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Electrical Resistivity of Semimetals in the Extreme Quantum Limit

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The electrical resistivity of semimetals in the extreme quantum limit is strongly affected by the presence of anomalous charge–density waves at low temperatures. Anomalous Green's function and perturbation theories of the charge–density wave regime are related and extended to calculation of the imaginary part of the electron self-energy. The electrical resistivity due to electron–hole scattering is calculated. The electron–hole scattering reaches a peak value near a scaling temperature T_c *with a* $\ln T$ *decrease at higher temperatures. When combined with electron–impurity and electron–phonon scattering contributions, the total resistivity behaves in a manner which has been observed experimentally in Bi–Sb alloys.*

1. INTRODUCTION

The term extreme quantum limit has been used extensively to refer to semiconductors or semimetals in magnetic fields sufficiently large so that $\hbar\omega_c \gg kT, kT_f$, where $\hbar\omega_c$ is the Landau level spacing. In this case the energy spectrum of the electrons is a quasicontinuous function of a single quantum number $k_z, z \parallel H$. The extreme quantization of the energy dependence on the remaining quantum numbers means that the dynamics of the electron gas may be regarded as quasi-one-dimensional. We will be discussing a one-dimensional semimetal with particular regard to the high magnetic field regime for bulk solids, although the mathematical model is also pertinent to linear molecules or any other quasi-one-dimensional electron gas with attractive interactions. For example, the model we use with equal

numbers of electrons and holes displaced in wave-vector space, and with only one spin state, is in the absence of an electromagnetic field mathematically identical to the model of a one-dimensional metal with both spins, considered by Bychkov *et al.*[1] in regard to the possibility of superconductivity in long linear molecules. However, for the most part we will discuss semimetals, since in this case electron–hole scattering affects the resistivity, and because relevant experimental work has been done for this case.

During the past several years a large body of theoretical work has been done on the "excitonic insulator,"[2] a condensate of the electrons of a semimetal or semiconductor in the form of an electron crystal in either charge or spin density. Paralleling this theory for bulk solids, theories of charge–density or spin–density waves have been developed for quasi-one-dimensional long polyenes.[3] With only one exception to our knowledge, the excitonic insulator has not been observed in three-dimensional bulk solids, and in particular has not been observed near the condition of zero bandgap or band overlap envisaged in the excitonic insulator theory. It now appears beyond reasonable doubt that the antiferromagnetic phase of chromium is a representation of the three-dimensional excitonic insulator,[4] however, this is associated with a peculiar and perhaps unique Fermi surface topology in which electron and hole sheets are almost perfectly "nested." In the ideal nested case, electron and hole sheets are of identical shape and a point-to-point correspondence exists according to a single translation vector in **k** space. In this case, anomalous coherent spin or charge density waves can occur with maximum amplitude at the Fermi energy over the entire Fermi surface.

As discussed by the author[5] and by Abrikosov,[6] at very large magnetic fields the electron crystal phase is more likely to occur than at zero field due to the one-dimensional nature of the electron spectrum, where electron and hole sheets of a semimetal must be perfectly nested, and because of the increased binding energy for the corresponding two-particle problem, the exciton. (For arbitrary but attractive interactions, in the infinite magnetic field limit the exciton bound state must always occur.[7]) Recently, experimental evidence of the quasi-one-dimensional excitonic insulator has been obtained by Brandt and Chudinov,[8] and also in Ref. 9 for Bi–Sb alloys in extreme quantum limit magnetic fields. The electrical conductivity shows anomalous depletion of the electron quasiparticle spectrum at the Fermi energy near the zero-bandgap condition, including a region of band overlap where the "normal" state would not be a degenerate Fermi liquid.

It is evidently not possible to calculate the electrical resistivity of a semimetal in the extreme quantum limit without including the effects of a low-temperature quasicondensate. In Ref. 5 the excitonic insulator in the

extreme quantum limit was treated using the Gor'kov anomalous Green's functions of superconductivity theory, assuming a weak-coupling theory. Abrikosov applied the "parquet algebra" perturbation techniques to a one-dimensional model of the semimetal.[6] Near the same critical or scaling temperature appearing in the anomalous Green's function theory, the vertex function diverges at zero frequency. This divergence, peculiar to the perturbation theory, can be removed by some renormalization procedure; however, a temperature range of strong coupling near T_c must remain. A similar situation occurs for the perturbation theory of the t matrix for electrons in a metal with attractive interactions in three dimensions; however, in this case, after the renormalization of superconductivity theory the temperature region of strong coupling is assumed narrow in accordance with the singular nature of the second-order phase transition, as required by the experimental results, e.g., the specific heat. In a one-dimensional Fermi liquid, as has been discussed extensively in recent years, no phase transition can occur.[10] In this case the temperature region of strong coupling must be of considerable extent. As in the Kondo alloy with magnetic impurities, we expect a temperature regime where coupling is strong and where logarithmic functions of frequency and temperature enter into the electrical resistivity and other effects. In this regime, the equations of Ref. 5 are invalid, e.g., the gap equation assuming weak coupling is linear in the interaction. However, we will show that when $T \ll T_c$ the theory becomes weak coupling again, as at very high temperatures, and the equations of Ref. 5 are not invalidated for this regime. Extending the work by Abrikosov,[6] we calculate the imaginary part of the self-energy for electrons or holes in the logarithmic approximation and determine the electrical resistivity due to electron–hole scattering. The strong-coupling intermediate regime near T_c will be discussed only qualitatively, since we do not at present have a mathematical solution for the vertex functions or self-energies which is valid in this regime. It will be assumed that in one dimension all quantities including the electrical resistivity are continuous functions of the temperature.

2. HIGH-TEMPERATURE PERTURBATION REGIME

In our model for a one-dimensional metal with only one spin state, the bare interaction is taken as[6]

$$H_{\text{int}} = -g \int \psi_1^+(z)\psi_2^+(z)\psi_1(z)\psi_2(z)\, dz \tag{1}$$

i.e., an attractive interaction with g positive. $\psi_1(z)$ and $\psi_2(z)$ are the Heisenberg operators for electrons and holes of two different fields, propagating in a single dimension z with Fermi velocity $\pm V_f$. The apparently artificial

notation of two fermion fields for the electron assembly simply eliminates the displacement wave vector **Q** between the two local centers of symmetry of two regions in **k** space characterized by equal but opposite curvature of E vs. **k** near the Fermi energy, i.e., the E vs. **k** spectrum of the electron assembly is "multivalleyed." The wave-vector transform of H_{int} is

$$[\gamma_{\alpha,\beta,\gamma,\delta}(k_1, k_2; k_3, k_4)]_0 = -g(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \tag{2}$$

where we have dropped the subscript of k_z, and the subscripts α, β, γ, δ assume values 1 or 2 to denote states of the electron or hole fields. The Kroneker delta terms require that the bare interaction act only between unlike particles, since the integral of Eq. (1) must vanish if the particles are identical. g positive corresponds to the sign of the coulomb interaction. The absence of a second spin state corresponds to the high magnetic field regime where the Zeeman energy is very large and where only one spin state occurs near the Fermi energy. (For a quasi-one-dimensional *metal*, where E vs. k_z can be regarded as a single-valley spectrum, the second fermion field refers to another spin state as in the linear molecule model of Bychkov *et al.*)

Before writing the results for dressed vertices obtained in Refs. 1 and 6, we will first discuss briefly the origin of the logarithmic functions which appear in them. Consider the graph of Fig. 1 representing part of the lowest order vertex correction. Using thermal Green's functions for free particles, with thermal frequencies $\omega_n = (2n + 1)\pi T$, this contribution is

$$\delta\gamma(\omega) = g^2 T \sum_{\omega_n} \int [i\omega_n - \varepsilon_1(k_f + k) + i\varepsilon_1\delta]^{-1} \times [2\omega - i\omega_n - \varepsilon_2(q - k - k_f) + i\varepsilon_2\delta]^{-1}(dk/2\pi) \tag{3}$$

where $\varepsilon_1\delta$ and $\varepsilon_2\delta$ are infinitesimal. The entrance energy is $2\omega = \omega_1 + \omega_2$ and $\varepsilon_1(k) = \varepsilon_2(k) = \varepsilon(k)$ since the electron and hole masses are equal. The summation over thermal frequencies can be continued analytically to real frequencies using the poles of the Fermi function and the residue theorem. Taking $q \approx 0$ for one of the dominant logarithmic parts of the integral (another logarithmic region of integration occurs for $q = 2k_f$),

$$\delta\gamma(\omega) = (-g^2/4\pi) \int_{-\infty}^{\infty} \{\tanh [\varepsilon(k)/2T]/[\varepsilon(k) - \omega]\}\, dk \tag{4}$$

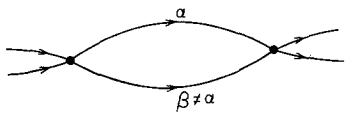

Fig. 1. Part of the lowest order vertex correction.

In Refs. 1 and 6 an arbitrary cutoff is imposed at $\pm D$ with a constant density of states through the band. We will retain both the entrance energy and finite temperature (which was not done in either of those calculations) and use a smooth cutoff according to a Lorentzian density of states function

$$n(\varepsilon) = n(0)D^2/(\varepsilon^2 + D^2) \tag{5}$$

where $n(0) = V_f^{-1}$, and D is the half-width of the band, approximately equal to the degeneracy energy. With $dk \to n(\varepsilon)\, d\varepsilon$ the integral can be evaluated exactly.

$$\delta\gamma(\omega) = [-g^2 n(\omega)/2\pi]\{\psi_d[\tfrac{1}{2} + (D/2\pi T)] - \psi_d[\tfrac{1}{2} - (i\omega/2\pi T)]\} \tag{6}$$

where ψ_d is the digamma function. Using an approximate expression for the digamma function in accordance with the logarithmic approximation where coefficients near unity are taken equal to unity and constant terms of order g^2 are dropped,

$$\delta\gamma(\omega) \approx [-g^2 n(0)/2\pi] \ln [D/(i\omega + \pi T)] \tag{7}$$

$|\omega|, \pi T \ll D$. The same result is obtained with a flat density of states and sharp cutoff when both ω and T are retained. The logarithmic functions obtained in Refs. 1 and 6 are not dependent on the cutoff procedure as long as $n(\varepsilon)$ falls off rapidly enough at infinity and is analytic in the neighborhood of the real axis. When $|\omega|, \pi T \approx D\, e^{-\pi/gn(0)}$ the lowest order corrections to the interaction are equal in magnitude to g. With a similar contribution from each higher order term in the parquet perturbation series, the theory must enter a strong-coupling regime for arbitrarily small g.

We will not discuss the various graphs giving dominant logarithmic contributions to the vertex corrections in each order g^n, referring only to Refs. 1 and 6. Using the form of Eq. (6) for the logarithmic variable, the results for the "symmetric" vertex functions with entrance energies $|\omega_1| \approx |\omega_2|$ and exit energies $|\omega_1'| \approx |\omega_2'|$ are

$$\gamma_{\alpha\alpha}(\omega) = g^2 n(0)(x/2\pi)/\{1 - [gn(0)/\pi]x\} \tag{8}$$

$$\gamma^{2k_f}_{\alpha,\beta\neq\alpha}(\omega) = -g/\{1 - [gn(0)/\pi]x\}$$

$$\gamma^{0}_{\alpha,\beta\neq\alpha}(\omega) = -g\{1 - [gn(0)/2\pi]x\}/\{1 - [gn(0)/\pi]x\}$$

The superscripts denote the momentum transfer 0, $\pm 2k_f$ from the α fermion to the β fermion. Here $x = \{\psi_d[\tfrac{1}{2} + (D/2\pi T)] - \psi_d[\tfrac{1}{2} - (i\omega/2\pi T)]\}$. With $\omega = 0$, all vertices diverge at

$$T_c = (2\gamma_E D/\pi)\, e^{-\pi/gn(0)} \tag{9}$$

where γ_E is the Euler constant, 1.78. The denominators of the vertex functions may be written as $\ln [(\omega + i\pi T)/i\pi T_c]$. A pole of the vertex occurs in

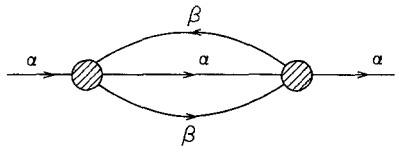

Fig. 2. Renormalized self-energy diagram summing the dominant self-energy graphs.

the lower half ω plane with $\omega = i\pi(T_c - T)$ for $T > T_c$. When this pole crosses the real axis into the upper half-plane at $T = T_c$, the electron spectrum is unstable if the logarithmic approximation is valid. This is not in fact the case since the logarithmic approximation breaks down near T_c, as is well known in, e.g., the Kondo alloy problem. Viewed as a resonance with damping rather than an instability, when $T = T_c$ the resonance energy coincides with the Fermi energy where incoming and outgoing states are most available.

We calculate now the imaginary part of the self-energy. Consider the diagram of Fig. 2 where the vertices and propagators are dressed. This diagram sums all diagrams in the perturbation series for the self-energy which can be cut across three lines between each vertex. We can guess immediately that this is the dominant contribution to the self-energy because opening one internal line to obtain $d\Sigma(\omega)/d\omega$ yields the graph summing the dominant parquet series for the Abrikosov calculation of the vertex function. This is expected for the dominant self-energy graph since the derivative of the self-energy and the vertex function are related by the Ward identity.[11] Closing the third internal line results in integration over k, from which we retain the part approximately equal to $[-in(0)/2]\delta[\omega - \varepsilon(k')]$ sign ω, if the self-energy in the dressed propagators is small. With reference to our discussion of Fig. 1, at zero temperature the remaining frequency integration is of the form

$$\Sigma(\omega) = [in^2(0)/2] \operatorname{sign} \omega |\gamma(\omega)|^2 \int \ln [(\omega' - |\omega|)/D] \, d\omega'$$

$$\approx [-in(0)/2] \langle \gamma(\omega) \rangle^2 \operatorname{sign} \omega$$

when $|\omega| \ll D$. With regard to graphs not included in Fig. 2, it can be verified by direct calculation in each order of g^n that graphs such as, e.g., Fig. 3

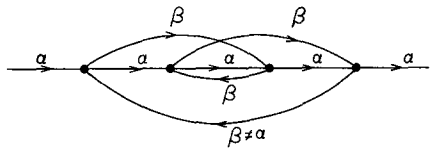

Fig. 3. A self-energy diagram not included in Fig. 2.

where internal lines cross, etc., yield powers of the logarithmic variable lower than $n-2$. In the logarithmic approximation where only highest powers of the logarithmic variable are retained in each order, these terms can be regarded as negligible.

Evaluation of Fig. 2 at finite temperature is more difficult since, as apparently first discussed by Eliashberg,[12] the analytic continuation of the thermal frequencies to real frequency is not simple. Using the Eliashberg continuation, we obtain the expression

$$\begin{aligned}\Sigma_\alpha(\omega) = & -i\int\frac{d\omega_1\,d\omega_2}{(2\pi)^2}\int\frac{dk_1\,dk_2}{(2\pi)^2}\operatorname{Im}G_\alpha^{\mathrm{Ret}}(\omega_1,k_1)\operatorname{Im}G_\beta^{\mathrm{Ret}}(\omega_2,k_2)\\ &\times\operatorname{Im}G_\beta^{\mathrm{Ret}}(\omega+\omega_1-\omega_2,k+k_2-k_1)\\ &\times\gamma_{\alpha\beta}(k,k_2;k_1,k+k_2-k_1)\gamma_{\alpha\beta}(k+k_2-k_1,k_1;k_2,k)\\ &\times\left[\tanh\frac{\omega_2}{2T}-\coth\frac{\omega+\omega_2}{2T}\right]\left[\tanh\frac{\omega_1}{2T}-\tanh\frac{\omega_1-\omega-\omega_2}{2T}\right]\end{aligned}\tag{10}$$

where the total imaginary part for the α fermion requires summation over β, i.e., two of the internal lines may be from either the same fermion field or from the other fermion field. As is well known, in the usual three-dimension calculation this graph is proportional to $[\omega^2+(\pi T)^2]/D^2$.[12] For the one-dimensional semimetal a dominant region of integration can be found where the differences between the frequency arguments (modulus) are much less than the frequencies. With $|\omega_2|-(\omega,\omega_1)=\Delta\omega$, for $|\omega|\ll T$ the thermal factors for this logarithmic region become

$$\lim_{\Delta\omega\to 0}\coth\frac{\Delta\omega}{2T}\left[\frac{\Delta\omega}{2T}\operatorname{sech}^2\frac{\omega}{2T}\right]\approx 1\tag{11}$$

which is the same as would be obtained in the $T=0$ calculation. Consistent with the logarithmic approximation where factors near unity are taken as unity, we use this factor when $|\omega|\approx T$ and when $|\Delta\omega|\approx|\omega|$ as well. Taking $\operatorname{Im}G^{\mathrm{Ret}}(\omega,k)\approx-\pi\delta[\omega-\varepsilon(k)]$ when the self-energy is small, performing the integrations and collecting all factors, we obtain

$$\Sigma_\alpha(\omega)=-i\pi\operatorname{sign}\omega[n(0)/2\pi][|\gamma^0_{\alpha\beta}(\omega)|^2+|\gamma^{2k_f}_{\alpha\beta}(\omega)|^2]\tag{12}$$

The vertices here are the symmetric vertices of Eq. (8) since $|\omega_1|\approx|\omega_2|\approx|\omega|$ is required in the dominant region of integration in Eq. (10). Identifying the imaginary part of the self-energy with the imaginary part of the diagonal t matrix, we see that this result is exactly what would be obtained using the optical theorem.[13]

Both Abrikosov[6] and Bychkov *et al.*[1] have shown within the logarithmic approximation that anomalous quasiaverages occur for the perturbation regime which are of the form

$$\Delta_\alpha^{Q,0} \approx \Delta_\alpha^{2k_f} \approx aT_c\{1 - b[gn(0)x/\pi]\}^{-1/4} \tag{13}$$

Here x is the same variable as we have used for the vertex functions, and the parameters a and b have been included to emphasize that these coefficients are arbitrary and undetermined in the logarithmic approximation and, moreover, are not necessarily the same for both quasiaverages. The quasiaverages are defined by

$$\begin{aligned}\Delta_\alpha^{Q,0}(\omega) &= \int_{-\infty}^{\infty} dt \int [dk\, dk'/(2\pi)^2] \int dz\, dz' \langle T[\psi_\alpha(z,t)\psi_{\beta\neq\alpha}(z',0)]\rangle \\ &\quad \times e^{ikz}\, e^{-ik'z'}\, e^{i\omega t}\delta_{kk'} \\ \Delta_\alpha^{2k_f}(\omega) &= \int_{-\infty}^{\infty} dt \int [dk\, dk'/(2\pi)^2] \int dz\, dz' \langle T[\psi_\alpha(z,t)\psi_\alpha^+(z',0)]\rangle \\ &\quad \times e^{-ikz}\, e^{i(k'\pm 2k_f)}\, e^{i\omega t}\delta_{kk'}\end{aligned} \tag{14}$$

For the semimetal, the annihilation operator for the α fermion is charge equivalent to the creation operator for the β fermion. Δ_α^Q refers to a charge–density wave with wave vector Q_z, where $\mathbf{Q}$ is the displacement between the electron and hole distributions as discussed earlier. In the calculation by Bychkov *et al.* the quasiaverages Δ_α^0 denotes correlation of an electron with spin α at k_z with an electron of opposite spin $\beta \neq \alpha$ at $-k_z$. The second quasiaverage $\Delta_\alpha^{2k_f}$ denotes an Overhauser charge–density wave for either semimetal or metal cases, with wave vector in the z direction $2k_f$. Abrikosov has shown for the semimetal with only one spin state, that all other quasiaverages which might be associated with, e.g., superconductivity, must vanish, and, in addition, that when the electron and hole masses are not equal, Δ^Q and Δ^{2k_f} do not vanish for either sign of the bare interaction.[6] Actually, in the Kroneker delta sense of Eq. (14), all anomalous quasiaverages must vanish at finite temperature for a one-dimensional electron assembly.[10] However, in a less restricted sense where near $T = 0$ the Kroneker delta is replaced by some highly peaked function, to our knowledge there is no "proof" that a damped, anomalous quasiaverage with finite line width cannot occur. The expressions of Eq. (13) diverge at T_c, and since the magnitude of the anomalous quasiaverage (or the integral over its line width) must be finite, the logarithmic approximation breaks down, as mentioned earlier. We include this discussion of the anomalous quasiaverages in the perturbation regime, however, since in one dimension such averages cannot vanish discontinuously as temperature increases at T_c,

but must become negligible in some asymptotic sense. We impose as a physical boundary condition

$$\Delta^{Q}(T \approx D) \approx \Delta^{2k_f}(T \approx D) \approx 0$$

In this case, since the functional forms of Eq. (13) are more slowly varying than the vertex functions or the imaginary part of the self-energy

$$\Delta^{Q}(T), \Delta^{2k_f}(T) \ll T \qquad \text{when} \qquad D \gg T \gg T_c$$

In this perturbation regime the energy gap at the Fermi energy, indicated by the square-root dependence of $|\Delta^{Q}|^2$, $|\Delta^{2k_f}|^2$ on the frequency, can be regarded as negligible and will be ignored when we calculate the electrical resistivity of the semimetal for $T \gg T_c$. Of course, in contrast to the anomalous quasiaverages of superconductivity theory, the charge density wave is inert with respect to response to a small static electric field.

3. LOW TEMPERATURES

When $|\omega|$, $T \ll T_c$, we take as an ansatz the anomalous quasiaverage of Ref. 5 and, in addition, a second quasiaverage of the form of Δ^{2k_f}, which was ignored in our earlier paper. As in Ref. 5, the Gor'kov decoupling scheme in the equations of motion for the Heisenberg operators leads to Green's functions of the form familiar from superconductivity theory:

$$\begin{aligned}
\hat{G}_\alpha(k, \omega) &= \{v_\alpha(k)/[\omega - E(k) + i\delta]\} + \{u_\alpha(k)/[\omega + E(k) - i\delta]\} \\
&= [\omega + \varepsilon(k)]/[\omega^2 - E^2(k)] \\
\Delta_\alpha^{Q,0} F_\alpha^{Q,0}(k, \omega) &= |\Delta_\alpha^{Q,0}|^2/[\omega^2 - E^2(k)] \\
\Delta_\alpha^{2k_f} F_\alpha^{2k_f}(k, \omega) &= |\Delta_\alpha^{2k_f}|^2/[\omega^2 - E^2(k)]
\end{aligned} \tag{15}$$

where $E^2(k) = \varepsilon^2(k) + \Delta^2$, $|\Delta|^2 = |\Delta^{Q,0}|^2 + |\Delta^{2k_f}|^2$, and $v_\alpha^2 = 1 - u_\alpha^2 = \frac{1}{2}\{1 - [\varepsilon(k)/E(k)]\}$. For equal electron and hole masses, $\varepsilon_1(k) = \varepsilon_2(k) = \varepsilon(k)$ as discussed earlier, and $E_{11}(k) = E_{22}(k) = E_{\alpha\beta}(k) = E(k)$. The gap function is given by

$$|\Delta_\alpha^{Q,0}(k)|^2 = -gT \sum_{\omega_n, k'} \Delta_\alpha^{Q,0} F_\alpha^{Q,0}(k', i\omega_n) \tag{16}$$

which yields the usual result $\Delta_\alpha^{Q,0}(T = 0) \simeq 1.7T_c$. In this weak-coupling approximation, $|g| \ll 1$, the quasiaverage $\Delta_\alpha^{2k_f}$ is zero because the bare interaction is zero for scattering by like particles. The gap function of Eq. (16) can be represented symbolically by the graph of Fig. 4. Bychkov *et al.* stress that when starting at high temperatures, as T approaches T_c graphs of higher order in g such as Fig. 5 must also be considered.[1] With

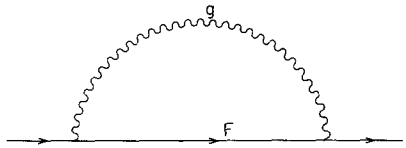

Fig. 4. The gap function in the weak-coupling approximation.

internal propagators given by the G_0 in Eq. (3), these higher order terms are comparable to the first-order term when $[gn(0)/\pi] \ln [iD/(\omega + i\pi T)] \approx 1$.

Bychkov *et al.* suggest that for the entire range $T \lesssim T_c$, the gap function equation (16) is invalid,[1] based on their results for the perturbation regime and using the Green's functions for free electrons for the internal propagators of, e.g., Fig. 5. For the $T \ll T_c$ low-temperature regime, we assume that anomalous quasiaverages do exist, and in all graphs use the renormalized Green's functions of Eq. (15). The results, as will be shown below, are not similar to those obtained by Bychkov *et al.* using the bare propagators and cutting off all integrations at a lower limit Δ.[1]

Consider the imaginary part of the electron self-energy at low temperatures, which is zero for the theory linear in g. The dominant terms in the perturbation expansion are summed as for the high-temperature regime (and as for a three-dimensional assembly[12]) by the self-energy diagram of Fig. 2, and the same analytic continuation of thermal frequencies [Eq. (10)] is valid. Consider the dominant region of integration discussed earlier, where symmetric vertices were required. The renormalized functions $\hat{G}_\alpha(k, \omega)$ are descriptive of a linear combination of two quasiparticles of equal amplitude when $u_\alpha^2 \approx v_\alpha^2$ for $T \ll T_c$. In this case, in Eq. (10)

$$\text{Im } G_\alpha^{\text{Ret}}(k, \omega) = \text{Im } \hat{G}_\alpha^{\text{Ret}}(k, \omega) = -(\pi/2)\{\delta[\omega - E(k)] - \delta[\omega + E(k)]\}$$

The symmetric vertices do not depend on the signs of the frequency arguments. The integrations over frequency and momenta in Eq. (10) yield a null result for this "dominant" region of integration where the thermal factors are

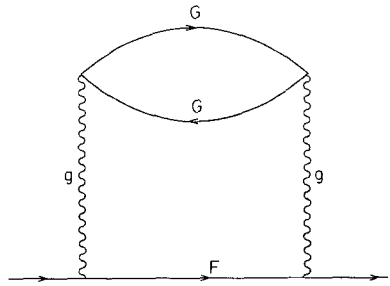

Fig. 5. A gap function diagram not included in the weak-coupling approximation.

approximately unity. When $T \ll T_c$, there is no dominant region of integration in Eq. (10), in contrast to the logarithmic approximation at high temperatures. In this case the diagram of Fig. 2 must be evaluated in a similar manner to the three-dimensional calculation by Eliashberg[12]:

$$\operatorname{Im}\sum(\omega) \approx -\operatorname{sign}\omega\langle|\gamma|^2\rangle \cosh\frac{\omega}{2T}\int \frac{d\varepsilon_1\, d\varepsilon_2 \delta(\omega - E_3 + E_1 - E_2)}{\cosh\dfrac{E_1}{2T}\cosh\dfrac{E_2}{2T}\cosh\dfrac{E_3}{2T}}$$

$$\times\left[1 - \frac{\Delta^2}{E_1 E_2}\right]$$

$$\approx -\operatorname{sign}\omega\langle|\gamma|^2\rangle n(0)\, e^{-\Delta/T}\theta(|\omega| - \Delta) \tag{17}$$

where the θ function is unity when $|\omega| > \Delta$ and zero otherwise, and $E_i = E(k_i)$. The renormalized quasiparticle energies $E(k)$ have replaced the $\varepsilon(k)$ of the Eliashberg calculation, and the delta function is approximately satisfied for the important part of the integral by $|\omega| \approx E_3$. The resulting integration is converted to an integration over quasiparticle energy using the BCS density of states and is similar to the integral occurring in the calculation of ultrasonic attenuation in a bulk superconductor. Physically, the result in Eq. (17) means that collisions between quasiparticles are exponentially infrequent because the number of excitations across the energy gap 2Δ is exponentially small.

A similar exponentially small result occurs evaluating, e.g., the vertex correction term represented by Fig. 1. In the gap function equation (16), in Eq. (17), and in general for $T \ll T_c$ the bare interaction can be taken as the complete interaction. The weak-coupling theory is valid for $T \ll T_c$, and the results of Ref. 5 for that regime are not invalidated by the results obtained for the high-temperature perturbation regime in Refs. 1 and 6. Similar to the high-temperature regime, however, the anomalous Green's function theory becomes invalid as T approaches the strong-coupling regime near T_c. The low-temperature theory is somewhat more than an ansatz, the expression used earlier in this section, since the high-temperature perturbation theory indicates that some renormalization procedure is required for $T \ll T_c$, and the renormalization must involve an energy gap at the Fermi energy.

For a real semimetal complete cancellation of direct and exchange interactions between like particles is not expected, and the absence of the Δ^{2k_f} quasiaverage when $T \ll T_c$ is an artifact of the structureless interaction. In a more general model the amplitudes of $\Delta^{Q,0}$ and Δ^{2k_f} may be comparable at $T = 0$. In the regime $T \ll T_c$ these amplitudes may be regarded as nearly constant. The small temperature dependence can be determined from the

gap equation (16), as discussed in Ref. 5; however, in a one-dimensional model determination of such small quantities cannot be regarded as accurate due to departures from the weak-coupling approximation which increase with temperature.

4. ELECTRICAL RESISTIVITY OF THE SEMIMETAL

The causal electromagnetic linear response function is defined by

$$j_\alpha(\omega) = K_\alpha(\omega + i\delta)A(\omega) \tag{18}$$

The electromagnetic response function is given by the current–current correlation function according to

$$\begin{aligned} K_\alpha(iv) = & -e^2T \int (dk/2\pi) \sum_{\omega_n} [V(k)G_\alpha(k, i\omega_n)\Gamma_\alpha(k, i\omega_n, i\omega_n - iv)G_\alpha(k, i\omega_n - iv) \\ & - V(k)G_\alpha(k, i\omega_n)V(k)G_\alpha(k, \omega_n)] \\ \equiv & -e^2T \int (dk/2\pi) \sum_{\omega_n} [P^p_\alpha(k, i\omega_n, i\omega_n - iv) - P^D_\alpha(k, i\omega_n)] \end{aligned} \tag{19}$$

where $V(k)$ is the velocity, and P^p and P^D are paramagnetic and diamagnetic parts. The second term in the integrand, the diamagnetic component, is obtained from the usual Ne^2/m form by using $m_k^{-1} = \nabla^2\varepsilon(k)$, performing a partial integration in k, and then using the Green's function representation for the particle number

$$N = N_1 = N_2 = T \sum_{k,\omega_n} e^{i\omega_n\delta}G_{1,2}(k, i\omega_n) = \sum_k n(k)$$

Γ is the dressed electromagnetic vertex. Since the symmetric vertex functions do not contain k as an argument, it might be expected that no corrections to the electromagnetic vertex occur. However, this is not exactly true since in the temperature range where vertex corrections occur, the electron–hole vertex does depend on the momentum transfer, as shown in Eq. (8), and corrections to the electromagnetic vertex will occur as well. Consistent with the parquet calculation of the dressed vertices, the sum of all graphs of the electromagnetic vertex in which electron lines do not cross is required, and Γ satisfies the integral equation summing such graphs:

$$\begin{aligned} \Gamma_\alpha(k, i\omega_n, i\omega_n - iv) = V(k) + \tfrac{1}{2} \int (dk'/2\pi)[|\gamma^0_{\alpha\beta}(i\omega_n)|^2 - |\gamma^{2k_f}_{\alpha\beta}(i\omega_n)|^2] \\ \times G_\alpha(k', i\omega_n)\Gamma_\alpha(k', i\omega_n, i\omega_n - iv)G_\beta(k', i\omega_n - iv) \end{aligned} \tag{20}$$

Here the arguments in the integrand are $i\omega_n$ rather than a summation over $i\omega_n'$, since, as discussed earlier, in the temperature regime where vertex corrections are important the imaginary part of the self-energy is dominated by the symmetric vertices, which is also true of the electromagnetic vertex.

We define the electromagnetic vertex consistent with the logarithmic approximation:

$$\Gamma_\alpha(i\omega_n, i\omega_n - iv) = \left[V_f + \operatorname{Im} \Sigma_\alpha'(i\omega_n) \int (d\varepsilon(k)/2\pi) G_\alpha(k, i\omega_n) \times \Gamma_\alpha(i\omega_n, i\omega_n - iv) G_\beta(k, i\omega_n - iv) \right] \tag{21}$$

where

$$\operatorname{Im} \Sigma_\alpha'(i\omega_n) = [n(0)/2][|\gamma^0_{\alpha\beta}(i\omega_n)|^2 - |\gamma^{2k_f}_{\alpha\beta}(i\omega_n)|^2] \tag{22}$$

We will not evaluate Γ immediately since this requires a knowledge of the proper analytic continuation of the summation over thermal frequencies in Eq. (19) to real frequencies. We are interested in the dc conductivity defined by

$$j(0) = \sigma(\omega) E(\omega)|_{\omega=0} \quad \text{or} \quad j_\alpha(0) = \lim_{\omega \to 0} [K_\alpha(\omega)/i\omega] E(\omega)$$

where the electric field is $E(\omega) = i\omega A(\omega)$. We require then $\lim_{\omega \to 0} K_\alpha(\omega)/i\omega$, which can be obtained after the analytic continuation for the self-energy graph of Fig. 2 discussed earlier. Analytic continuation of the summation in Eq. (19) is complicated by the fact that two cuts of the function occur, the cut due to the first Green's function along the real axis, and, for P^p, the cut from the second Green's function which is displaced from the real axis to $+iv$. The frequency summation is written as a contour integral in the complex plane encompassing the poles of $\tanh \omega'/2T$ at $\omega' = i\omega_n$. The contour integral is then transformed into an integral of the discontinuities of P along the two cuts. This procedure results in

$$\lim_{\omega \to 0} [K(\omega)/i\omega] = \lim_{\omega \to 0} [n(0)e^2/\omega] \int_{-\infty}^{\infty} [d\varepsilon(k)/2\pi] \int_0^{\infty} d\omega' \tanh(\omega'/2T) \times [P(k, \omega' + i\delta, \omega' - \omega - i\delta) - P(k, \omega' - i\delta, \omega' - \omega - i\delta) + P(k, \omega' + i\delta, \omega' + \omega + i\delta) - P(k, \omega' - i\delta, \omega' + \omega + i\delta)] \tag{23}$$

The second and third terms in the square brackets of the integrand do not contribute in the limit $\omega \to 0$ due to the fact that the self-energy parts of the two Green's functions in the $\varepsilon(k)$ integrands of Eq. (21) are of the same sign. After a change of variable in the first term of Eq. (23), $\omega' \to \omega' + \omega$, combining

with the fourth term and then differentiating the entire expression which results, exclusive of the ω^{-1} factor, the limit is obtained by L'Hospital's rule:

$$\lim_{\omega \to 0} \sigma_\alpha(\omega) = (Ne^2/2T) \int_0^\infty d\omega' \int_{-\infty}^\infty d\varepsilon(k) \operatorname{sech}^2(\omega'/2T) \times P_\alpha^p(\omega' + i\delta, \omega' - i\delta) V_f^{-2} \tag{24}$$

The order of integration has been reversed from Eq. (23), a procedure which can lead to difficulties when the paramagnetic or diamagnetic currents are taken individually in the integrand. Here we have reversed the order before taking the $\omega \to 0$ limit, and no problems in this regard occur.

We now require $\Gamma(\omega' + i\delta, \omega' - i\delta)$ in Eq. (21). Using the residue technique, the integral can be evaluated and the resulting algebraic equation solved for Γ:

$$\Gamma_\alpha(\omega' + i\delta, \omega' - i\delta) = V_f \operatorname{Im} \Sigma_\alpha(\omega')/[\operatorname{Im} \Sigma_\alpha(\omega') - \operatorname{Im} \Sigma'_\alpha(\omega')] \tag{25}$$

$P^p(\omega + i\delta, \omega - i\delta)$ is obtained directly, substituting in Eq. (19). Performing $d\varepsilon(k)$ integration of Eq. (24), we obtain

$$\sigma_\alpha(0) = (Ne^2/mT) \int_0^\infty d\omega' \operatorname{sech}^2(\omega'/2T) \tau_\alpha(\omega') \tag{26}$$

where

$$\tau_\alpha^{-1}(\omega) = \operatorname{Im} \Sigma_\alpha(\omega) - \operatorname{Im} \Sigma'_\alpha(\omega) = n(0) |\gamma^{2k_f}_{\alpha,\beta \neq \alpha}(\omega)|^2 \tag{27}$$

using Eqs. (12) and (22). Not surprisingly, no contribution from the $\gamma^0_{\alpha\beta}$ vertex occurs. We have been consistent throughout with the logarithmic calculations of the vertex functions and self-energies wherein all momentum values are taken as $\pm k_f$, which means that there can be no contribution to the resistivity unless a particle is scattered to the opposite Fermi velocity. In Eq. (27) we have included only electron–hole scattering, since in this approximation two initial momenta $(\pm k_f, \pm k_f)$ may remain the same or be interchanged, and with scattering of identical particles the final momenta are indistinguishable from the initial state. This condition arising from the logarithmic approximation in one dimension is somewhat more stringent than the usual result for three dimensions, that there can be no electron–electron resistivity for equal-mass particles due to conservation of the *total* momentum (in the absence of Umklapp processes).

To the accuracy of the logarithmic approximation, the integration in Eq. (26) leads to

$$\sigma_{eh}(T) = \sigma_1(T) + \sigma_2(T) = 2Ne^2\tau_{eh}(T)/m \tag{28}$$

where $\tau_{eh}(T) = \tau_1(T) = \tau_2(T)$ is given by Eqs. (27) and (8) with $\omega = 0$. The factor two in Eq. (28) accounts for the two types of charge carriers. If we had considered the two-component *metal* where the indices α, β refer to the spin,[1] the electrical resistivity must be calculated using a two-component Nambu notation for the Green's functions. As might be expected in an electron assembly where all collisions must conserve velocity as well as momentum (not true for the semimetal where, regarding all charge carriers as electrons, one component can be regarded as having the reverse mass of the other), the electrical resistivity resulting from this procedure is zero.

In contrast to the T^2 "electron–electron" resistivity of bulk solids, the electron–hole resistivity in one dimension increases with decreasing temperature for $T \gg T_c$. Near T_c neither the perturbation nor anomalous Green's function theories for the self-energies are valid. When $T \ll T_c$ the typical self-energies are proportional to the number of quasiparticle excitations, which results in a factor which cancels $N(T)$ in $\sigma_{eh}(T)$. Near $T = 0$ the vertices are given by the bare interaction of Eq. (2). As the temperature increases, $\gamma(T)$ increases toward the strong-coupling regime near T_c. At low temperatures, $0 \ll T \ll T_c$, where $N(T) \propto e^{-\Delta/T}$, the electron–hole resistivity is given by

$$\rho_{eh} \propto |\gamma(T)|^2 \tag{29}$$

which increases with T. For $T_c \ll T \ll D$, the electron–hole resistivity is given by

$$\rho_{eh} = [mn(0)g^2/2Ne^2]\{1 - [gn(0)/\pi] \ln (D/\pi T)\}^{-2} \tag{30}$$

which decreases with increasing T. Since, as stated in the introduction, we assume all physical variables to be continuous in the temperature, the electron–hole resistivity reaches a smooth maximum in the strong-coupling regime near T_c intermediate to the temperature ranges of Eqs. (29) and (30). Near $T = 0$, the total resistance is dominated by impurity scattering in one dimension as in three dimensions, since[9] $\tau_{\text{Imp}}(T) \propto T^{1/2}$. Apart from the electron–impurity scattering, the term $i\delta$ in the argument of the causal response function of Eq. (18) ensures that the charge–density wave regime of the semimetal is an insulator when $N(T) \to 0$ as $T \to 0$. At sufficiently high temperatures, scattering by phonons contributes a term in the resistivity proportional to T^1, which saturates to a constant value when $T \approx T_f \approx D$, the degeneracy energy.[9] In Figs. 6 and 7 we illustrate qualitatively the

temperature dependence of the total resistivity for the one-dimensional semimetal model considered here.

5. DISCUSSION

There is some question as to whether a model theory of a one-dimensional semimetal is descriptive of a semimetal in the high magnetic field, extreme quantum limit. No problem exists for the logarithmic approximation, where integrations over the harmonic oscillator functions of the transverse coordinates contribute factors of order unity in each order of g^n. Since contributions to each order which are of lower power in the logarithmic variable than the dominant term also contribute such factors, which in the logarithmic approximation are taken as unity, qualitatively no new errors have been introduced by neglecting the transverse coordinates. A more fundamental question concerns the possibility of a phase transition in the high magnetic field regime, which cannot occur in the one-dimensional model. (Abrikosov has suggested that the poles in the vertex functions of the logarithmic approximation do indicate a phase transition in the high-field regime, but not in one dimension.[14]) A similar question occurs for the relation of one-dimensional theories to physical systems in partially finite geometry, i.e., physical specimens with two microscopic dimensions and the remaining dimension macroscopic.[15] The experimental results on fine superconductor wires show that for this case[16] the absence of a phase transition is not peculiar to the one-dimensional model. Although the opposite could be true for the high-field case, it appears that the absence of a phase transition is directly attributable to the extreme quantization of energy levels in quantum numbers related to transverse coordinates, and that the extreme quantum limit is qualitatively the same whatever the nature of the agent responsible for that quantization, i.e., no phase transition occurs.

Construction of a mathematical solution for the intermediate temperature range near T_c will probably not be simple to achieve, since this problem is similar in magnitude to the difficult regime near T_k for the Kondo alloy. By analogy with the Kondo problem, it might be expected that an integral equation solution exact to the Gor'kov (Nagaoka) decoupling procedure would be possible. If this is true, the anomalous Green's functions of Eq. (15) must be generalized so that for finite temperature the constraint $k = k'$ in Eq. (14) is abandoned. At finite temperatures, in a one-dimensional electron assembly, the anomalous correlations are undoubtedly damped and have finite line widths.

The behavior of the electrical resistivity in Fig. 6 is similar to that observed for Bi–Sb alloys in high magnetic fields, e.g., the regime near 50 kG of Figs. 8 and 9 in Ref. 9. In particular, the electrical resistivity proceeds

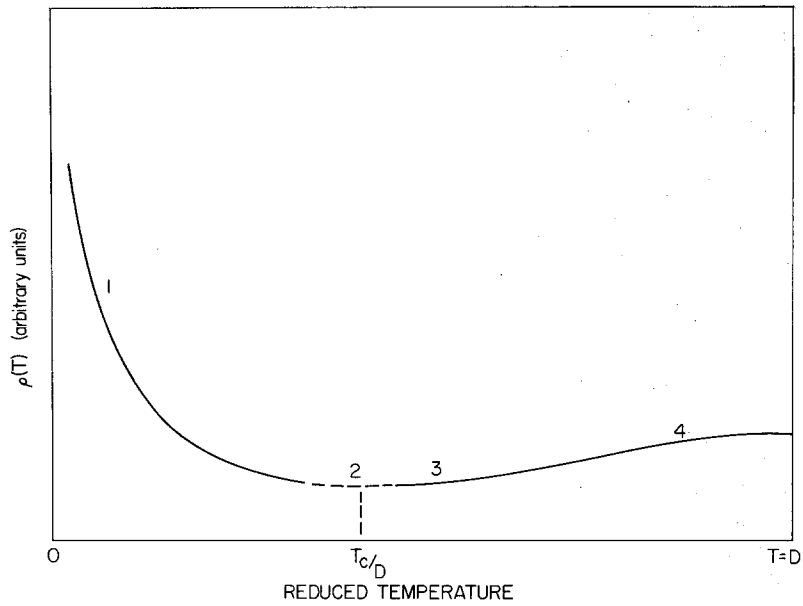

Fig. 6. Total resistivity vs. reduced temperature when $\tau_{\text{Imp}}(T_c) \ll \tau_{eh}(T_c)$: (1) charge-density wave regime where $\rho \approx \rho_{\text{Imp}}(T) \approx \frac{1}{4}\tau_{\text{Imp}}^{-1}(T)N^{-1}\, e^{\Delta/T}$; (2) intermediate-temperature range where strong-coupling occurs; (3) $T_c \ll T \ll D$ regime where $\rho_{eh}(T)$ is given by Eq. (30); (4) high-temperature regime where, in addition to $\rho_{eh}(T)$ and $\rho_{\text{Imp}}(T)$, electron–phonon scattering contributes a term proportional to T^1.

continuously from semimetal to "semiconductor" temperature dependence as the temperature is lowered. Similar measurements by N. B. Brandt and associates have usually been confined to only one or two temperatures,[17] with the magnetic field sweeping very rapidly in a pulse-field apparatus. However, Fig. 17 of Ref. 17 shows quite definitely that the boundary between semimetal and apparently semiconductor regions is defined by a critical value $(H/T)_c$, rather than by some critical field H_c. Brandt and Svistova have suggested for this figure that the band overlap (which has been increasing at low fields) vanishes at H_c, and bandgaps increasing with H occur at higher fields.[17] There can be no appreciable dependence of H_c on temperature in this mechanism of the "normal" band structure. On the other hand, since the band overlap is approximately proportional to field at lower fields, $D = aH$, these results are consistent with Fig. 6 (or perhaps Fig. 7). The *scaled* temperature decreases as D increases with H, and in this case beyond a field given by $(T/aH)_c = (T/D)_c \approx e^{-\pi/gn(0)}$, the charge–density wave regime occurs where the electrical resistivity displays a semiconductor-type dependence. To confirm this interpretation, detailed measurements through a range of both temperature and field would be required.

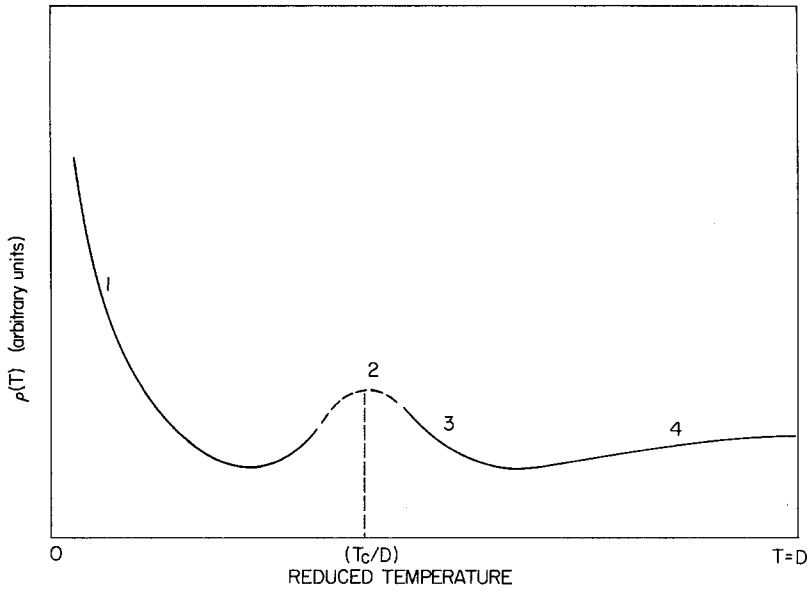

Fig. 7. Total resistivity vs. reduced temperature, when $\tau_{eh}(T_c) \ll \tau_{\text{Imp}}(T_c)$. In this case a peak in the resistivity due to $\rho_{eh}(T)$ occurs near T_c. On the high-temperature side of this peak, regime 3, and before electron–phonon scattering becomes appreciable, the total resistivity is described by Eq. (30).

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