UNIFORM RADIAL MOTION OF SOUND IN A RELATIVISTIC FLUID BALL

NEERAJ PANT
B.S.F Polytechnic S.T.S-1 Tigri Camp New Delhi – 62 India

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Abstract. We present a new class of spherically symmetric exact solutions of the general relativistic field equations. These solutions describe perfect fluid balls with infinite central pressure and central density though their ratio is finite. A member of the class has been studied in detail from which we have constructed a model of causal fluid ball with constant sound speed.

1. Introduction

On account of the non linearity of the field equations of general relativity, not many realistic, analytic solutions are known for the description of relativistic perfect fluid balls. For a meaningful model of relativistic star, a solution must correspond to monotonically decreasing positive expressions for pressure and density and must obey the principle of causality everywhere right from the centre region upto the boundary. If one discovers a new exact solution which corresponds to monotonically decreasing positive expressions for pressure and density but principle of causality is not obeyed in some region within the ball, the effort is wasted. On the other hand, if one succeeds in obtaining a parametric class of solutions with physical meaningful expressions for pressure and density then in all probability for some values of the parameter there may result models obeying the principle of causality and also satisfying some reasonable equations of state. It is in this context that attempts to obtain new classes of exact solutions have assumed significance (Wyman, 1949; Kuchowicz, 1968; Pant and Sah, 1982, 1985; Pant and Pant, 1993, 1995). In this paper we present a new parametric class of exact solutions giving rise to physically reasonable models of perfect fluid balls for certain range of values of the parameter.

2. Field Equation and Method of Obtaining Analytic Solutions

We consider the static, spherically symmetric metric in the standard form

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu} dt^2 \tag{1}$$

where λ and ν are functions of the radical coordinate r. Accordingly, the field equations of general relativity for a perfect fluid ball of pressure $p(r)$ and density $\rho(r)$ are (Tolman, 1939).

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$$\frac{8\pi Gp}{c^4} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{2}$$

$$\frac{8\pi G\rho}{c^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{3}$$

$$\frac{d}{dr}\left(\frac{e^{-\lambda}-1}{r^2}\right) + \frac{d}{dr}\left(\frac{\nu' e^{-\lambda}}{2r}\right) + e^{-\lambda-\nu}\frac{d}{dr}\left(\frac{\nu' e^{-\nu}}{2r}\right) = 0 \tag{4}$$

where a prime denotes differentiation with respect to r. The problem consists of solving (4) for λ and ν by assuming an *adhoc* relationship between λ and ν which would correspond to the equation of state of the fluid.

Using the substitution (Pant and Pant, 1993)

$$r^2 e^{\nu} = U, \; e^{-\lambda} = V \tag{5}$$

the equation (4) reduces to the following linear differential equation in V.

$$V' - 2\left\{\left(\log\frac{2r^3\sqrt{U}}{U'}\right)' - \frac{4U}{r^2U'}\right\}V = -\frac{4U}{r^2U'}. \tag{6}$$

On integration, we get

$$e^{-\lambda} = V = \frac{4r^6U}{(U')^2}\left[A - \int\frac{U'}{r^8}\exp\left(\int\frac{8U}{r^2U'}dr\right)dr\right]\exp\left(-\int\frac{8U}{r^2U'}dr\right) \tag{7}$$

where A is an arbitrary constant. Thus by assuming U suitably one can make right hand side of the last equation integrable. In the present paper, we assume

$$\exp\left(\int\frac{8U}{r^2U'}dr\right) = r^a(U')^n \tag{8}$$

where 'a' and 'n' are arbitrary constants. It result into a second order homogeneous differential equation in U:

$$nr^2U'' + arU' - 8U = 0. \tag{9}$$

Its solution is

$$U = c_1 r^{2(c-b+1)} + c_2 r^{2(c+b+1)} \tag{10}$$

where

$$b = \frac{\sqrt{(n-a)^2 + 32n}}{4n}, \; c = \frac{a+3n}{-4n} \tag{11}$$

provided $n \neq 0$. Also (7) is simplified into

$$V = \frac{r^{6-(a+n+2(c-b)(n+1))}[c_1 + c_2 r^{4b}]\left(\frac{A}{2^n} - 2I\right)}{[(c-b+1)c_1 + (c+b+1)c_2 r^{4b}]^{n+2}} \tag{12}$$

where

$$I \equiv \int r^{a-8+[2(c-b)+1](n+1)}[(c-b+1)c_1 + (c+b+1)c_2 r^{4b}]^{n+2} dr. \tag{13}$$

The solution is complete if (13) is solved. In the foregoing sections we shall discuss a method of solving (13) and present a detailed study of the resulting solution. It may be mentioned here that for $n = 0$, we rediscover the class of solutions due to Tolman, usually referred as Tolman's V solution. Tolman's V solution is also obtainable if either of the constants c_1 and c_2 vanishes.

3. New Class of Solutions

The equation (13) can be integrated by the method of substitution, if we assume

$$4b - 1 = a - 8 + (2c - 2b + 1)(n + 1). \tag{14}$$

In view of (11), the equation (14) yields a quadratic equation in a:

$$(n+1)a^2 + 2n(n-3)a + n^3 - 3n^2 + 36n = 0 \tag{15}$$

which solves into

$$a = \frac{-n(n-3) + 2(n+3)\sqrt{-n}}{(n+1)}. \tag{16}$$

Here we have considered only the positive radical sign; for, corresponding to negative radical sign, e^ν becomes singular at the origin.

We thus obtain a new class of solutions of the equation (4) as follows:

$$e^\nu = c_1 r^{2(c-b)} + c_2 r^{2(c+b)} \tag{17}$$

$$e^{-\lambda} = (c_1 r^{-4b} + c_2)\left(\frac{\bar{A}}{(\bar{c}_1 + \bar{c}_2 r^{4b})^{n+2}} - \bar{B}\right) \tag{18}$$

where

$$\bar{A} = \frac{A}{2^n} \tag{19}$$

$$\bar{B} = \frac{1}{2b(c+b+1)(n+2)C_2} \tag{20}$$

$$\bar{c}_1 = (c-b+1)c_1 \tag{21}$$

$$\bar{c}_2 = (c + b + 1)c_2. \tag{22}$$

We observe that (e^ν) $r = 0$ becomes singular for all values of 'n' save $-1 \leq n < 0$. For $n = -1$ the solution reduces to Tolman's IV solution. It may be pointed out here that the class of solutions obtained by Pant and Sah (Pant and Sah, 1982) has an expression for e^ν similar to (17), however, the two classes of solutions are disjoint.

4. Properties of the New Class of Solutions

In view of (17) and (18), we obtain from (2) and (3) the pressure and the density distributions:

$$\frac{8\pi Gp}{c^4} =$$

$$\frac{1}{r^2}\left[\{2(c-b)+1C_1 r^{-4b} + (2(c+b)+1)c_2\}\left\{\frac{\bar{A}}{(\bar{c}_1+\bar{c}_2 r^{4b})^{n+2}} - \bar{B}\right\} - 1\right] \tag{23}$$

$$\frac{8\pi G\rho}{c^2} =$$

$$\frac{1}{r^2}\left[\frac{\{(4b-1)c_1\bar{c}_1 r^{-4b} + [(n+3)4b-1]\bar{c}_2 c_1 - c_2\bar{c}_1 + [(n+2)4b-1]c_2\bar{c}_2 2r^{4b}\}\bar{A}}{(\bar{c}_1+\bar{c}_2 r^{4b})^{n+3}}\right.$$

$$\left. + \bar{B}\{(1-4b)c_1 r^{-4b} + c_2\} + 1\right]. \tag{24}$$

By using (23) and (24) we can easily derive

$$\frac{1}{c_2}\frac{dp}{d\rho} = \frac{g_1+g_2}{f_1+f_2} \tag{25}$$

where

$$g_1 \equiv \frac{-2\bar{A}}{(\bar{c}_1+\bar{c}_2 r^{4b})^{n+3}}\Big[\{2(c-b)+1\}(1+2b)c_1\bar{c}_1 r^{-4b} + [2b(n+3)$$

$$+1][2(c-b)+1]c_1\bar{c}_2 + [2(c+b)+1]c_2\bar{c}_1 + [2b(n+2)$$

$$+1][2(c+b)+1]c_2\bar{c}_2 r^{4b}\Big] \tag{26}$$

$$g_2 \equiv 2\Big[\{(1+2b)[2(c-b)+1]c_1 r^{-4b} + [2(c+b)+1]c_2\}\bar{B} + 1\Big] \tag{27}$$

$$f_1 \equiv \frac{\bar{A}}{(\bar{c}_1 + \bar{c}_2 r^{4b})^{n+4}} \Big[(1-4b)(2+4b)\bar{c}_1^2 c_1 r^{-4b} + [(1-4b)\{2$$

$$+4b(n+4)\}) - 2\{(n+3)4b - 1\}]c_1\bar{c}_1 c_2 + 2\bar{c}_1^2 c_2$$

$$+[2-(n+3)4b\{1+(n+3)4b\}]c_1\bar{c}_2^2 r^{4b} + [4b\{(n+2)4b$$

$$-(n+1)\} + 4]\bar{c}_1\bar{c}_2 c_2 r^{4b} + [(4b-2)\{(n+2)4b-1\}$$

$$-(n+3)4b\{(n+3)4b\{(n+2)4b-1\}]c_2\bar{c}_2^2 r^{8b}\}\Big] \tag{28}$$

$$f_2 \equiv \bar{B}\{-(1-4b)(2+4b)c_1 r^{-4b} - 2c_2\} - 2. \tag{29}$$

In addition to the parameter n, the solution (17), (18) contains three arbitrary constants c_1, c_2 and $\bar{A}$. These are to be determined by the boundary conditions.

$$P(r_b) = 0 \tag{30}$$

$$e^{\lambda(rb)} = (1-2u)^{-1} \tag{31}$$

$$e^{\nu(rb)} = 1 - 2u \tag{32}$$

where

$$u \equiv \frac{GM}{c^2 r_b}. \tag{33}$$

Consequently,

$$c_1 = \frac{c+b-u(2c+2b+1)}{2br_b^{2(c-b)}} \tag{34}$$

$$c_2 = \frac{b-c+u(2c-2b+1)}{2br_b^{2(c+b)}} \tag{35}$$

$$\bar{A} = \Bigg[\frac{1}{(2(c-b)+1)c_1 r_b^{-4b} + (2(c+b)+1)c_2}$$

$$+ \frac{1}{(2b(c+b+1)(n+2)c_2}\Bigg](\bar{c}_1 + \bar{c}_2 r_b^{4b})^{n+2}. \tag{36}$$

For e^{ν} to be definitely positive in the region $0 < r \leq r_b$ we must have $c_1, c_2 > 0$. That is

$$\frac{c+b}{2c+2b+1} < u < \frac{c-b}{2c-2b+1}. \tag{37}$$

Table I

March of pressure, density, pressure-density ratio and square of the ratio of sound speed to light speed within the fluid ball corresponding to $n = -0.38198$ and $u = 0.3333278$

$x = \frac{r}{r_b}$	$\frac{8\pi G}{c^4} p r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$\frac{p}{\rho c^2}$	$\left(\frac{dp}{d\rho}\right) \frac{1}{c^2}$
0	∞	∞	0.99989	0.99989
0.1	49.4951	50.49989	0.9801	0.9999
0.2	11.99884	12.99993	0.92299	0.9999
0.3	5.05508	6.0555	0.83479	0.9999
0.4	2.62476	3.62495	0.72408	0.9999
0.5	1.49986	2.49957	0.60004	0.99991
0.6	0.88812	1.88885	0.47055	0.99992
0.7	0.52636	1.52037	0.34226	0.99992
0.8	0.28122	1.28121	0.21949	0.99993
0.9	0.11727	1.11725	0.10496	0.99994
1.0	0	.99997	0	0.99994

The central values of pressure and density are infinite, however, the limiting value of their ratio at the centre is finite and equals the limiting value of $dp/d\rho$:

$$\left(\frac{p}{\rho}\right)\gamma \to 0 = \left(\frac{dp}{d\rho}\right)\gamma \to 0 = -\left(\frac{\frac{2(c+b)+1}{2b(c+b+1)(n+2)} + 1}{\frac{1}{2b(c+b+1)(n+2)} + 1}\right). \tag{38}$$

It has been calculated that for values of n in the interval [–0.2, –0.147) the right hand side of (38) is negative thus making the adiabatic sound speed imaginary. Again we observe that the limiting central value of $dp/d\rho$ violates the causality principle for n in the interval [–0.381966, –0.115]. Hence for physically meaningful solutions, n ranges in the intervals [–1, –0.381966) and (0.115, 0). It is to be noted that the only known solution which corresponds to constant sound speed within the ball (equal to c, the speed of light in vacuo) is due to Leibovitz (Leibovitz, 1969; Matese and Whitman, 1980). We further note that the new class of solutions gives rise to Leibovitz's solution for $\frac{1}{c^2}\left(\frac{dp}{d\rho}\right)\gamma \to 0 = 1$ or $n = -0.381966049$, and $u = \frac{1}{3}$. It is therefore, interesting to study a member of the class which asymptotically approaches to Leibovitz's solution.

We consider a particular member corresponding to $n = -0.38198$. In view of (37) we fine for this value of n,

$$0.3333277 < u < 0.3819643. \tag{39}$$

It is found that not all fluid balls which satisfy (39) are causal, however, we have calculated that corresponding to values of u close to the lower limit in (39) one obtains perfectly causal fluid balls. One such ball corresponding to $u = 0.3333278$

has been found to be of particularly significant interest. It corresponds to monotonically decreasing pressure, density and pressure density ratio from the central region to the boundary. $dp/d\rho$, the square of adiabatic sound speed, has almost a stationary value throughout within the ball (Table 1) as expected. The new class of solution is therefore useful in studying stellar models where the variation in the sound speed is insignificant.

We now present here a model of neutron star based on the particular solution discussed above. The neutron star is supposed to have a surface density equivalent to the typical nuclear density: $\rho_b = 2 \times 10^{14}$ gm cm^{-3}. The resulting causal model has the mass. $M = 3.7 M_{\Theta}$ and the linear dimension, $2r_b \approx 32.76$ Km. The surface red shift $Z_b = (1 - 2u)^{-1/2} - 1$ has been calculated for this model and we obtain $Z_b \approx 0.73202$.

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