutrinos and antineutrinos, emitted from the photosphere of the 'neutrino core', penetrate through the envelope and their energy densities, calculated per unit volume and per unit energy, can be approximated by

$$U_\nu = 4\pi\left(\frac{m_e c}{h}\right)^3 \frac{(e_\nu/m_e c^2)^3 W_\nu}{1 + \exp\left(\frac{e_\nu}{kT_{\nu p}} - \psi_{\nu p}\right)}, \tag{47}$$

$$U_{\bar\nu} = 4\pi\left(\frac{m_e c}{h}\right)^3 \frac{(e_{\bar\nu}/m_e c^2)^3 W_{\bar\nu}}{1 + \exp\left(\frac{\varepsilon_{\bar\nu}}{kT_{\nu p}} + \psi_{\nu p}\right)}, \tag{48}$$

where W_ν and $W_{\bar\nu}$ are determined by Equations (34) and (35), respectively. The deposition of energy due to the absorption of the photospheric neutrinos and antineutrinos in the envelope can be estimated now by introducing of U_ν and $U_{\bar\nu}$, as defined by Equations (47) and (48), into Equation (9). In this case, the free paths of neutrinos $\tilde{l}_\nu$ and antineutrinos $\tilde{l}_{\bar\nu}$, entering in Equation (9), have to be substituted for those given by Equations (43) and (44) and multiplied in addition by factor S^{-1}. After some transformations, Equation (9) is reduced to

$$\varepsilon_{\nu da}(r) = \frac{4\pi\sigma_0 k^6}{m_p h^3 m_e^2 c^6} \frac{S T_{\nu p}^6}{1 + \theta} [W_\nu I_\nu + W_\nu F_5(-\psi_{\nu p})], \tag{49}$$

where

$$I_\nu = \int_0^\infty \frac{\xi^5\left[\theta + \exp\left(\psi_e - \frac{T_{\nu p}}{T}\xi\right)\right] d\xi}{[1 + \exp(\xi - \psi_{\nu p})]\left[1 + \exp\left(\psi_e - \frac{T_{\nu p}}{T}\xi\right)\right]}. \tag{50}$$

The dependence of $\varepsilon_{\nu da}$ on r takes place due to the changes of W_ν, $W_{\bar\nu}$, T, ψ_e, and θ within the envelope. While deriving Equation (49), we have neglected the stimulated absorption of antineutrinos and the Pauli principle for positrons resulted from the second reaction (31). As a result, the respective integral $I_{\bar\nu}$ was reduced to the Fermi-Dirac function $F_5(-\psi_{\nu p})$. This approximate treatment of the absorption of antineutrinos is justified because of: (a) the positrons are nondegenerated in the envelope, and (b) the main contribution to the energy deposition is due to the absorption of neutrinos, but not of antineutrinos, as the number density of neutrons is greater than that of protons. The integral I_ν, given by Equation (50), was evaluated with the aid of the Gauss-Laguerre's four-point formula with the weight function of $x^5 e^{-x}$ (Rabinowitz and Weiss, 1959).

Another source of energy deposition is the neutrino- and antineutrino-electron scattering. It is of importance for the external layers of the envelope, where the absorption is suppressed by a low beta-activity of the iron-group nuclei. The neutrino-positron scattering may be neglected inasmuch as the number density of positrons in the envelope is much less than that of electrons. We are interested in the case of neutrino and antineutrino energies being much greater than electron energy (with allowance for rest energy). In this case, according to Bahcall (1964), the angle-averaged cross-section of (ν, e^-) and $(\tilde{\nu}, e^-)$ scattering are given by

$$\sigma_\nu = \tfrac{1}{2}\sigma_0 \frac{e_\nu}{m_e c^2} \frac{e_e}{m_e c^2}, \qquad \sigma_{\tilde{\nu}} = \tfrac{1}{6}\sigma_0 \frac{e_{\tilde{\nu}}}{m_e c^2} \frac{e_e}{m_e c^2}. \tag{51}$$

In the external layers of the envelope, where scattering dominates over the absorption, the electron gas is nondegenerate. Bearing this in mind, we can obtain the mean free paths of neutrinos and antineutrinos with respect to scattering in the form

$$l_\nu = 2 \frac{m_p}{\sigma_0} \frac{\mu_e}{\varrho} \frac{m_e c^2}{e_\nu} \frac{1}{\langle \varepsilon_e \rangle}, \tag{52}$$

$$l_{\tilde{\nu}} = \frac{2}{3} \frac{m_p}{\sigma_0} \frac{\mu_e}{\varrho} \frac{m_e c^2}{e_{\tilde{\nu}}} \frac{1}{\langle \varepsilon_e \rangle}, \tag{53}$$

where $\langle \varepsilon_e \rangle = \langle e_e/m_e c^2 \rangle$ is electron energy averaged over Maxwell distribution; and μ_e, the mean molecular weight per electron, was taken as $\mu_e = 2$ (oxygen) in our calculations.

As the neutrinos and antineutrinos lose only a part of their energies at every elementary scattering, the energy distributions (47) and (48) should be multiplied by the coefficients of energy conversion before use of them for the estimation of the energy deposition. For $e_\nu \gg e_e$, the coefficients of energy conversion are about $\frac{1}{2}$ and $\frac{1}{4}$ for neutrinos and antineutrinos, respectively (Bahcall, 1964). Inserting all above data in Equation (9), we obtain the rate of the energy deposition due to scattering in the final form

$$\varepsilon_{\nu\, ds}(r) = \frac{\pi \sigma_0 k^5}{m_p h^3 m_e c^4} \frac{T_{\nu p}^5}{\mu_e} \left[W_\nu F_4(\psi_{\nu p}) + \tfrac{1}{6} W_{\tilde{\nu}} F_4(-\psi_{\nu p}) \right] \langle \varepsilon_e \rangle, \tag{54}$$

where F_4 is the Fermi-Dirac function. For calculation of $\langle \varepsilon_e \rangle$, we made use of a crude interpolation formula

$$\langle \varepsilon_e \rangle = \begin{cases} 1 & (x \leqslant 1), \\ -0.25x^3 + 1.75x^2 - 2.75x + 2.25 & (1 < x < 3), \\ x & (3 \leqslant x), \end{cases} \tag{55}$$

where $x = 3kT/(m_e c^2)$. The cubic spline in Equation (55) gives a smooth fitting between low-temperature and high-temperature asymptotes. In the case of $kT \ll m_e c^2$, the main part of e_e is due to the rest energy and therefore $\langle \varepsilon_e \rangle \approx 1$. On the contrary, for $kT \gtrsim m_e c^2$, the rest energy is small in comparison with the mean energy of nondegenerated electrons which equals about to $3kT$; and, therefore, $\langle \varepsilon_e \rangle \approx 3kT/(m_e c^2)$. The overall rate of energy deposition, which was used in calculations at the opaque stage of the collapse, is given by

$$\varepsilon_{\nu d}(r) = \varepsilon_{\nu\, da}(r) + \varepsilon_{\nu\, ds}(r). \tag{56}$$

Let us consider now the effect of attenuation of the neutrino-antineutrino flux in the envelope surrounding the 'neutrino core'; i.e., the nature of the factors α_ν and $\alpha_{\bar{\nu}}$ which were included in Equation (35). At first sight, it seems that, for a reasonable choice of the 'neutrino core' boundary, the attenuation of the neutrino-antineutrino flux in the 'optically' thin envelope is insignificant. However, it is not the case. First of all, the criterion of the 'neutrino core' boundary position is based on the *mean* neutrino-antineutrino 'optical' depth, which is calculated with the use of the neutrino-thermal-conductivity coefficients, averaged properly. Just around the 'neutrino core' there is a layer consisting of a hot neutron-proton gas. The concentration of neutrons in this layer is about an order of magnitude greater than that of protons. Therefore, the 'optical' depth of this layer with respect to neutrinos is several times the 'optical' depth with respect to antineutrinos. Besides, the mean neutrino-antineutrino 'optical' depth in the 'neutrino core' corresponds to the averaging of the mean free paths, l_ν and $l_{\bar{\nu}}$, of neutrinos and antineutrinos over the derivatives of Fermi-Dirac distribution (an analogy with the Rosseland mean). However, in the case of the energy deposition law, we deal with the averaging of the reciprocals, l_ν^{-1} and $l_{\bar{\nu}}^{-1}$, over the Fermi-Dirac distribution itself (an analogy with the Planckian mean) with the photospheric values of temperature and neutrino chemical potential, but not with the local ones.

Taking into account the above discussion and the fact that the cross-section of neutrino-antineutrino interaction with matter is proportional to the square of the energy, we conclude that the main contribution to the energy transport in the case of neutrino thermal conductivity is due to the neutrinos and antineutrinos with the energies of about $2kT$, and in the case of the energy deposition the relevant energies are about $5kT$. In essence, we discuss here the structure of the intermediate layer which separates the 'neutrino core' from the transparent envelope. On the one hand, the neutrino-thermal-conductivity approximation is violated in this layer but, on the other hand, the attenuation of the antineutrino (and especially neutrino fluxes, generated in 'neutrino core') cannot be neglected on careful examination. Fortunately, this layer proves to contain a small fraction of the total mass of a collapsing star, and influences but weakly the overall dynamics of the collapse. However, it may influence the details of the neutrino and antineutrino energy spectra radiated by a collapsing star.

We restrict ourselves to a crude allowance for the attenuation of the neutrino and antineutrino fluxes. Let us neglect the deformation of the neutrino and antineutrino spectra due to the high-energy tail of the Fermi-Dirac distribution being absorbed in the first place. Then, we may describe the attenuation with the aid of certain coefficients α_ν and $\alpha_{\bar{\nu}}$ diminishing the intensity of the Fermi-Dirac distribution uniformly at all energies (grey-body approximation). The dependence of α_ν and $\alpha_{\bar{\nu}}$ on the Lagrangian mass-coordinate m is given then by

$$\frac{\mathrm{d}\alpha_\nu}{\mathrm{d}m} = -\frac{\varepsilon_{\nu d\nu}^{(0)}}{L_{\nu c}}\alpha_\nu, \tag{57}$$

$$\frac{\mathrm{d}\alpha_{\tilde{\nu}}}{\mathrm{d}m} = -\frac{\varepsilon^{(0)}_{\nu d\tilde{\nu}}}{L_{\tilde{\nu}c}}\alpha_{\tilde{\nu}}, \tag{58}$$

where $\varepsilon^{(0)}_{\nu d\nu}$ and $\varepsilon^{(0)}_{\nu d\tilde{\nu}}$ stand for neutrino and antineutrino component of the overall deposition rate (56) without allowance for the factors α_ν and $\alpha_{\tilde{\nu}}$, correspondingly; $L_{\nu c}$ and $L_{\tilde{\nu}c}$ are the neutrino and antineutrino luminosities of the 'neutrino core'. Equations (57) and (58) can be solved easily. For α_ν, for example, we find out

$$\alpha_\nu(m) = \exp\left(-\int_{M_\nu}^{m} \frac{\varepsilon^{(0)}_{\nu d\nu}}{L_{\nu c}}\,\mathrm{d}m\right), \qquad (M_\nu \leqslant m \leqslant M_0), \tag{59}$$

where M_ν is the mass of the 'neutrino core'.

5. Muon Neutrinos

At $T_9 \gtrsim 150$, muon pairs and at somewhat higher temperatures pion pairs are being created. The decays of positive and negative muons and pions give rise to muon neutrinos and antineutrinos. The 'optical' depth of a collapsing star for muon neutrinos and antineutrinos is smaller than for electron neutrinos and antineutrinos. However, the difference is not so high as Arnett (1967) claimed. According to Domogatsky (1969), the collapsing stellar core becomes opaque in respect to muon neutrinos and antineutrinos when $T_9 \gtrsim 200$. In the case of the central region of a collapsing star being opaque for both electron and muon neutrinos and antineutrinos, Equations (10) and (11) should be supplemented with another diffusion equation, like Equation (11), which ensues from the conservation of muon lepton charge, and with an expression for the muon-lepton-charge flux. Besides that, another additive term, proportional to the gradient of muon neutrino chemical potential, should be added in energy flux H which enters in Equation (10); and coefficient A_2 in Equation (15) should be appropriately modified.

However, these modifications would highly complicate the problem. Therefore, we have treated muon neutrinos by an approximate method, which has been applied formerly to electron neutrinos by Ivanova *et al.* (1969) (see also Imshennik and Nadyozhin, 1974). According to this method, the volume energy losses are multiplied all over a star by the same factor, $\exp(-\tau_{\mu c})$, where $\tau_{\mu c}$ is the mean 'optical' depth of the collapsing star with regard to muon neutrinos and antineutrinos. For the rate of volume energy losses ε^0_μ, and the mean free path of muon neutrinos and antineutrinos, l_μ, we made use (Domogatsky, 1970) of the equations

$$\varepsilon^0_\mu = \frac{1.05\times10^{39}}{\varrho}\left(\frac{kT}{m_\mu c^2}\right)^{3/2} \times \\ \times\left[\exp\left(-\frac{m_\mu c^2}{kT}\right) + 52.7\exp\left(-\frac{m_\pi c^2}{kT}\right)\right]\text{erg g}^{-1}\,\text{s}^{-1}, \tag{60}$$

$$l_\mu^{-1} = 2.58 \times 10^{-3}\left[1 + \frac{21}{22}\frac{kT}{m_e c^2}\right] \times \times \left(\frac{kT}{m_\mu c^2}\right)^{3/2} \exp\left(-\frac{m_\mu c^2}{kT}\right) \mathrm{cm}^{-1}, \tag{61}$$

where m_μ is the mass of the muon; and m_π, that of the pion. The first and the second terms in the square brackets of Equation (60) give the contributions of muon and pion decays, respectively. Equation (61) takes into account the absorption of muon neutrinos and antineutrinos by positive and negative muons. We took the eventual calculating formula for ε_μ in Equation (10) in the form

$$\varepsilon_\mu = \varepsilon_\mu^0 \exp\left(-\int_0^{R_\nu} \frac{\mathrm{d}r}{l_\mu}\right), \tag{62}$$

where R_ν is the radius of the 'neutrino core'.

6. Equation of State

Consider first the equation of state for the 'neutrino core' where temperature is high enough ($T_9 \geqslant 40$–50) for matter to consist of free nucleons. The total pressure and specific energy are the functions of the three independent variables, temperature T, density ϱ, and neutrino chemical potential ψ_ν: i.e.,

$$P = \frac{k}{m_p}\varrho T + \tfrac{1}{3}aT^4[1 + \tfrac{7}{4}B(\psi_e) + \tfrac{7}{8}B(\psi_\nu)], \tag{63}$$

$$E = \tfrac{3}{2}\frac{k}{m_p}T + \frac{aT^4}{\varrho}[1 + \tfrac{7}{4}B(\psi_e) + \tfrac{7}{8}B(\psi_\nu)] + C_{\mathrm{Fe}}. \tag{64}$$

The implicit dependence of electron chemical potential, ψ_e, on T, ϱ, and ψ_ν is given by supplementary Equations (16) and (17). The function $B(x)$ in Equations (63) and (64) is a sum of the third-order Fermi-Dirac functions and can be expressed analytically (Rhodes, 1950; see also Nadyozhin, 1974) in the form

$$B(x) = \frac{F_3(x) + F_3(-x)}{2F_3(0)} = \frac{15}{7\pi^4}\left(x^4 + 2\pi^2 x^2 + \frac{7\pi^4}{15}\right). \tag{65}$$

The three terms in square brackets of Equations (63) and (64) account for the photon radiation, electron-positron gas, and neutrino-antineutrino gas, respectively. The first sums in the right-hand sides of Equations (63) and (64) represent the pressure and specific energy of nondegenerate nucleons. During the collapse of iron-oxygen stars of masses $M \gtrsim 2\,M_\odot$, nucleon gas appears to be practically nondegenerate excluding, perhaps, the latest stage when a hot neutron star is formed in hydrostatic equilibrium. A constant $C_{\mathrm{Fe}} = 8.77 \times 10^{18}$ erg g^{-1} in Equation (64) is the energy of dissociation of iron-group nuclei into free nucleons. It was introduced for the equation of state in the 'neutrino core' to be fitted continuously to the equation of state in transparent envelope. The thermodynamical properties of the above equation of state were briefly discussed by Zentsova and Nadyozhin (1975).

Consider now the equation of state in the envelope, which is 'optically' thin in respect to neutrinos and antineutrinos. At the transparent and semitransparent stages of the collapse this equation of state was applied to all the star. It should be noted that the equation of state is of special importance only at the beginning of the collapse – i.e., at the moment of instability onset. However, soon after the loss of stability (when the velocity of collapsing matter approaches that of sound) the motion of nearly free-falling matter is only slightly affected by the special features of the equation of state. For massive stars, the main reason of the loss of stability is the dissociation of iron-group nuclei into alpha-particles and free nucleons (Fowler and Hoyle, 1964). Consequently, this process should be taken into account in full details. The equation of state, with allowance for dissociation of iron, has been calculated by Imshennik and Nadyozhin (1965). We extended the range of densities, investigated in this work, up to $\varrho = 10^{13}$ g cm^{-3} and tabulated P and E in a rectangle on ϱT-plane, determined by the inequalities $10^5 \leqslant \varrho \leqslant 10^{13}$ g cm^{-3} and $1 \leqslant T_9 \leqslant 20$.

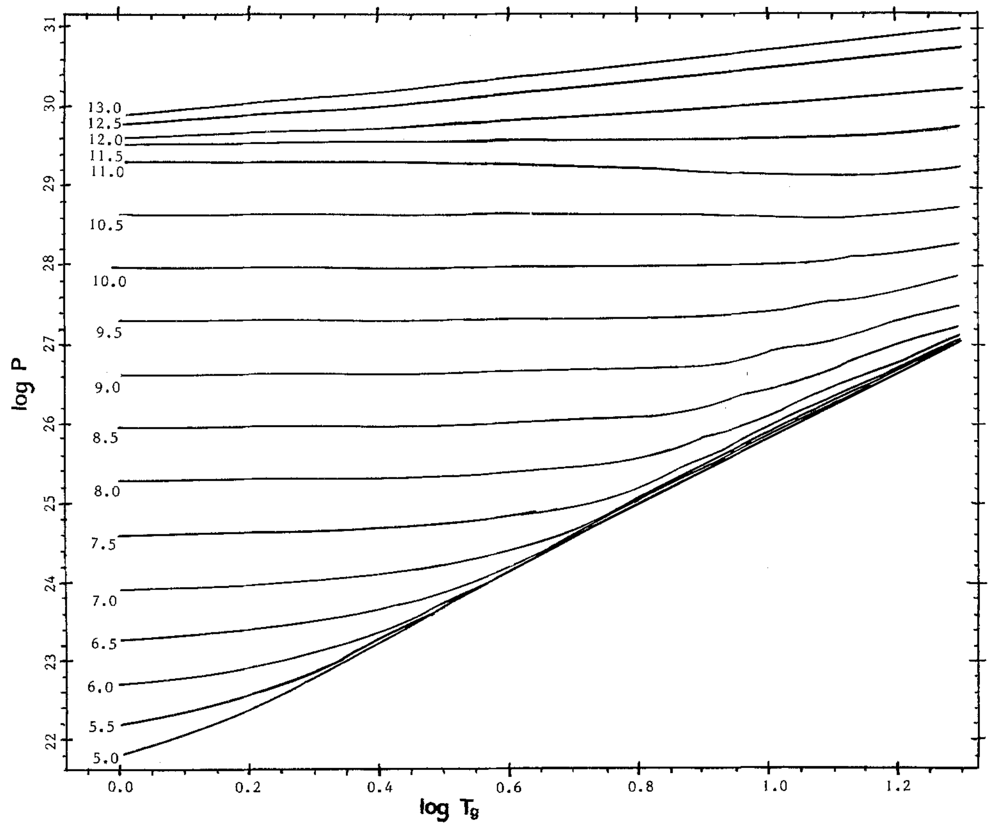

Fig. 4. The pressure versus temperature for different values of $\log_{10} \varrho$.

The tabulation steps were chosen as follows. The density step is the same in logarithmic scale and equals to $\Delta \log_{10} \varrho = 0.25$. However, the temperature step is varied in order for the majority of mesh points to fall in the strip of violent dissociation of iron-group nuclei ($6 \lesssim T_9 \lesssim 10$). The total amount of points for tabulation in density and temperature are equal to 33 and 46, respectively. The resultant tables for $\log P$ and $\log E$ of dimensions of $33 \times 46 = 1518$ mesh points each were disposed in a computer memory. When the specific values of T and ϱ get into the tabulation rectangular on $T\varrho$-plane, the four-point Bessel's interpolation formula (Korn and Korn, 1961) is applied to estimate the pressure and the specific energy by interpolation of $\log P$ and $\log E$ in respect to $\log \varrho$ and $\log T$. Figures 4 and 5, plotted just with the use of above-mentioned tables, show the dependences of P and E on T for a number of values of ϱ.

Outside the tables the equation of state is described by the various asymptotic expansions involving the contributions of nondegenerated nuclei, black-body photon radiation, positrons, and electrons (Figure 6). The regions on $T\varrho$-plane, labelled in Figure 6 with integers (1) to (6), are characterized by the following features: (1) for electron component, the perfect gas law with small corrections due to degeneracy and

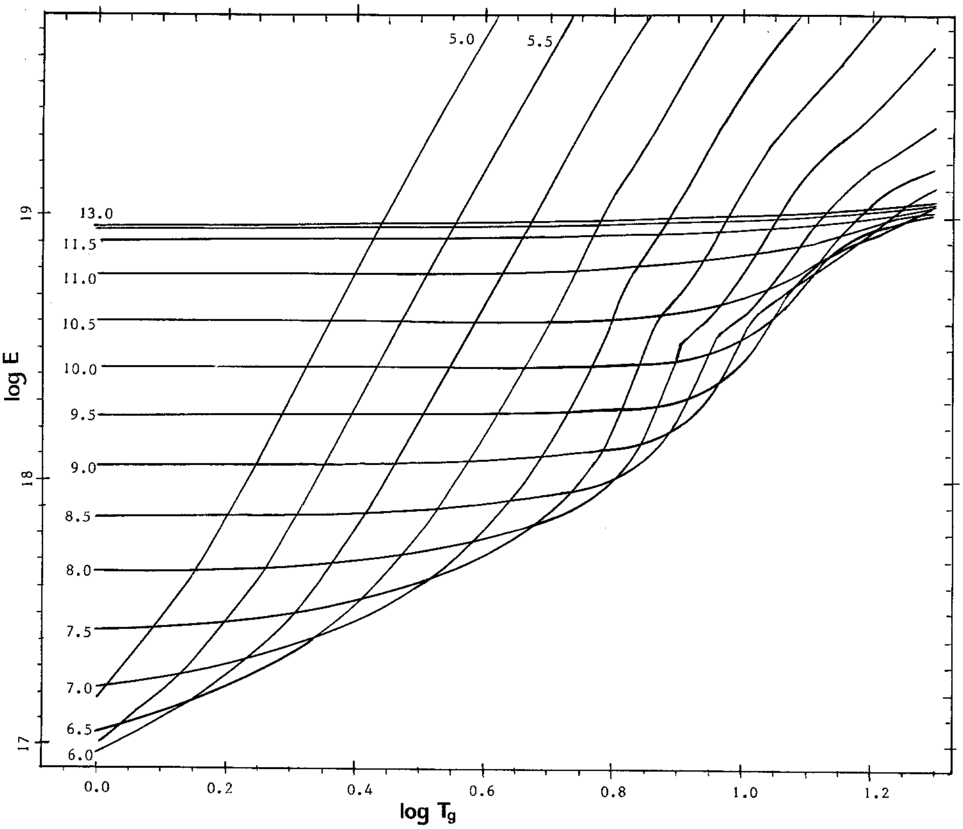

Fig. 5. The specific energy versus temperature for different values of $\log_{10} \varrho$.

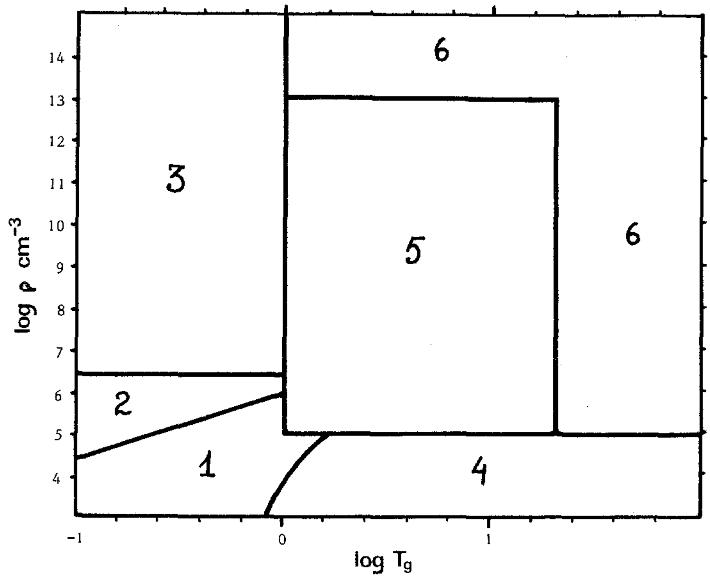

Fig. 6. The (T, ϱ)-plane showing the regions with the various approximations of equation of state (see text, Section 6).

pair creation is taken here; (2) the positrons are neglected, and for electrons an expansion in half-integer Fermi-Dirac functions is used; (3) the well-known Chandrasekhar's expansion for degenerate electron gas is used with three temperature terms being included; (4) in this region, an intensive creation of electron-positron pairs takes place and the two-term expansion for electron-positron gas is used here (Nadyozhin, 1974); it should be noted that collapsing matter never passes through this region but it would be of importance if external low-density matter were heated up by a powerful shock wave; (5) the above-mentioned tables operate here; (6) the following equation of state for matter consisting of free nondegenerated nucleons and ultrarelativistic electrons and positrons is used

$$P = \frac{k}{m_p}\varrho T + \tfrac{1}{3}aT^4[1 + \tfrac{7}{4}B(\psi_e)], \tag{66}$$

$$E = \frac{3}{2}\frac{k}{m_p}T + \frac{aT^4}{\varrho}[1 + \tfrac{7}{4}B(\psi_e)] + C_{\mathrm{Fe}} - \frac{C_{pn}}{1+\theta}, \tag{67}$$

where the function $B(\psi_e)$ is determined by Equation (65). The dependence of ψ_e on T and ϱ is specified by Equation (17) and either Equation (29) or (32); Equation (29) being used at the initial, transparent, stage of the collapse while Equation (32) comes into action at the opaque stage. The last term in the right-hand side of Equation (67) takes into account *an increase* of specific energy due to the transformation of protons into neutrons. The constant C_{pn} equals to 1.2×10^{18} erg g^{-1}.

The different forms of equation of state, presented in Figure 6, fit each other quite well along the lines of demarcation. There are, nevertheless, very small gaps in P and E across each line of demarcation. To remove these gaps, which are undesirable in calculations, the buffer bands were introduced (for simplicity, they are not depicted in Figure 6). The values of P and E inside each buffer band are estimated with the aid of logarithmic interpolation between the two nearest forms of the equation of state.

7. Neutrino Optical Depth, Neutrino Core Boundary, and Neutrino Photosphere

In order to control the position of the 'neutrino core' boundary during the calculations it is necessary to specify the distribution of the neutrino 'optical' depth all over the collapsing star. As the 'neutrino core' occupies a region in which the equations of neutrino thermal conductivity operate, the neutrino optical depth should be expressed by the mean free path, averaged in conformity with neutrino thermal conductivity law (an analogy with the Rosseland mean). In theory of photon radiation conductivity, the energy flux H is related to the mean free path l by

$$H = -\tfrac{1}{3}lc\frac{\partial U}{\partial r}, \tag{68}$$

where U is the bulk energy density. Let us try to make use of an analogous relation for neutrino thermal conductivity. We meet, however, with a difficulty resulting from the fact that the energy flux in the theory for the thermal conductivity of neutrinos is affected by gradients both of temperature and of neutrino chemical potential.

The calculations of the collapse show that, in the external layers of the 'neutrino core', the chemical neutrino potential is small enough and, therefore, the lepton charge flux F is small too; i.e., $FkT \ll H$. In the case of $F=0$, Equation (14) yields

$$\frac{\partial \psi_\nu}{\partial m} = -\frac{A_1}{T}\frac{\partial T}{\partial m}. \tag{69}$$

Introducing U and H from Equations (20) and (13) into Equation (68) and eliminating $\partial\psi_\nu/\partial m$ with the aid of Equation (69), we get

$$l = \frac{m_p}{2\sigma_0}\frac{(1+\theta)^2}{\theta\varrho S}\left(\frac{m_e c^2}{kT}\right)^2 \frac{\dfrac{\pi^2}{3} - \left(\dfrac{\theta-1}{\theta+1}\right)^2}{(\psi_\nu^2+\pi^2)\left(\dfrac{\theta-1}{\theta+1}\psi_\nu+\pi^2\right) - \dfrac{8\pi^4}{15}}. \tag{70}$$

In Equation (70), the additional factor S is included for an increase of the mean free path in matter, consisting of iron-group nuclei, to be properly taken into account. The dependence of the 'optical' neutrino depth on Lagrangian mass-coordinate m is given by

$$\tau_\nu(m) = \int_m^{M_0} \frac{\mathrm{d}m'}{4\pi r^2 \varrho l}. \tag{71}$$

The function $\tau_\nu(m)$ was estimated at each time-step of numerical calculations; outside the 'neutrino core', ψ_ν and θ being computed with the aid of Equations (16) and (32).

The automatic advance of the 'neutrino core' boundary was being carried out in the course of calculations by the following method. Let τ_1 and τ_2 be some fixed numbers satisfying to inequalities $\tau_2 < \tau_1 < 1$. When, at a certain time-step of calculations, the value of $\tau_\nu(M_\nu)$ estimated from Equation (71) with $m = M_\nu$ becomes greater than τ_1 (i.e., $\tau_\nu(M_\nu) > \tau_1$), all layers which are located just above the 'neutrino core' and satisfying the inequality $\tau_\nu(m) > \tau_2$, are immediately adjoined to the 'neutrino core', with a resulting increase of M_ν. Thus the position of the 'neutrino core' boundary corresponds to the inequality $\tau_2 \lesssim \tau_\nu(M_\nu) \lesssim \tau_1$. A number of special trial calculations have shown that the most favourable values of τ_1 and τ_2 are $\tau_1 \approx 0.03$ and $\tau_2 \approx 0.01$. It is the kinetic boundary conditions (19) that make the solution of neutrino thermal conductivity equations be correct qualitatively in the outer region of the 'neutrino core' in spite of $\tau_\nu(M_\nu) \ll 1$.

The concept of neutrino photosphere is more complex than the usual notion of photon photosphere. This is due to the fact that, in the case of neutrino radiation, there are the fluxes of both energy and lepton charge and, besides the temperature gradient, another gradient of chemical potential bears on the problem. We restrict ourselves here with a crude approximation only. According to the boundary conditions (19), the fluxes of energy and lepton charge at the boundary of the 'neutrino core' are twice as large as those following from the Stefan's law. The energy density U increases rapidly with depth and, at a certain level the energy flux H determined by Equation (13), attains just Stefan's value of $\frac{1}{4}cU$. It is this level that we assume to represent the neutrino photosphere. It should be emphasized that the flux F of lepton charge is not generally bound to be equal to $\frac{1}{4}cN$ at this level. The characteristics of the neutrino photosphere $R_{\nu p}$, $T_{\nu p}$, and $\psi_{\nu p}$ were estimated by a linear interpolation of r-, T-, and ψ_ν-distributions between those two mesh points where the difference $H - \frac{1}{4}cU$ changes its sign.

8. An Outline of a Computational Scheme

Equations which we used at various stages of the collapse are all listed in Table II. A star was divided into 151 mass shells. The mass-step Δm is varied with the number of mesh point in order for the central region of the collapsing star, where the steep gradients of physical quantities occur, to be treated in more detail. The distribution of Δm over the mesh point number is presented in Table III. According to Table III, in the internal half of the star's mass M_0 the relative mass-step $\Delta m/M_0$ is of the order of 5×10^{-3}, and in the outer region it is about twice as large.

Shock waves were treated with the method of an artificial viscosity Q (Richtmyer, 1957). The difference equations were written out in terms of integer and half-integer mesh points (Richtmyer, 1957). The variables r, u, m, and fluxes H, and F are estimated at integral mesh points, while T, ϱ, P, E, ε_ν, X_{16}, ε_{16}, $\varepsilon_{\nu d}$, θ, ψ_ν, ψ_e, and Q, are related to half-integral mesh points. The difference equations were taken in explicit form,

TABLE II

The equations in question

The stage of the collapse	The total neutrino optical depth of the star τ_ν, Equations (70), (71)	The differential equations	The boundary conditions	The additional physical relations	Remarks
Transparent	$\tau_\nu < 0.001$	(1), (2), (3), (4), (5)	(7)	(23), (25), (27), (28), (29), (17). For P and E, Figures 3, 4, 5 (Section 6) and Equations (66), (67) are used.	Equation (5) and ε_{16} in Equations (4) and (4)′ are used for $m > M_{Fe}$ only.
Semi-transparent	$0.001 < \tau_\nu \lesssim 1$	(1), (2), (3), (4)′, (5)	(7)	The relations are all the same as for transparent stage with Equation (46) being added for accounting of $\varepsilon_{\nu d}$	
		Inside of the neutrino core, for $m \leqslant M_\nu$.			
Opaque	$\tau_\nu > 1$	(1), (2), (3), (10), (11), (13), (14)	For $m=0$ conditions (19)′ and the first one from (7). For $m = M_\nu$ conditions (19), (20), (21)	(15), (16), (17), (18), (60), (61), (62), (63), (64), (65)	The boundary of the neutrino core, M_ν, is a function of time in accordance with Section 7.
		Outside the neutrino core within the envelope, for $m > M_\nu$.			
		(1), (2), (3), (4), (5)	For $m = M_0$, the second condition from (7)	(17), (23), (25), (27), from (33) to (37), (49), (50), (54), (55), (56), (59). For P and E Figures 3, 4, 5 and Equations (66), (67)	Equation (5) and ε_{16} in (4)′ are used for $m > M_{Fe}$. For neutrino photosphere see Section 7

TABLE III

The distribution of mass intervals adopted in calculations (the arrows indicate a smooth variation of the values)

The number of mesh point	The relative mass step $\Delta m/M_0$	The current relative mass m/M_0
1	2.50×10^{-4}	2.50×10^{-4}
2	2.00×10^{-3}	2.25×10^{-3}
3–93	4.99×10^{-3}	$7.24 \times 10^{-3} \to 0.4763$
99–103	$5.99 \times 10^{-3} \to 9.98 \times 10^{-3}$	$0.4823 \to 0.5162$
104–141	1.198×10^{-2}	$0.5282 \to 0.9713$
142–151	$9.98 \times 10^{-3} \to 9.98 \times 10^{-6}$	$0.9813 \to 1.0000$

excluding only the equations of energy and lepton charge diffusion in 'neutrino core' (see below). We do not have the space to write down the full system of difference equations and, therefore, give only a short account of the main features of numerical procedure. Let all the quantities be specified at a moment t_n. It is required now to estimate the quantities at the next moment $t_{n+1}=t_n+\Delta t_n$. In the first place, for all space mesh points we evaluate new velocities U^{n+1} from Equation (2) (from its difference representative, rather), then from Equation (1) we obtain radii r^{n+1}, and Equation (3) gives densities ϱ^{n+1} at the moment t_{n+1}. It remains temperature and neutrino chemical potential in the 'neutrino core' at the opaque stage of the collapse to be determined. The difference approximation of Equation (4)′ is given by

$$\frac{E^{n+1} - E^n}{\Delta t_n} + \frac{1}{\Delta t_n}\left(\frac{P^{n+1}}{2} + P^n + Q^n\right)\left(\frac{1}{\varrho^{n+1}} - \frac{1}{\varrho^n}\right) = = -\frac{\varepsilon_\nu^{n+1} + \varepsilon_\nu^n}{2} + \varepsilon_{\nu d}^{n+1/2} + \varepsilon_{16}^n. \tag{72}$$

The subscripts, denoting mesh point number, are omitted in Equation (72) for the sake of simplicity. The rate ε_{16}^n of energy generation due to oxygen burning is different from zero for $m > M_{\mathrm{Fe}}$ only; the rate of energy deposition $\varepsilon_{\nu d}^{n+1/2}$ is absent at the transparent stage of the collapse. At the semitransparent stage $\varepsilon_{\nu d}^{n+1/2}$ was estimated by the numerical evaluation of integral (46) with the use of new distributions r^{n+1}, ϱ^{n+1} and preceding distribution T^n. At the opaque stage, however, the rate of energy deposition was taken in time-centred form (just as ε_ν did)

$$\varepsilon_{\nu d}^{n+1/2} = \tfrac{1}{2}(\varepsilon_{\nu d}^{n+1} + \varepsilon_{\nu d}^n). \tag{73}$$

In the difference equations, representatives of Equations (10) and (11) for the 'neutrino core', coefficients before the derivatives of T and ψ_ν with respect to m and t were taken at preceding moment t_n, while the derivatives themselves were considered at the current moment t_{n+1}. Thus, the implicit difference scheme was obtained in which the values of T and ψ_ν at every mesh point were calculated with the aid of the method of successive exclusion (Godunov and Ryabenzhkii, 1964). In this method, the values of vector (T, ψ_ν) at all successive pairs of mesh points are connected by recurrent linear relations; the coefficients of these relations being matrices of the second order.

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