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Nonlinear Waves in the Pitaevskii–Gross Equation

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Nonlinear waves, solitary and periodic, are studied exactly in the Pitaevskii–Gross equation for the wave function of the condensate of a superfluid. We also study the relationship between these two waves and Bogoliubov's phonon, and the energies associated with these waves. The creation energy of a solitary wave with amplitude A *is proportional to* $A^{3/2}$. *Solitary waves show interesting behavior on their collision due to their localized character. The effect of collision on solitary waves can be described by the phase shift. We give a formula of the phase shift on a collision of two solitary waves. We further discuss the decay of an arbitrary initial disturbance into solitary waves.*

1. INTRODUCTION

The solitary wave (or soliton) appears to be one of the interesting modes of motion in some sorts of nonlinear media, for example, in a low-density plasma,[1–4] an anharmonic crystal,[5] and so on. A solitary wave is a nonlinear and pulselike wave and propagates without decay through the media with a velocity which depends on its amplitude. Further, in spite of a nonlinear interaction during their collisions two solitary waves restore their initial wave shapes at large spatial separation; new effects on various properties may be expected which could not be found by the usual perturbational considerations.

We can expect the existence in a superfluid of such solitary waves and of the nonlinear waves in general for the following reason. In the linear approximation, with respect to the deviation from equilibrium, a wave (a phonon, first sound or second sound) propagates with a definite velocity, its wave shape remaining unchanged. When the amplitude of the wave is not small the velocity of propagation is different for different points in the wave. This results in a steepening of the wave and, finally, in its decay. However, in a nonlinear dispersive medium there exists another mechanism which competes with the above steepening effect and tends to keep the wave shape unchanged. This is a dispersive effect and in our case appears as a nonlinear dependence of the dispersion relation on the wave vector. Thus the nonlinear wave can exist stably when the two above-mentioned mechanisms balance each other. The nonlinear wave phenomenon occurs commonly in nonlinear dispersive media.

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As a first attempt to study such nonlinear waves in a superfluid we examine the Pitaevskii–Gross (PG) equation[6] for the condensate at absolute zero temperature. Although this equation is rather an academic one and has no relation to real superfluid helium, this model has in the past provided some qualitative understanding about superfluid helium. In addition we can treat it rigorously, as its mathematical structure is simple.

In Section 2 we write down the PG equation in terms of the density and velocity fields. We can apply the reductive perturbation method[7] to the case of a wave with small but finite amplitude. Our basic equations are reduced to the Korteweg–de Vries (KdV) equation,[8,9] which has historically been shown to describe the asymptotic development of a shallow-water wave.[10] The KdV equation has two types of stationary solution, that is, the solitary wave and the nonlinear periodic wave. We review these perturbational solutions in Section 3 to show the similarities of our system with others studied previously. However, we can solve exactly our basic equations for both types of waves. Our solutions and their properties are discussed in Section 4. We also discuss the relationship between the present nonlinear waves and Bogoliubov's phonon. The energies and the local currents associated with the waves are studied in Section 5.

In subsequent sections we explain further properties of the solitary wave. For this purpose we discuss in Section 6 which quantities are conserved during the evolution of the system, and derive two types of conservation law from our basic equations. We study the collision of two solitary waves[4] in Section 7. By means of the conservation laws we can show not only that the solitary waves restore their initial wave shapes after the collision but also that the phases of the restored waves are shifted by the interaction during the collision. We further discuss the conversion of an initial disturbance into solitary waves[3] in Section 8. In conclusion we extend the results found from the KdV equation to our system, and obtain a few new findings. Some discussion is given in the final section.

2. BASIC EQUATIONS

We start from the Pitaevskii–Gross equation[6]

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi + g|\psi|^2\psi \tag{1}$$

where $\psi(\mathbf{r}, t)$ is the wave function of the condensate, μ is the chemical potential assumed to be constant, m is the mass of a particle, and g is a repulsive coupling constant. Let us restrict ourselves to the case of unidirectional motion and introduce the density field $\rho(x, t)$ and the velocity field $S(x, t)$ which are related to $\psi(x, t)$ by

$$\psi(x, t) = \sqrt{\rho(x, t)}\exp\left[i\frac{m}{\hbar}\int^x S(x', t)\,dx'\right] \tag{2}$$

Here ρ and S are real functions. Substituting (2) into (1) and separating the real and imaginary parts from the resulting equations we get

$$\int^x \frac{\partial S(x', t)}{\partial t} dx' + \frac{1}{2}S^2 + 2\alpha\beta\rho + \frac{1}{2}\alpha^2\rho^{-2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \alpha^2\rho^{-1}\frac{\partial^2\rho}{\partial x^2} = \frac{\mu}{m} \tag{3}$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho S) = 0 \tag{4}$$

where

$$\alpha \equiv \hbar/2m \qquad \beta = g/\hbar$$

Differentiating (3) with respect to x we find

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{1}{2}S^2 + 2\alpha\beta\rho + \frac{1}{2}\alpha^2\rho^{-2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \alpha^2\rho^{-1}\frac{\partial^2\rho}{\partial x^2}\right\} = 0 \tag{5}$$

Equations (4) and (5) are the equations of continuity and acceleration. The chemical potential is determined by (3). These three equations construct a set of the basic equations of our problem.

3. PERTURBATIONAL TREATMENT OF NONLINEAR WAVES

When we apply an infinitesimal disturbance to our system, we find Bogoliubov's phonon as the normal mode of our set of equations (4) and (5). We want to know the result of applying a small but finite disturbance to our system. Let us expand $\rho(x, t)$ and $S(x, t)$ in the following forms:

$$\begin{aligned} \rho(x, t) &= \rho_0 + \varepsilon\rho_1(x, t) + \varepsilon^2\rho_2(x, t) + \cdots \\ S(x, t) &= \varepsilon S_1(x, t) + \varepsilon^2 S_2(x, t) + \cdots \end{aligned} \tag{6}$$

where ρ_0 is a constant and ε is a small but finite expansion parameter. We expect the existence of waves which propagate with definite velocities in spite of the nonlinear character of our system, and therefore we transform the space and time coordinates to new ones,[8] that is,

$$\begin{aligned} \xi &= \varepsilon^{1/2}(x - vt) \\ \tau &= \varepsilon^{3/2}t \end{aligned} \tag{7}$$

where v is a constant velocity to be determined later. Equations (4) and (5) are reduced to

$$\frac{\partial}{\partial\xi}\rho(S - v) = \varepsilon\frac{\partial\rho}{\partial\tau} \tag{8}$$

$$(S - v)\frac{\partial S}{\partial\xi} + 2\alpha\beta\frac{\partial\rho}{\partial\xi} = -\varepsilon\alpha^2\frac{\partial}{\partial\xi}\left\{\frac{1}{2}\rho^{-2}\left(\frac{\partial\rho}{\partial\xi}\right)^2 - \rho^{-1}\frac{\partial^2\rho}{\partial\xi^2}\right\} - \varepsilon\frac{\partial S}{\partial\tau} \tag{9}$$

where the τ dependences are now in higher order of ε compared to the ξ dependences. Substituting (6) into the above equations and equating the terms of first order in ε to zero we get

$$v\frac{\partial \rho_1}{\partial \xi} - \rho_0 \frac{\partial S_1}{\partial \xi} = 0$$
$$v\frac{\partial S_1}{\partial \xi} - 2\alpha\beta \frac{\partial \rho_1}{\partial \xi} = 0 \tag{10}$$

Nontrivial solutions exist only when the velocity v is given by

$$v^2 = 2\alpha\beta\rho_0 = \frac{g}{m}\rho_0 \tag{11}$$

Equating the terms of second order in ε to zero we get

$$\frac{\partial \rho_1}{\partial \tau} + \frac{\partial}{\partial \xi}(\rho_1 S_1) = v\frac{\partial \rho_2}{\partial \xi} - \rho_0 \frac{\partial S_2}{\partial \xi}$$
$$\frac{\partial S_1}{\partial \tau} + S_1 \frac{\partial S_1}{\partial \xi} - \frac{\alpha^2}{\rho_0}\frac{\partial^3 \rho_1}{\partial \xi^3} = v\frac{\partial S_2}{\partial \xi} - 2\alpha\beta \frac{\partial \rho_2}{\partial \xi} \tag{12}$$

Although these equations involve ρ_2 and S_2 we can easily eliminate them because of the relation (11), and we find that

$$\frac{\partial}{\partial \tau}(v\rho_1 + \rho_0 S_1) + \rho_0 S_1 \frac{\partial S_1}{\partial \xi} + v\frac{\partial}{\partial \xi}(\rho_1 S_1) - \alpha^2 \frac{\partial^3 \rho_1}{\partial \xi^3} = 0 \tag{13}$$

This expression together with either of (10) constructs a set of equations for the first-order quantities ρ_1 and S_1. Now we have to set the boundary condition on ρ and S. We choose them in the general forms such that for localized solutions

$$\rho_1 \to 0\,(\rho \to \rho_0) \qquad S_1 \to S_{1\infty} \geq 0 \qquad \text{for } |\xi| \to \infty \tag{14}$$

and their derivatives also tend to zero, and such that for periodic solutions ρ_1 and $S_1 - S_{1\infty}$ are periodic functions. Since we obtain from (10) that

$$v\rho_1 = \rho_0(S_1 - S_{1\infty}) \tag{15}$$

for both cases, (13) is reduced to

$$\frac{\partial S_1}{\partial \tau} + \frac{3}{2}S_1 \frac{\partial S_1}{\partial \xi} - \frac{1}{2}S_{1\infty}\frac{\partial S_1}{\partial \xi} - \frac{\alpha^2}{2v}\frac{\partial^3 S_1}{\partial \xi^3} = 0 \tag{16}$$

This is one variation of the Korteweg–de Vries (KdV) equation* which has been studied in other nonlinear systems. This equation has two types of stationary solution.[3] One is the solitary wave and the other is the periodic wave.

*In fact, when we operate another transformation defined by the relations that $\eta = 2^{1/3}(\xi - S_{1\infty}\tau)$, $\zeta = -S_{1\infty}\tau$, and $S_1 = S_{1\infty}\{1 - 2^{2/3}\cdot 3^{-1}Z(\eta,\zeta)\}$, Eq. (16) is reduced to the KdV equation that

$$\frac{\partial Z}{\partial \zeta} + Z\frac{\partial Z}{\partial \eta} + \left(\frac{\alpha^2}{vS_{1\infty}}\right)\frac{\partial^3 Z}{\partial \eta^3} = 0$$

Let us first study the single solitary wave. Our solution is given by

$$S_1(\xi, \tau) = S_{1\infty} - av\,\mathrm{sech}^2\left\{\frac{v}{2\alpha}\sqrt{a}(\xi - u\tau - \theta)\right\} \tag{17}$$

and therefore ρ_1 is expressed by

$$\rho_1(\xi, \tau) = -a\,\mathrm{sech}^2\left\{\frac{v}{2\alpha}\sqrt{a}(\xi - u\tau - \theta)\right\} \tag{18}$$

where $u = S_{1\infty} - \frac{1}{2}av$, a is a positive constant of integration ($a > 0$), and θ is an arbitrary phase constant which has the dimension of length. The phase velocity of the wave changes from v to $v + \varepsilon u$ due to the nonlinear effect.* In order to complete our solution let us relate ρ_0 to the average density of the system n. If we assume that the total number of particles and the volume $\Omega = L\sigma$ of the system are fixed (where L and σ are the length and the cross section of the system, respectively), we find that

$$\frac{n}{\rho_0} = 1 - \frac{2\hbar}{m|v|}\cdot\frac{1}{L}\sqrt{\varepsilon a} \tag{19}$$

where we neglect the terms of the order of L^{-2}. This expression means that ρ_0 is larger than n. This relation with (11) determines ρ_0 and v in terms of n, a, and L. However, ρ_0 can be regarded as n and equivalently $|v|$ can be replaced by the velocity of Bogoliubov's phonon $c \equiv \sqrt{gn/m}$ as $\hbar/(m|v|)$ is of the order of the healing length $r \equiv \hbar/\sqrt{2mng}$ and L has a macroscopic value, that is,

$$[2\hbar/m|v|L)] \ll 1$$

Finally, the chemical potential in this case is given by

$$\mu = g\rho_0 - mv\cdot\varepsilon S_{1\infty} \simeq gn - mc\cdot\varepsilon S_{1\infty} \tag{20}$$

The periodic solution is expressed by the Jacobian elliptic functions[11] whose modulus k has an arbitrary value between 0 and 1 ($0 \leq k \leq 1$).† Requiring that S_1 is a periodic function around $S_{1\infty}$, we get

$$S_1(\xi, \tau) = S_{1\infty} - ac\left[dn^2\left\{\frac{c}{2\alpha}\sqrt{a}(\xi - w\tau - \theta)\right\} - \frac{E}{K}\right] \tag{21}$$

where $w = S_{1\infty} - \{(2 - k^2)/2 - 3E/2K\}ac$, and we have used the simplified notation that $dn\,x = dn(x, k)$, and K and E are the first and second complete

*Taniuti and Yajima discussed a special case that $u = 0$ in Ref. 8.

†The general solution is given by

$$S_1 = S_{1\infty} - av\left[dn^2\left\{\frac{v}{2\alpha}\sqrt{a}(\xi - \omega\tau - \theta)\right\} - \gamma\right]$$

where $w = S_{1\infty} - \{(2 - k^2)/2 - (\frac{3}{2})\gamma\}av$ and a, θ, and γ are arbitrary constants.

elliptic integrals:

$$K = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$
$$E = E(k) = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2 \theta} \tag{22}$$

The density field is given by

$$\rho_1(\xi, \tau) = -a\left[dn^2\left\{\frac{c}{2\alpha}\sqrt{a}(\xi - w\tau - \theta)\right\} - \frac{E}{K}\right] \tag{23}$$

which describes a perodic wave around the average density n. The wavelength in the present case is given by $(4\alpha/c\sqrt{\varepsilon a})K$.

We have explained the nonlinear stationary waves in our system by reductive perturbation in the lowest order. However, as we can find the exact stationary solutions, we stop to discuss the details of our system on the basis of the perturbational solutions.

4. EXACT STATIONARY SOLUTIONS FOR (4) AND (5)

Let us choose the boundary condition for a solitary wave so that

$$\begin{aligned} \rho(x, t) &\to \rho_0 \\ S(x, t) &\to S_\infty \end{aligned} \quad \text{for} \quad |x| \to \infty \tag{24}$$

and the derivatives of ρ and S also vanish up to the second order as $|x| \to \infty$. Putting

$$\rho(x, t) = \rho_0 F(x, -Vt - \theta) \tag{25}$$

we find from (4) that

$$S(x, t) = V - \frac{V - S_\infty}{F} \tag{26}$$

Equation (5) is reduced to

$$\alpha^2 F^{-1/2} \frac{\partial}{\partial x}\left\{F^{-1/2}\frac{\partial F}{\partial x}\right\} = \tfrac{1}{2}(S_\infty - V)^2\{F^{-2} - 1\} + v^2\{F - 1\} \tag{27}$$

For a simple solitary wave we find

$$F(x) = 1 - A\,\mathrm{sech}^2\left(\frac{Q}{2}x\right) \tag{28}$$

where

$$A = \frac{\alpha^2 Q^2}{v^2} = 1 - \left(\frac{V - S_\infty}{v}\right)^2 \tag{29}$$

The quantity v is defined by (11). Equation (29) determines the width and velocity of the solitary wave in terms of the amplitude A. Further, these relations require that

$$1 \geq A > 0 \tag{30}$$

We exclude the case $A = 0$ because this case does not represent the solitary wave. If we assume that A is small, our solution is reduced to the one obtained by the perturbational treatment in Section 3.

Our solution has the following interesting properties. Firstly, it is a pulselike wave which propagates without decay and so it is called a solitary wave. Secondly, the waveshape becomes steep as A becomes large. Thirdly, the phase velocity V is determined by the amplitude A and is limited within the range that

$$-v\sqrt{1-A} \leq V - S_\infty \leq v\sqrt{1-A}$$

When the wave has the maximum amplitude ($A = 1$), it propagates with the velocity S_∞, that is, it is at rest relative to the fluid. Lastly, the wave is rarefaction wave. As we approach the center of the wave, the relative fluid velocity $S - S_\infty$ increases negatively (or positively) in the wave propagating toward the positive (or negative) x direction.

The chemical potential is given by

$$\mu_S = g\rho_0 + \frac{1}{2m}(S_\infty^2 - 2S_\infty V) \tag{31}$$

which should be compared with that of the uniform flow state with the velocity S_∞, that is,

$$\mu_U = gn + \frac{1}{2m}S_\infty^2$$

The density ρ_0 is related to n by

$$\frac{n}{\rho_0} = 1 - \frac{4A}{QL} \tag{32}$$

Here the system has the length $L(-L/2 \leq x \leq L/2)$. Again, ρ_0 and v can be regarded as n and c, respectively, as discussed in Section 3.

Next let us study a periodic wave under periodic boundary conditions on ρ and S oscillating around n and S_∞, respectively. We can find (25), (26), and (27) again where V is replaced by W to avoid confusion. The solution is given by

$$F(x) = 1 - A\left\{dn^2\left(\frac{Q}{2}x\right) - \Gamma\right\} \tag{33}$$

where Q is real, $\Gamma = E/K$, and

$$A = \frac{\alpha^2 Q^2}{c^2} \tag{34}$$

$$\left(\frac{W - S_\infty}{c}\right)^2 = (1 + \Gamma A)\{1 - (1 - \Gamma)A\}\{1 - (1 - k^2 - \Gamma)A\} \tag{35}$$

The modulus of the Jacobian elliptic function k is an arbitrary number between 0 and 1. The amplitude A is again limited within the range $0 < A \leq 1$ because of (34) and the requirement that the density should be nonnegative. The wavelength λ is given by

$$\lambda = \frac{4K}{Q} = \frac{4\alpha K}{c\sqrt{A}} \tag{36}$$

which depends on the amplitude. The variation of $\rho - n$ over one period is described by a wave whose compressed part is rather spread out and flat, and whose rarefied part is steep. This periodic wave propagates much faster than the solitary wave relative to the fluid, if A, n, and S_∞ have the same values in both cases, because the relative velocity of the former, $|W - S_\infty|$, is larger than that of the latter, $|V - S_\infty|$, for every allowed value of k (Fig. 1). The present chemical potential is given by

$$\begin{aligned}\mu_P = gn &+ \tfrac{1}{2}m(S_\infty^2 - 2S_\infty W) \\ &+ \tfrac{1}{2}mc^2A^2[\{\Gamma(2 - k^2 - 2\Gamma) - (1 - \Gamma)(1 - k^2 - \Gamma)\} \\ &- \Gamma(1 - \Gamma)(1 - k^2 - \Gamma)A]\end{aligned} \tag{37}$$

This is larger than the chemical potential of the solitary wave given by (31).

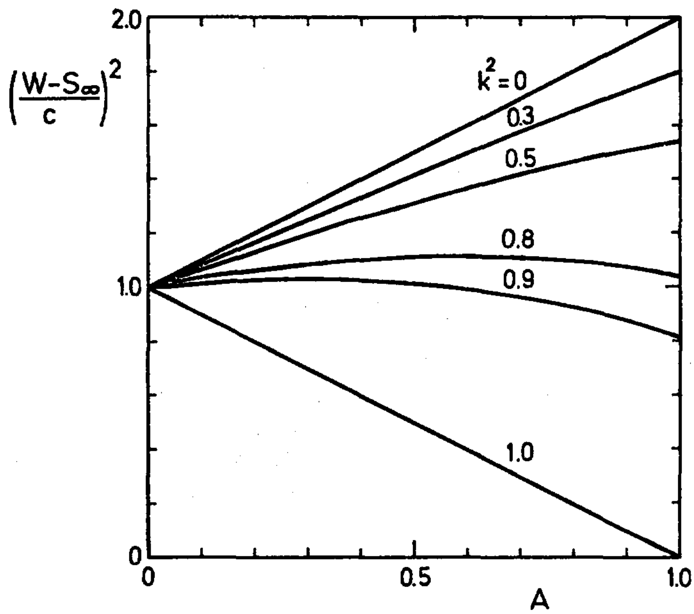

Fig. 1. The relative wave velocity $\{(W - S_\infty)/c\}^2$ as the function of the wave amplitude A. The quantity $k(0 \leq k \leq 1)$ is the modulus of the Jacobian elliptic function. The line $k = 0$ is the dispersion relation of Bogoliubov's phonon, and the line $k = 1$ is that of the solitary wave if we neglect the correction of the order of $1/L$.

Now let us discuss the mutual relations among the present two types of nonlinear waves and Bogoliubov's phonon. In the limit $k \to 1$ we find that (33), (34), and (35) are reduced to (28) and (29) because $dn\, x \to \operatorname{sech} x$, $K \to \ln(4/\sqrt{} - k^2)$, $E \to 1$, and $\Gamma \to 0$. Further, since $\lambda \to \infty$ the wave in this limit is nothing but the solitary wave. Next we consider the limit $k \to 0$. Since $dn^2 x \to 1 + (k^2/2)\cos 2x$ and $\Gamma \to 1$ in this limit, we get that

$$\rho(x, t) = n[1 - \tfrac{1}{2}k^2 A \cos\{Q(x - Wt - \theta)\}] \tag{38}$$

The quantities $k^2 A$ and Q are now independent parameters because of the arbitrary k. Thus the dispersion relation is given by

$$\hbar\omega(Q) = \hbar Q W = \hbar c Q\sqrt{1 + A} = \sqrt{\hbar^2 c^2 Q^2 + \left(\frac{\hbar^2 Q^2}{2m}\right)^2} \tag{39}$$

in the case $S_\infty = 0$. This is just Bogoliubov's phonon. Hence it is clear that the present periodic wave reduces to the solitary wave or to Bogoliubov's phonon in the appropriate limits.

5. ENERGIES AND LOCAL CURRENTS

Let us discuss the energies associated with a single solitary wave and a periodic wave, which are propagating towards the positive x direction, in the case $S_\infty = 0$. The energy of the system is given by

$$E[A] = \frac{\Omega}{L}\int_{-L/2}^{L/2} dx\left[\frac{\hbar^2}{8m}\rho^{-1}\left(\frac{\partial\rho}{\partial x}\right)^2 + \tfrac{1}{2}m\rho S^2 + \tfrac{1}{2}g\rho^2\right] \tag{40}$$

where L is the length of the system ($-L/2 \le x \le L/2$). For the case of a single solitary wave we find by the use of (26) and (28) that

$$E_S[A] = \frac{\Omega}{L}\cdot\frac{1}{2}m\rho_0\left[v^2 L - \frac{4A}{Q}\{(1 - \tfrac{1}{3}A)v^2 + V^2\}\right]$$

Using (29) and (32) we can reduce E_S to

$$\begin{aligned} E_S[A] &= E_G + \Omega\varepsilon_S[A] \\ \varepsilon_S[A] &= \frac{r}{L}\cdot\frac{4\sqrt{2}}{3}nmc^2 A^{3/2} \end{aligned} \tag{41}$$

where $E_G = \Omega\cdot(nmc^2/2)$ is the ground-state energy and ε_S is calculated in the order of r/L. The quantity ε_S can be interpreted as the creation energy of one solitary wave with an amplitude A, and is proportional to $A^{3/2}$.

Next we consider the case of one periodic wave. The energy is expressed by

$$\begin{aligned} E_P[A] &= E_G + \Delta E_P[A] \\ \Delta E_P[A] &= \Omega nmc^2 A^2[\{\tfrac{2}{3}(2 - k^2)\Gamma - \tfrac{1}{3}(1 - k^2) - \Gamma^2\} \\ &\quad -\tfrac{1}{2}\Gamma(1 - \Gamma)(1 - k^2 - \Gamma)A] \end{aligned} \tag{42}$$

It is convenient to rewrite (42) in terms of the energy per period of the wave in order to compare it with the case of the solitary wave. Since the number of oscillations in the system is L/λ, where λ is given by (36), we find that

$$\Delta E_P[A] = \Omega \cdot \frac{L}{\lambda}\,\varepsilon_P[A]$$

$$\begin{aligned}\varepsilon_P[A] &= \frac{r}{L}\cdot\frac{4\sqrt{2}}{3}nmc^2A^{3/2}\\ &\quad\times[\{(2-k^2)E-\tfrac{1}{2}(1-k^2)K-\tfrac{3}{2}\Gamma E\}\\ &\quad-\tfrac{3}{4}E(1-\Gamma)(1-k^2-\Gamma)A]\end{aligned} \tag{43}$$

Figure 2 shows the forms of the terms in the square bracket of (43) in the cases $A = 1$ and $A = 0$. The curves for other values of $A(0 < A < 1)$ lie between these two curves. The energy $\varepsilon_P[A]$ is naturally reduced to the solitary wave energy $\varepsilon_S[A]$ in the limit $k \to 1$ where $E \to 1$, $\Gamma \to 1$, and $(1 - k^2)K \to 0$. In the opposite limit $k \to 0$, $\varepsilon_P[A]$ tends to zero, but we have to take account of the k dependence because the amplitude of the wave is proportional to k^2A in this case. Then we find that

$$\begin{aligned}\Delta E_P[A] &= \Omega\cdot\tfrac{1}{2}nmc^2(\tfrac{1}{2}k^2A)^2(1+A)\\ &= \Omega\cdot\tfrac{1}{2}nm(\tfrac{1}{2}k^2A)^2\left\{\frac{\hbar\omega(Q)}{\hbar Q}\right\}^2\end{aligned} \tag{44}$$

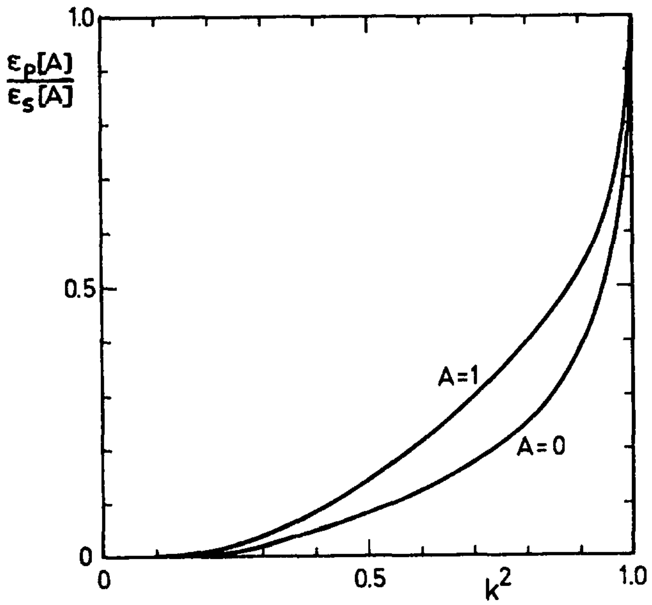

Fig. 2. The curves $\varepsilon_P[A]/\varepsilon_S[A]$ as the function of k^2 in the cases that $A = 0$ and 1. The curves for other values of A between 0 and 1 run between the two curves indicated. The value of $\varepsilon_P[A]/\varepsilon_S[A]$ is 0 at $k = 0$ and 1 at $k = 1$ for every value of A.

where $\hbar\omega(Q)$ is the energy of Bogoliubov's phonon given by (39). Since the amplitude of the wave is related to the number of phonons with wave number Q by

$$(\tfrac{1}{2}k^2A)^2 = \frac{(\hbar Q)^2}{nm\hbar\omega(Q)} N_Q$$

$\Delta E_P[A]$ is rewritten as

$$\Delta E_P[A] = \Omega \cdot \tfrac{1}{2}\hbar\omega(Q) N_Q \tag{45}$$

This is the familiar expression for the phonon energy contribution. The factor $\frac{1}{2}$ comes from the fact that we have considered only the wave propagating towards the positive x direction. The zero-point energy is neglected in the present calculation.

Finally, we show the local current J associated with the waves. They are given by

$$J(x, t) = \rho_0[S_\infty - V(1 - F)] \tag{46}$$

We find that

$$J_S = \rho_0\left[S_\infty - AV\operatorname{sech}^2\left\{\frac{Q}{2}(x - Vt - \theta)\right\}\right] \tag{47}$$

for the case of the solitary wave and

$$J_P = n\left[S_\infty - AW\left(dn^2\left\{\frac{Q}{2}(x - wt - \theta)\right\} - \Gamma\right)\right] \tag{48}$$

for the case of the periodic wave. The current J_S is smaller than $\rho_0 S_\infty$ for a wave propagating towards the positive direction, and larger for a wave in the negative direction. The current J_P oscillates periodically around nS_∞.

6. CONSERVATION LAWS

Until now we have only studied the properties of a single wave. One naturally asks whether more than two waves can coexist stably or not. Nonlinear periodic waves cannot coexist stably since their linear combination does not satisfy our basic equations. However, the situation is different for solitary waves. Due to their localized character the superposition principle holds asymptotically for solitary waves at large spatial separation. Therefore we are interested in studying the collision of solitary waves which are asymptotically independent of each other initially and discussing the possibility of the conversion of an arbitrary initial disturbance into solitary waves. In subsequent sections we study the simplest case of two solitary waves.

For the above purpose we derive here two types of conservation laws in integral form. In order to find them we write the first few differential equations which are derived by making use of (4) and (5):

$$\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x}\left\{\rho S^2 + \alpha\beta\rho^2 + \alpha^2\rho^{-1}\left(\frac{\partial \rho}{\partial x}\right)^2 - \alpha^2\frac{\partial^2 \rho}{\partial x^2}\right\} = 0 \tag{49}$$

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\rho S^2 + \alpha\beta\rho^2 + \frac{1}{2}\alpha^2\rho^{-1}\left(\frac{\partial \rho}{\partial x}\right)^2\right\}$$
$$+ \frac{\partial}{\partial x}\left\{\frac{1}{2}\rho S^3 + 2\alpha\beta\rho^2 S + \frac{1}{2}\alpha^2\rho^{-1}\left(\frac{\partial \rho}{\partial x}\right)^2 - \alpha^2 S\frac{\partial^2 \rho}{\partial x^2} - \alpha^2\rho^{-1}\frac{\partial \rho}{\partial t}\cdot\frac{\partial \rho}{\partial x}\right\} = 0 \tag{50}$$

The first type of conservation law includes the familiar cases of density, current, and energy conservation:

$$I_\rho = \int dx\{\rho(x,t) - \rho_0\} \tag{51}$$

$$I_S = \int dx\{S(x,t) - S_\infty\} \tag{52}$$

$$I_{\rho S} = \int dx\{\rho S - \rho_0 S_\infty\} \tag{53}$$

$$I_{\rho S}^2 = \int dx\left[\frac{1}{2}\{\rho S^2 - \rho_0 S_\infty^2\} + \alpha\beta\{\rho^2 - \rho_0^2\} + \frac{1}{2}\alpha^2\rho^{-1}\left(\frac{\partial \rho}{\partial x}\right)^2\right] \tag{54}$$

These quantities are conserved during the evolution of the system. Their common feature is their explicit independence of x and t. This means that they tell us nothing about the phases of the waves. As is well known the phases are shifted by the interaction during the collision. In order to obtain some information about the phase shift we have to find another type of conservation laws which depends on x and/or t explicitly. One such law is given by

$$P = \int dx[x\{G(x) - G(\infty)\} - t\{H(x) - H(\infty)\}] \tag{55}$$

if $G(x)$ and $H(x)$ satisfy

$$\frac{\partial G(x)}{\partial t} + \frac{\partial H(x)}{\partial x} = 0$$

$$\frac{\partial H(x)}{\partial t} + \frac{\partial M(x)}{\partial x} = 0$$

that is, if $H(x)$ is the current of $G(x)$ in a general sense and further both $H(x)$ and $G(x)$ are locally conserved quantities. As seen from the sequence of our differential equations, we can find only one set of such conserved quantities: ρ and ρS. In this case (55) is written by*

$$P_\rho = \int dx[x\{\rho - \rho_0\} - t\{\rho S - \rho_0 S_\infty\}] \tag{56}$$

Needless to say there remains the possibility that we might be able to construct another one depending on x and/or t in a more complicated way.

7. COLLISION OF TWO SOLITARY WAVES

Hereafter we confine ourselves to discussion of the problems of two solitary waves. We assume that in the initial stage there are two solitary waves with different amplitudes propagating in the positive x direction. The wave with larger amplitude is moving ahead of the one with smaller amplitude and they are widely separated so that we can neglect the effect of their overlap. The initial field is expressed by

$$F(x, t; A_1, \theta_1; A_2, \theta_2) = F(x, t; A_1, \theta_1) + F(x, t; A_2, \theta_2) - 1 \tag{57}$$

$$F(x, t; A_i, \theta_i) = 1 - \operatorname{sech}^2\left\{\frac{Q_i}{2}(x - V_i t - \theta_i)\right\} \tag{58}$$

where $A_1 > A_2$ and Q_i and V_i are related to A_i by (29). Some time later the wave A_2 catches up with the wave A_1. We want to know what waves we can find after their interaction. Zabusky[12] examined this problem in the KdV equation in a plasma, and found that two solitary waves are restored. Lax[13] discussed the same case analytically and showed that the overlap of the two solitary waves vanishes exponentially when their separation becomes large. On the basis of these studies we can expect similar behavior in our exact case. So the field at the final stage is described by

$$F(x, t; A_1', \theta_1'; A_2', \theta_2') = F(x, t; A_1', \theta_1') + F(x, t; A_2', \theta_2') - 1 \tag{59}$$

Now the small wave A_2' is moving ahead of the large one A_1'. Let us use the conservation laws I_ρ, $I_{\rho S}$, and P_ρ in order to find relations between $(A_1, \theta_1, A_2, \theta_2)$ and $(A_1', \theta_1', A_2', \theta_2')$. Neglecting the overlap integrals in $I_{\rho S}$ and P_ρ we get

$$\sqrt{A_1'} + \sqrt{A_2'} = \sqrt{A_1} + \sqrt{A_2} \tag{60}$$

$$\sqrt{A_1'(1 - A_1')} + \sqrt{A_2'(1 - A_2')} = \sqrt{A_1(1 - A_1)} + \sqrt{A_2(1 - A_2)} \tag{61}$$

$$\sqrt{A_1'}\theta_1' + \sqrt{A_2'}\theta_2' = \sqrt{A_1}\theta_1 + \sqrt{A_2}\theta_2 \tag{62}$$

Remembering that $1 \geq A_1, A_2, A_1', A_2' > 0$ we can derive an equation

$$A_1' A_2' = A_1 A_2 \tag{63}$$

*Namiki has found the same type of conservation law as this in the case of the KdV equation (private communication). The relation (65) is also derived in his case.

from (60) and (61). Thus we find that

$$A'_1 = A_1$$
$$A'_2 = A_2 \tag{64}$$

and

$$\frac{\theta'_1 - \theta_1}{\theta'_2 - \theta_2} = -\sqrt{\frac{A_2}{A_1}} \tag{65}$$

These results tell us that the two resultant waves are never the same ones that we found at the initial stage. Their phases are shifted by the interaction. This result is just the one that Zabusky found in his computer experiment. It is interesting that the solitary wave appears to behave like a particle.

8. INITIAL VALUE PROBLEM

Next we treat the decay of an arbitrary initial disturbance into solitary waves.[3] As an example let us consider the decay into two waves. We write the initial disturbance as

$$\rho(x, 0) = \rho_0\{1 - f(x)\}$$
$$\rho(x, 0)S(x, 0) = \rho_0\{S_\infty - vg(x)\} \tag{66}$$

where $f(x)$ and $g(x)$ are dimensionless functions.* Our problem is to find under what conditions on $f(x)$ and $g(x)$ we can observe two solitary waves after a sufficiently long time. The final field is expressed by (59). Since I_ρ, $I_{\rho S}$, and P_ρ are conserved during the evolution of the system, we find that

$$\sqrt{A'_1} + \sqrt{A'_2} = \frac{v}{4\alpha}\int dx f(x) \equiv \delta \tag{67}$$

$$\sqrt{A'_1(1 - A'_1)} + \sqrt{A'_2(1 - A'_2)} = \frac{1}{4\alpha}\int dx\{vg(x) - S_\infty f(x)\} \equiv \kappa \tag{68}$$

$$\sqrt{A'_1}\theta'_1 + \sqrt{A'_2}\theta'_2 = \frac{v}{4\alpha}\int dx x f(x) \tag{69}$$

Let us first examine (68). The value of the left-hand side is positive and its maximum value is one. Therefore κ should lie in the range $1 \geq \kappa \geq 0$. The curve (68) is expressed in Fig. 3. Combining (67) and (68) we can find two solitary waves at the final stage in the following cases: for the case $1 \geq \kappa \geq \frac{1}{2}$

$$\sqrt{2}\{1 + \sqrt{1 - \kappa^2}\}^{1/2} \geq \delta \geq \sqrt{2}\{1 - \sqrt{1 - \kappa^2}\}^{1/2} \tag{70}$$

and for the case $\frac{1}{2} > \kappa > 0$

*According to the equation of continuity $(\partial\rho/\partial t)_{t=0} = \rho_0 v(\partial g/\partial x)$.

$$\sqrt{2}\{1 + \sqrt{1 - \kappa^2}\}^{1/2} \geq \delta \geq 1 + \frac{1}{\sqrt{2}}\{1 + \sqrt{1 - 4\kappa^2}\}^{1/2} \tag{71}$$

$$1 + \frac{1}{\sqrt{2}}\{1 - \sqrt{1 - 4\kappa^2}\}^{1/2} \geq \delta > \frac{1}{\sqrt{2}}\{1 + \sqrt{1 - 4\kappa^2}\}^{1/2} \tag{72}$$

$$\frac{1}{\sqrt{2}}\{1 - \sqrt{1 - 4\kappa^2}\}^{1/2} > \delta \geq \sqrt{2}\{1 - \sqrt{1 - \kappa^2}\}^{1/2} \tag{73}$$

The phases of these two waves are related to one another by (69). The two waves have the same amplitude when δ is equal to either of the two limits in (70), to the upper limit in (71), and to the lower limit in (73). They propagate with the same velocity. However, they should be regarded as different waves, since their phase planes are different because of (69). This point was overlooked in previous work where the phases were not considered. The case (73) includes the corresponding criterion in the case of the KdV equation which was obtained by Berezin and Karpman.[3] They also discussed numerically the decay into up to six waves. We do not write down the amplitudes of the resultant solitary waves

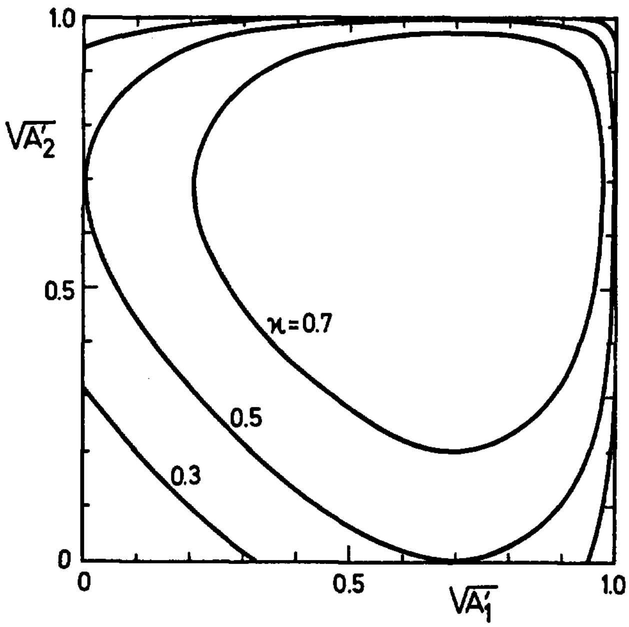

Fig. 3. The curves described by the equation $\sqrt{A_1'(1 - A_1')} + \sqrt{A_2'(1 - A_2')} - \varkappa = 0$. They are symmetric with respect to the line $\sqrt{A_2'} = \sqrt{A_1'}$. For $\varkappa \geq \frac{1}{2}$ the curve is a closed one, while the curve is divided into four parts for $\varkappa < \frac{1}{2}$. An initial disturbance can convert into two solitary waves when there exists a cross point of the curve and the line $\sqrt{A_1'} + \sqrt{A_2'} - \delta = 0$. Here we consider that $1 \geq A_1' \geq A_2' > 0$.

as they are such complicated functions of δ and κ that it is not easy to understand their context. The important point is that an initial disturbance can be converted into solitary waves under certain conditions.

9. CONCLUDING REMARKS

We have studied the solitary wave and the nonlinear periodic wave solutions of the Pitaevskii–Gross equation. They are described by the same functional forms as those of the KdV equation. The nonlinear periodic wave reduced to the solitary wave and to Bogoliubov's phonon in the appropriate limits. This fact does not mean that the present periodic waves can describe arbitrary excited states which deviate by a large amount from equilibrium, as the superposition principle does not hold for them. However, this principle holds asymptotically for the solitary waves. This fact together with their properties discussed in Sections 7 and 8 suggests that the solitary waves may play an important role in irreversible processes. For this reason we consider the solitary wave is of great interest. Before we predict its effect on the properties of the system, we have to extend our theory to the three-dimensional case. This will be studied in subsequent work.

We will also study nonlinear wave solutions of the two-fluid hydrodynamic equations of superfluid helium. The nonlinear waves will be able to exist stably at very low temperatures where the dissipative effects are small.

We suggest that superfluid helium may be one of the best systems in which to study experimentally the present types of nonlinear wave. Up to now there are no experimental results for real systems except two which show the propagation of the solitary wave in a low-density plasma[14] and in an anharmonic crystal.* "Experimental" information is at present obtained from computers. Experiments in superfluid helium would promote our understanding of nonlinear waves in general and the solitary wave in particular.

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