COSMOLOGICAL MODELS IN CERTAIN SCALAR-TENSOR THEORIES

SHRI RAM and J.K. SINGH
Department of Applied Mathematics, Institute of Technology, Banaras, Hindu University, Varanasi, India

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Abstract. Exact solutions of Einstein field equations are obtained in the scalar-tensor theories developed by Saez and Ballester (1985) and Lau and Prokhovnik (1986) when the line-element has the form

$$ds^2 = \exp(2h)dt^2 - \exp(2A)(dx^2 + dy^2) - \exp(2B)dz^2$$

where h, A and B are functions of t only. The solutions are spatially homogeneous, locally rotationally symmetric and admit a Bianchi I group of motions on hypersurfaces t = constant. The dynamical behaviours of these models have also been discussed.

1. Introduction

The cosmological problem within the framework of general relativity consists in finding a model of the physical universe which correctly predicts the result of astronomical observations and which is determined by those physical laws which describe the behaviour of matter on scales upto those of clusters of galaxies. It is held that the long-range forces in the universe are produced by scalar fields. Scalar-tensor theories of gravitation have become a focal point of interest in many areas of gravitational physics and cosmology. They provide the most natural generalizations of general relativity and thus provide a convenient set of representations for the observational limits on possible deviations from general relativity. The most widely accepted and possibly the best motivated theory in which a scalar field shares the stage of gravitation is that of Brans and Dicke (Brans and Dicke, 1961). The role of the scalar field in the Brans–Dicke theory is confined to its effects on gravitational field equations and the scalar field has the dimension of the inverse of the gravitational constant. Scalar-tensor theories of gravitation of the other type involving a dimensionless scalar field have also been extensively studied by fairly a large number of workers (Bergmann, 1968; Nordvedt, 1970; Wagoner, 1970 etc.).

Saez and Ballester (1985) have proposed a theory in which the metric is coupled with a dimensionless scalar field in a simple manner. This coupling gives a satisfactory description of weak fields. In spite of the dimensionless character of the scalar field, an antigravity regime appears. This theory suggests a possible way to solve the missing matter problem in non-flat FRW cosmologies. Singh

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and Agrawal (1991a) have investigated models of Bianchi types I, III, V, VI_0 and Kantowski–Sachs models in this theory.

Lau and Prokhounik (1986) have proposed a scalar-tensor theory in terms of an action principle. This theory is a generalization of Lau's (1985) theory. Lau (1985) proposed, as a natural consequence of the Dirac's large number hypothesis, the field equations with time-dependent cosmological term λ and the gravitational constant G. Any variation of the Newtonian gravitational constant with time may produce unusual physical effects if the black holes are formed in the very early Universe (Barrow, 1992). The implication of time-varying λ and G is important when the history and evolution of the Universe is considered, particularly in its early stages. Maharaj and Beesham (1988) have pointed out an error in the equations obtained by Lau and Prokhounik (1986) and obtained a vacuum solution to the generalized field equations for the Robertson–Walker space-time. Singh and Agrawal (1991b) have obtained the vacuum Bianchi type I, III, V, VI_0 and Kantowski–Sachs model in this theory.

In this paper we obtain exact solutions of Einstein field equations in the scalar-tensor theories developed by Saez and Ballester (1985) and Lau and Prakhovnik (1986) when the line-element has the form

$$ds^2 = \exp(2h)dt^2 - \exp(2A)\,(dx^2 + dy^2) - \exp(2B)dz^2 \tag{1.1}$$

where h, A and B are functions of the time variable t only. Carminati and MacIntosh (1980) considered the line-element (1.1) and solved completely the Einstein–Maxwell field equations. The form (1.1) is found by transforming the locally rotationally symmetric Bianchi type I metric

$$ds^2 = dt^2 - \exp(2A)\,(dx^2 + dy^2) - \exp(2B)dz^2 \tag{1.2}$$

and by redefining the t coordinate. There is a freedom to choose h equal to any function of A and B. The metric (1.1) is spatially homogeneous, locally rotationally symmetric and admits a Bianchi I group of motions on hypersurface $t =$ constant (Cahen and Defrise, 1968). We also discuss the physical features of the solutions.

2. Models in Saez and Ballester Theory

The field equations in the scalar-tensor theory developed by Saez and Ballester (1985) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \omega\phi^n\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right) = -KT_{\mu\nu}. \tag{2.1}$$

The scalar field ϕ satisfies the equation

$$2\phi^n \phi^{;\mu}_{;\mu} + n\phi^{n-1}\phi_{,\alpha}\phi^{,\alpha} = 0, \tag{2.2}$$

where n is an arbitrary exponent and ω is a dimensionless coupling constant. For a perfect fluid the energy-momentum tensor $T_{\mu\nu}$ is of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}. \tag{2.3}$$

As the consequence of the field equations (2.1) and (2.2), the equations of motion are

$$T^{\mu\nu}_{,\nu} = 0. \tag{2.4}$$

Here a comma and a semicolon denote ordinary and covariant differentiation respectively.

In comoving coordinates, the field equations (2.1), (2.2) and (2.4) for the metric (1.1) lead to

$$\ddot{A} + \dot{A}^2 + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} - \dot{h}(\dot{A} + \dot{B}) = -Kpe^{2h} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.5}$$

$$2\ddot{A} + 3\dot{A}^2 - 2\dot{A}\dot{h} = -Kpe^{2h} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.6}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = K\rho e^{2h} - \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.7}$$

$$\ddot{\phi} + \dot{\phi}(2\dot{A} + \dot{B} - \dot{h}) + \frac{n}{2\phi}\dot{\phi}^2 = 0, \tag{2.8}$$

$$\dot{\rho} + (\rho + p)\,(2\dot{A} + \dot{B}) = 0. \tag{2.9}$$

Here a dot denotes differentiation with respect to t. Thus, we have five equations to determine seven unknowns A, B, h, ρ, p, ϕ and ω. To obtain physically realistic solutions of the Equations (2.5)–(2.9), we need two more relations connecting the field variables A, B, h, ρ, p, ϕ and ω.

2.1. $h = 2A$

In this case field equations (2.5)–(2.9) reduce to

$$\ddot{A} - \dot{A}^2 + \ddot{B} + \dot{B}^2 - \dot{A}\dot{B} = -Kpe^{4A} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.10}$$

$$2\ddot{A} - \dot{A}^2 = -Kpe^{4A} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.11}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = K\rho^{4A} - \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.12}$$

$$\ddot{\phi} + \dot{\phi}\dot{B} + \frac{n}{2\phi}\dot{\phi}^2 = 0, \tag{2.13}$$

$$\dot{\rho} + (\rho + p)\,(2\dot{A} + \dot{B}) = 0. \tag{2.14}$$

Adding (2.10), (2.11) and (2.12), we obtain

$$3\ddot{A} - \dot{A}^2 + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} = K(\rho - 2p)e^{4A} + \frac{1}{2}\omega\phi^n\dot{\phi}^2. \tag{2.15}$$

A linear combination of Equations (2.10)–(2.12) and (2.15) provides

$$2\ddot{A} + 2\dot{A}\dot{B} = K(\rho - p)e^{4A}, \tag{2.16}$$

$$\ddot{A} + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} = K(\rho - p)e^{4A}, \tag{2.17}$$

$$3\ddot{A} + \ddot{B} + \dot{B}^2 + 3\dot{A}\dot{B} = 2K(\rho - p)e^{4A}, \tag{2.18}$$

$$3\ddot{A} - 2\dot{A}^2 + \ddot{B} + \dot{B}^2 - \dot{A}\dot{B} = -2Kpe^{4A} + \omega\phi^n\dot{\phi}^2. \tag{2.19}$$

It is difficult to solve these equations in general. So we consider some cases of physical interest.

2.1.1. *Case I: Vacuum* $(\rho = p = 0)$
Equations (2.16)–(2.18) give the solutions

$$\exp(A) = \delta(\alpha t + \beta)^{\gamma/\alpha},$$

$$\exp(B) = \alpha t + \beta,$$

$$\exp(h) = \{\delta(\alpha t + \beta)^{\gamma/\alpha}\}^2, \tag{2.20}$$

where α, β, γ and δ are integration constants. The Equation (2.13) finally gives

$$\exp\{\phi^{(n/2)+1}\} = m_4(\alpha t + \beta)^{m_3}, \tag{2.21}$$

where

$$m_3 = \left(\tfrac{n}{2} + 1\right)\tfrac{m_1}{\alpha} \text{ and } m_4 = m_2^{(n/2)+1} \text{ and } m_1, m_2 \tag{2.22}$$

being integration constants. The Equation (2.19) gives a relation between the constants viz.

$$\omega m_1^2 + 2\gamma(\gamma + 2\alpha) = 0, \tag{2.23}$$

which gives the value of the coupling constant ω. Without loss of any generality we can take $\delta = 1$.

2.1.2. *Case II: Stiff-matter* $(\rho = p)$
In the presence of stiff-matter, Equations (2.16)–(2.18) and (2.13) have the same solutions as given by (2.20) and (2.21). Equation (2.14) yields

$$\rho = p = \frac{\mu_1}{(\alpha t + \beta)^{(4\gamma/\alpha)+2}}, \tag{2.24}$$

where μ_1 is an integration constant. The Equation (2.19) gives the constraint relation

$$\omega m_1^2 = 2K\mu_1 - 2\gamma(\gamma + 2\alpha). \tag{2.25}$$

The metric of the solutions can be written in the form

$$ds^2 = (\alpha t + \beta)^{4\gamma/\alpha} dt^2 - (\alpha t + \beta)^{2\gamma/\alpha}(dx^2 + dy^2) - (\alpha t + \beta)^2 dz^2. \tag{2.26}$$

The physical and kinematical quantities for the model (2.26) have the following expressions:

Spatial volume $\quad V^3 = (\alpha t + \beta)^{(2\gamma/\alpha)+1}$,

expansion scalar $\quad \theta = (2\gamma + \alpha)/(\alpha t + \beta)^{(2\gamma/\alpha)+1}$,

shear scalar $\quad \sigma = (\gamma - \alpha)/\sqrt{3}(\alpha t + \beta)^{(2\gamma/\alpha)+1}$,

Hubble parameter $\quad H = (2\gamma + \alpha)/3(\alpha t + \beta)$,

and

deceleration parameter $\quad q = 2(\alpha - \gamma)/(\alpha + 2\gamma). \qquad (2.27)$

The solution is singular at $t = -\beta/\alpha$.

2.2. $h = B$

In this case the field Equations (2.5)–(2.9) reduce to

$$\ddot{A} + \dot{A}^2 + \ddot{B} = -Kpe^{2B} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.28}$$

$$2\ddot{A} + 3\dot{A}^2 - 2\dot{A}\dot{B} = -Kpe^{2B} + \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.29}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = K\rho e^{2B} - \frac{1}{2}\omega\phi^n\dot{\phi}^2, \tag{2.30}$$

$$\ddot{\phi} + 2\dot{\phi}\dot{A} + \frac{n}{2\phi}\dot{\phi}^2 = 0, \tag{2.31}$$

$$\dot{\rho} + (\rho + p)\,(2\dot{A} + \dot{B}) = 0. \tag{2.32}$$

Adding Equations (2.28)–(2.30), we obtain

$$3\ddot{A} + 5\dot{A}^2 + \ddot{B} = K(\rho - 2p)e^{2B} + \frac{1}{2}\omega\dot{\phi}^2\phi^n. \tag{2.33}$$

A linear combination of Equations (2.28)–(2.30) and (2.33) provides

$$2\ddot{A} + 4\dot{A}^2 = K(\rho - p)e^{2B}, \tag{2.34}$$

$$\ddot{A} + 2\dot{A}^2 + \ddot{B} + 2\dot{A}\dot{B} = K(\rho - p)e^{2B}, \tag{2.35}$$

$$3\ddot{A} + 6\dot{A}^2 + \ddot{B} + 2\dot{A}\dot{B} = 2K(\rho - p)e^{2B}, \tag{2.36}$$

$$3\ddot{A} + 4\dot{A}^2 + \ddot{B} - 2\dot{A}\dot{B} = -2Kpe^{2B} + \omega\phi^n\dot{\phi}^2. \tag{2.37}$$

It is again difficult to solve these equations in general. So we consider some cases of physical interest.

2.2.1. *Case I: Vacuum* ($\rho = p = 0$)
Equations (2.34)–(2.36) give the solutions

$$\exp(A) = (at + b)^{\frac{1}{2}},$$

$$\exp(h) = \exp(B) = d(at + b)^{c/a}, \tag{2.38}$$

where a, b, c and d are integration constants. Without loss of any generality 'd' can be taken as unity.

The Equation (2.31) finally gives

$$\exp\{\phi^{(n/2)+1}\} = \ell_4(at + b)^{\ell_3}, \tag{2.39}$$

where $\ell_3 = \left(\frac{n}{2} + 1\right)\frac{\ell_1}{a}$ and $\ell_4 = \ell_2^{(n/2)+1}$ and ℓ_1, ℓ_2 being integration constants. The Equation (2.37) yields a relation between the constants,

$$2\omega\ell_1^2 + a(a + 4c) = 0, \tag{2.40}$$

which gives the value of coupling constant.

2.2.2. *Case II: Stiff-matter* ($\rho = p$)
In the presence of stiff-matter, Equations (2.34)–(2.36) and Equation (2.31) have the same solutions as given by (2.38) and (2.39). The Equation (2.32) yields

$$\rho = p = \frac{\mu_2}{(at + b)^{2(c/a)+2}}, \tag{2.41}$$

where μ_2 is an integration constant. The Equation (2.37) gives a constraint relation

$$2\omega\ell_1^2 = 4K\mu_2 - a(a + 4c). \tag{2.42}$$

The metric of the solutions can be written in the form

$$ds^2 = (at+b)^{2c/a}dt^2 - (at+b)(dx^2+dy^2) - (at+b)^{2c/a}dz^2. \qquad (2.43)$$

For the model (2.43), the physical and kinematical quantities have the expressions:

$$V^3 = (at+b)^{(c/a)+1},$$

$$\theta = (a+c)/(at+b)^{(c/a)+1},$$

$$\sigma = (a-2c)/2\sqrt{3}(at+b)^{(c/a)+1},$$

$$H = (a+c)/3(at+b),$$

and

$$q = (2a-c)/(a+c). \qquad (2.44)$$

The solution is singular at $t = -b/a$.

3. Models in Lau and Prokhovnik Theory

The field equations in the scalar-tensor theory proposed by Lau and Prokhovnik (1986) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi GT_{\mu\nu} - \psi_{,\mu}\psi_{,\nu}, \qquad (3.1)$$

where G is the gravitation constant and Λ a new cosmological term related to the gravitational term λ by

$$\Lambda = \lambda - \frac{1}{2}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}. \qquad (3.2)$$

The scalar field ψ satisfies the equation

$$\dot{\psi}g^{\mu\nu}\psi_{;\mu\nu} + \dot{\Lambda} + \frac{1}{2}\dot{g}^{44}\dot{\psi}^2 + g^{44}\dot{\psi}\ddot{\psi} + 8\pi\dot{G}L_m = 0, \qquad (3.3)$$

L_m being the mass Lagrangian density including all non-gravitational fields. In vacuum $L_m = 0$, $T_{\mu\nu} = 0$ and so G can be chosen arbitrary. For spatially homogeneous space-times

$$\psi = \psi(t), \ \Lambda = \Lambda(\psi), \ G = G(\psi). \qquad (3.4)$$

For the metric (1.1) the field equations in vacuum reduce to

$$\ddot{A} + \dot{A}^2 + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} - \dot{h}(\dot{A}+\dot{B}) = \Lambda e^{2h}, \qquad (3.5)$$

$$2\ddot{A} + 3\dot{A}^2 - 2\dot{A}\dot{h} = \Lambda e^{2h}, \tag{3.6}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = \Lambda e^{2h} + \dot{\psi}^2, \tag{3.7}$$

$$2\dot{\psi}\ddot{\psi} + \dot{\psi}^2(2\dot{A} + \dot{B} - 2\dot{h}) + \dot{\Lambda}e^{2h} = 0. \tag{3.8}$$

Thus, we have four equations to determine five unknowns A, B, h, Λ and ψ. To obtain physically realistic solutions of the Equations (3.5)–(3.8), we need two more relations connecting the field variables A, B, h, Λ and ψ.

3.1. $h = 2A$

In this case the field equations (3.5)–(3.8) reduce to

$$\ddot{A} - \dot{A}^2 + \ddot{B} + \dot{B}^2 - \dot{A}\dot{B} = \Lambda e^{4A}, \tag{3.9}$$

$$2\ddot{A} - \dot{A}^2 = \Lambda e^{4A}, \tag{3.10}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = \Lambda e^{4A} + \dot{\psi}^2, \tag{3.11}$$

$$2\dot{\psi}\ddot{\psi} + \dot{\psi}^2(\dot{B} - 2\dot{A}) + \dot{\Lambda}e^{4A} = 0. \tag{3.12}$$

Adding Equations (3.9)–(3.11), we obtain

$$3\ddot{A} - \dot{A}^2 + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} = 3\Lambda e^{4A} + \dot{\psi}^2. \tag{3.13}$$

For tractability of the equations we assume a relation between Λ and ψ, viz., $\Lambda = -\frac{1}{2}\dot{\psi}^2 e^{-2h}$. Note that the condition $\Lambda = -\frac{1}{2}\dot{\psi}^2$ is an arbitrary relation without any physical justification. A linear combination of Equations (3.9)–(3.11) and (3.13) and transformation of Equation (3.12) provide

$$2\ddot{A} + 2\dot{A}\dot{B} = 0, \tag{3.14}$$

$$\ddot{A} + \ddot{B} + \dot{B}^2 + \dot{A}\dot{B} = 0, \tag{3.15}$$

$$3\ddot{A} + \ddot{B} + \dot{B}^2 + 3\dot{A}\dot{B} = 0, \tag{3.16}$$

$$3\ddot{A} - 2\dot{A}^2 + \ddot{B} + \dot{B}^2 - \dot{A}\dot{B} = -\dot{\psi}^2, \tag{3.17}$$

$$\dot{\psi}\ddot{\psi} + \dot{\psi}^2\dot{B} = 0. \tag{3.18}$$

The Equations (3.14)–(3.16) have the same solutions as given by (2.20). The Equation (3.18) yields

$$\exp(\psi) = \lambda_2(\alpha t + \beta)^{\lambda_1/\alpha} \tag{3.19a}$$

where λ_1 and λ_2 are integration constants. Now using Equation (3.17), we obtain a relation between constants, viz.,

$$\lambda_1^2 = 2\gamma(\gamma + 2\alpha). \tag{3.19b}$$

The kinematical quantities for this model have the same expressions as given by (2.27).

3.2. $h = B$

In this case the field equations (3.5)–(3.8) reduce to

$$\ddot{A} + \dot{A}^2 + \ddot{B} = \Lambda e^{2B}, \tag{3.20}$$

$$2\ddot{A} + 3\dot{A}^2 - 2\dot{A}\dot{B} = \Lambda e^{2B}, \tag{3.21}$$

$$\dot{A}^2 + 2\dot{A}\dot{B} = \Lambda e^{2B} + \dot{\psi}^2, \tag{3.22}$$

$$2\dot{\psi}\ddot{\psi} + \dot{\psi}^2(2\dot{A} - \dot{B}) + \dot{\Lambda} e^{2B} = 0. \tag{3.23}$$

Adding Equations (3.20)–(3.22), we obtain

$$3\ddot{A} + 5\dot{A}^2 + \ddot{B} = 3\Lambda e^{2B} + \dot{\psi}^2. \tag{3.24}$$

A linear combination of Equations (3.20)–(3.22) and (3.24) and transformation of Equation (3.23), by putting $\Lambda = -\frac{1}{2}\dot{\psi}^2 e^{-2h}$ provide

$$2\ddot{A} + 4\dot{A}^2 = 0, \tag{3.25}$$

$$\ddot{A} + 2\dot{A}^2 + \ddot{B} + 2\dot{A}\dot{B} = 0, \tag{3.26}$$

$$3\ddot{A} + 6\dot{A}^2 + \ddot{B} + 2\dot{A}\dot{B} = 0, \tag{3.27}$$

$$3\ddot{A} + 4\dot{A}^2 + \ddot{B} - 2\dot{A}\dot{B} = -\dot{\psi}^2, \tag{3.28}$$

$$\dot{\psi}\ddot{\psi} + 2\dot{\psi}^2\dot{A} = 0. \tag{2.29}$$

The Equations (3.25)–(3.27) have the same solutions as given by (2.38). The Equation (3.29) finally gives

$$\exp(\psi) = \xi_2(at + b)^{\xi_1/a}, \tag{3.30}$$

where ξ_1 and ξ_2 being integration constants. The Equation (3.28) gives a constraint relation

$$2\xi_1^2 = a(a + 4c). \tag{3.31}$$

The kinematical quantities for this model have the same expressions as given by (2.44).

4. Discussion

For all models, expansion scalar θ and shear scalar σ tend to zero as $t \to \infty$ if $\gamma, \alpha > 0$ and $c, a > 0$. The ratio σ/θ tends to a constant as $t \to \infty$. These models do not admit rotation and acceleration.

The scalar field ϕ, for the models in Saez and Ballester theory and the scalar field ψ, for the models in Lau and Prokhovnik theory are monotonically increasing towards infinity as $t \to \infty$. The magnitude of Λ tends to zero as $t \to \infty$.

The mass energy density ρ and pressure p for the models in Saez and Ballester theory are infinite at singularities and are monotonically decreasing towards zero as $t \to \infty$.

Though the metric (1.1) is of Bianchi type I the solutions are different than that of Singh and Agrawal (1991a, 1991b). Some of the solutions in the two scalar-tensor theories seen to be very similar. It is due to the fact that some of the field equations of both theories are identical when the metric (1.1) and the condition $\Lambda = -\frac{1}{2}\dot{\psi}^2 e^{-2h}$ are assumed.

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