VARIETIES OF NEW CLASSES OF INTERIOR SOLUTIONS IN GENERAL RELATIVITY

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Abstract. In this paper we present a method of obtaining varieties of new classes of exact solutions representing static balls of perfect fluid in general relativity. A number of previously known classes of solutions has been rediscovered in the process. The method indicates the possibility of constructing a plethora of new physically significant models of relativistic stellar interiors with equations of state fairly applicable to the case of extremely compressed stars. To emphasize our point we have derived two new classes of solutions and discussed their physical importance. From the solutions of these classes we have constructed three causal interiors out of which in two models the outward march of pressure, density, pressure-density ratio and the adiabatic sound speed is monotonically decreasing.

1. Introduction

There have been a few attempts to obtain parametric classes of exact solutions of Einstein's field equations describing the interior field of perfect fluid balls in equilibrium (Tolman, 1939; Wyman, 1949; Kuchowicz, 1968, 1970; Pant and Sah 1982, 1985; Pant and Pant, 1993, 1993a,b). The importance of a parametric class of solutions over an ordinary solution lies in the flexibility of the associated parameter which brings out various models of relativistic star with physically realizable fluid properties. Moreover, given a class of solutions, by imposing realistic conditions on the parameter one may weed our unphysical as well as insignificant solutions. For instance, not all solutions in a class would correspond to causal models and therefore the causality principle shall limit the range of the associated parameter.

Methods have been suggested to obtain classes of solutions but the scope of such attempts is limited to generalize known particular solutions (Leibovitz, 1969; Goldman, 1978, Matese and Whitman, 1980; Whitman, 1983). In this paper we present a method of integrating Einstein's field equations which results into a plethora of parametric classes of physically sound solutions. The paper indicates as to how a host of new possibilities may emerge for the meaningful integration of the field equations.

2. Field Equations and Method of Integration

In canonical coordinates the metric of a static, spherically symmetric field is

$$\mathrm{d}s^2 = -\,\mathrm{e}^{\lambda(r)}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + c^2\,\mathrm{e}^{\nu(r)}\,\mathrm{d}t^2. \tag{1}$$

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It follows that the field equations of general relativity for a ball of perfect fluid with pressure $p(r)$ and density $\rho(r)$ are (Tolman, 1939)

$$\frac{8\pi G}{c^4}p = \mathrm{e}^{-\lambda}\left(\frac{1}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{2}$$

$$\frac{8\pi G}{c^2}\rho = \mathrm{e}^{-\lambda}\left(\frac{1}{r}\frac{\mathrm{d}\lambda}{\mathrm{d}r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{-\lambda}-1}{r^2}\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{-\lambda}}{2r}\frac{\mathrm{d}\nu}{\mathrm{d}r}\right) + \mathrm{e}^{-\lambda-\nu}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{\nu}}{2r}\frac{\mathrm{d}\nu}{\mathrm{d}r}\right) = 0. \tag{4}$$

By allowing one of the two field variables λ and ν as some known function of r the equation (4) transforms into a form which on integration determines the metric (1) completely, the fluid parameters are then calculated from (2) and (3).

By subjecting (4) to the transformation

$$U = r^m\,\mathrm{e}^{m\nu/2}, \qquad V = \mathrm{e}^{-\lambda}, \tag{5}$$

m being a non-zero arbitrary constant, we obtain a linear differential equation in V:

$$\frac{\mathrm{d}V}{\mathrm{d}r} - 2\left\{\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{r^3U^{1-1/m}}{\mathrm{d}U/\mathrm{d}r}\right) - \left(\frac{2mU}{r^2\,\mathrm{d}U/\mathrm{d}r}\right)\right\}V = -\frac{2mU}{r^2\,\mathrm{d}U/\mathrm{d}r}. \tag{6}$$

Integration yields

$$\begin{aligned}\mathrm{e}^{-\lambda} = V = &\left[\frac{r^6U^{2(1-1/m)}}{(\mathrm{d}U/\mathrm{d}r)^2}\right]\times\\ &\times\left[A - 2\int\frac{m}{r^8}\frac{\mathrm{d}U}{\mathrm{d}r}U^{(-1+2/m)}\,\mathrm{e}^{\int[4mU/r^2(\mathrm{d}U/\mathrm{d}r)]\,\mathrm{d}r}\,\mathrm{d}r\right]\times\\ &\times\mathrm{e}^{-\int[4mU/r^2(\mathrm{d}U/\mathrm{d}r)]\,\mathrm{d}r},\end{aligned} \tag{7}$$

where A is another arbitrary constant. Our aim is to explore the possibilities of choosing U such that the right hand side of (7) becomes integrable. A set of possibilities arises if one assumes for $\mathrm{e}^{\int[4mU/r^2(\mathrm{d}U/\mathrm{d}r)]\,\mathrm{d}r}$ some algebraic function of r, U and $\mathrm{d}U/\mathrm{d}r$. In this paper we assume

$$\mathrm{e}^{\int[4mU/r^2(\mathrm{d}U/\mathrm{d}r)]\,\mathrm{d}r} = r^l\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^n, \tag{8}$$

l and n being arbitrary constants. Equation (8) results into a second order homogeneous equation in U:

$$nr^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2} + lr\frac{\mathrm{d}U}{\mathrm{d}r} - 4mU = 0. \tag{9}$$

The solution is

$$U = c_1 r^{2(a+b)+m} + c_2 r^{2(a-b)+m}, \tag{10}$$

where c_1 and c_2 are arbitrary constants and

$$a = -\frac{(2m-1)n+l}{4n}, \qquad b = \frac{\sqrt{(n-l)^2+16mn}}{4n}, \tag{11}$$

provided $n \neq 0$. (12)

Equation (7) takes the form

$$e^{-\lambda} = \frac{r^{8+n-l-2\left(a-b+\frac{m}{2}\right)\left(n+\frac{2}{m}\right)}(A-2I)}{(c_1 r^{4b}+c_2)^{2[-1+(1/m)]}\left[2\left(a+b+\frac{m}{2}\right)c_1 r^{4b}+2\left(a-b+\frac{m}{2}\right)c_2\right]^{n+2}}, \tag{13}$$

where

$$I \equiv m\int r^{l-9-n+2(a-b+(m/2))(n+(2/m))}(c_1 r^{4b}+c_2)^{-1+(2/m)} \times$$
$$\times\left[2\left(a+b+\frac{m}{2}\right)c_1 r^{4b}+2\left(a-b+\frac{m}{2}\right)c_2\right]^{n+1} \mathrm{d}r. \tag{14}$$

Also from (5) and (10)

$$e^{\nu} = \left[c_1 r^{2(a+b)} + c_2 r^{2(a-b)}\right]^{2/m}. \tag{15}$$

We note that the expressions for the field variables given by (13) and (15) contain six arbitrary constants out of which three constants namely, c_1, c_2 and A are to be evaluated by the conditions resulting from the junction of (2.1) with the vacuum Schwarzschild's metric. Further, we shall see that the integration of (14) is possible if two conditions are imposed on the remaining three constants, namely l, m and n. Since this can be affected in various ways we are able to derive varieties of one parameter classes of solutions.

The integrability requirement of (14) gives rise to the following possibilities :

(i) Each of the exponents $(2/m)-1$ and $n+1$ takes nonnegative integral values. In such cases the integrand reduces to a power series with finite terms. We thus obtain a variety of classes of solutions with l as parameter.

TABLE I
Derivation of some known classes of solutions

Serial No.	Assumptions	Resulting classes of solutions
1	$b = 0$	Tolman's Class-usually referred as Tolman's V Solution (Tolman, 1939)
2	$m = 1, a = 1$	Wyman's Class (Wyman, 1949)
3	$m = 1, n = -2$	Kuchowicz's Class (Kuchowicz, 1970)
4	$m = 2, n = -1$	Pant and Sah Class (Pant and Sah, 1982)
5	$m = 2, a = \|b\|$	Pant and Pant Class (Pant and Pant, 1993)
6	$a = \|b\|$	Pant and Pant Class (Pant and Pant, 1993a)
7	$m = 2$, $l = \frac{-n(n-3)+2(n+3)\sqrt{-n}}{n+1}$	Pant and Pant Class (Pant and Pant, 1993b)

(ii) One of the exponents $(2/m) - 1$ and $n + 1$ is allowed to take nonnegative integral values and the exponent $l - n - 9 + 2(a - b + (m/2))(n + (2/m))$ is subjected to the following condition:

$$l - n - 9 + 2(a - b + \frac{m}{2})(n + \frac{2}{m}) = 4b\alpha - 1,$$

α being a positive integer.

In such cases the integrand in the right hand side of (14) is transformed into an exact differential equation and the result is a variety of parametric classes of solutions.

Some of the classes of solutions already known are rediscovered following the procedure discussed above (cf. Table I).

3. Non-Singular Solutions

A solution which does not give rise to singularity in the metric (1) may be called nonsingular. Nonsingular solutions may be applied to construct models of a relativistic star if the pressure and density are monotonic decreasing positive functions in the region $0 \leq r \leq r_b$ where r_b denotes the boundary of the star, and the principle of causality is obeyed. Such solutions may have infinite central density (ρ_0), as well. In view of (15), the requirement that e^{ν_0} be nonsingular implies that

$$a \geq |b|. \tag{16}$$

For $a = |b|$ we obtain a solution with

$$e^{\nu_0} = \text{constant}$$

in each class. A discussion on the family of such solutions has been given by Pant and Pant (1993b).

For $a > |b|$, we obtain solutions with $e^{\nu 0} = 0$, which are referred as those with quasistatic character (Tolman, 1939).

Solutions obtained under the assumption (15), in general, correspond to infinite central density. In particular, for $a = |b| = \frac{1}{2}$, one obtains a class of solutions with finite central density. Examples are Tolman's IV solution and the Wyman–Adler solution (Adler, 1974). A detailed study of this class of solutions is given by Durgapal (1982).

Zel'dovich and Novikov (1971) comment that for solutions with finite mass and $\rho_0 = \infty$, the curvature at the centre is infinite and $e^{\lambda 0} \neq 1$. One can easily verify that this is not absolutely true as there exist finite mass solutions with $\rho_0 = \infty$ and $e^{\lambda 0} = 1$ (e.g. solutions derived by Pant and Pant (1993), (1993a)).

Following the process outlined in the foregoing section one can easily derive a multitude of classes of nonsingular solutions. To demonstrate that such classes may provide physically sound stellar interiors, we shall present two new classes in the following section.

4. Two New Classes of Solutions

Class I. We assume

$$n + 1 = 0, \tag{17}$$

and

$$l - 8 + (2a - 2b + m)\left(\frac{2}{m} - 1\right) = 4b - 1.$$

The resulting class of solutions is

$$e^{\nu} = (c_1 r^{2a+2b} + c_2 r^{2a-2b})^{2/m}, \tag{18}$$

$$e^{-\lambda} = \frac{(c_1 + c_2 r^{-4b})^2}{(c_1' + c_2' r^{-4b})}\left\{\frac{A r^{-8b/m}}{(c_1 + c_2 r^{-4b})^{2/m}} - \frac{m^2}{4bc_1}\right\}, \tag{19}$$

where

$$\begin{aligned} c_1' &\equiv (2a + 2b + m)c_1, \\ c_2' &\equiv (2a - 2b + m)c_2 \end{aligned} \tag{20}$$

with

$$\begin{aligned} a &= \frac{4 + 12m - 3m^2}{8(m+2)}, \\ b &= -\frac{1}{8(m+2)}(16 - 96m + 152m^2 - 24m^3 + m^4)^{1/2}, \end{aligned} \tag{21}$$

provided that

$$m \neq 0, -2; \qquad m \notin (0.34310, 0.34334). \tag{22}$$

For $m = 2$, we rediscover Tolman's IV solution which is the only member of the class giving rise to finite central pressure and finite central density. The condition (16) requires that

$$0 < m \leq 0.3431, \qquad 0.34334 \leq m \leq 2. \tag{23}$$

The expressions for pressure and density are

$$\begin{aligned}\frac{8\pi G}{c^4}p = \frac{1}{r^2}\Bigg[-1 + \frac{(4a+4b+m)c_1 + (4a-4b+m)c_2 r^{-4b}}{m(c_1' + c_2' r^{-4b})(c_1 + c_2 r^{-4b})^{-1+(2/m)}} \times \\ \times \left\{ A r^{-8b/m} - \frac{m^2}{4bc_1}(c_1 + c_2 r^{-4b})^{2/m} \right\}\Bigg],\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{8\pi G}{c^2}\rho = \frac{1}{r^2}\Bigg[1 + \frac{8b}{m}\frac{c_1 A r^{-8b/m}}{(c_1 + c_2 r^{-4b})^{-1+(2/m)}(c_1' + c_2' r^{-4b})} + \\ + \frac{A r^{-8b/m} - \frac{m^2}{4bc_1}(c_1 + c_2 r^{-4b})^{2/m}}{(c_1 + c_2 r^{-4b})^{-1+(2/m)}(c_1' + c_2' r^{-4b})^2} \times \\ \times \Big\{ -(2a + 2b + m)c_1^2 + \\ +(8ab + 24b^2 + 4bm - 4a - 2m)c_1 c_2 r^{-4b} + \\ +(1 + 4b)(2a - 2b + m)c_2^2 r^{-8b} \Big\}\Bigg].\end{aligned} \tag{25}$$

We find that the central pressure and central density are infinite but their ratio is finite and equals the limiting value of $(\mathrm{d}p/\mathrm{d}\rho)_0$:

$$\frac{1}{c^2}\left(\frac{p}{\rho}\right)_0 = \frac{1}{c^2}\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 = -\frac{m^2 + 4m(a+2b) + 8b(a+b)}{m^2 + 4mb + 8b(a+b)} \tag{26}$$

The causality condition at the centre

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 \leq c^2 \tag{27}$$

further restricts the range of the parameter to the following intervals:

$$0.207 \leq m \leq 0.246; \qquad 0.66667 \leq m < 2. \tag{23a}$$

The constants c_1, c_2 and A are evaluated by matching the solution (18), (19) with the Schwarzschild exterior solution for a ball of mass M and linear dimension $2r_b$:

$$c_1 = \frac{1}{4b}\{ u(4a - 4b + m) - 2a + 2b\} r_b^{-2(a+b)}(1 - 2u)^{(m/2)-1},$$

$$c_2 = \frac{1}{4b}\{ - u(4a + 4b + m) + 2a + 2b\} r_b^{-2(a-b)}(1 - 2u)^{(m/2)-1}, \tag{28}$$

$$A = \left[\frac{m^2}{4bc_1 r_b^{2a+2b}} + (1 - 2u)^{1-m}(c_1' r_b^{2a+2b} + c_2' r_b^{2a-2b})\right] \times \\ \times (1 - 2u) r_b^{2a+2b+(4/m)(b-a)},$$

where

$$u \equiv \frac{GM}{c^2 r_b}. \tag{29}$$

Class II. We assume

$$n + 1 = 0, \qquad m = 1. \tag{30}$$

The resulting class of solutions is

$$\mathrm{e}^{\nu} = \left(c_1 r^{[(l-1)/2+2b]} + c_2 r^{[(l-1)/2-2b]}\right)^2, \tag{31}$$

$$\mathrm{e}^{-\lambda} = \frac{A r^{[(13-3l)/2-2b]} - c_1'' - c_2'' r^{-4b}}{c_1' + c_2' r^{-4b}}, \tag{32}$$

where

$$\begin{aligned} c_1' &\equiv \left[\frac{1}{2}(l + 1) + 2b\right] c_1, & c_2' &= \left[\frac{1}{2}(l + 1) - 2b\right] c_2, \\ c_1'' &= \frac{4c_1}{3l - 13 + 4b}, & c_2'' &= \frac{4c_2}{3l - 13 - 4b} \end{aligned} \tag{33}$$

and

$$b = -\frac{1}{4}(l^2 + 2l - 15)^{1/2}. \tag{34}$$

The class provides real non-singular solutions for

$$3 \leq l \leq 4. \tag{35}$$

The expressions for pressure and density are given by

$$\frac{8\pi G}{c^4}p = \frac{1}{r^2}\left\{\frac{[(l+4b)c_1+(l-4b)c_2r^{-4b}][Ar^{(13-3l)/2-2b}-c_1''-c_2''r^{-4b}]}{(c_1+c_2r^{-4b})(c_1'+c_2'r^{-4b})}-1\right], \tag{36}$$

$$\frac{8\pi G}{c^2}\rho = \frac{1}{r^2}\left[1+\frac{c_1''+(1-4b)c_2''r^{-4b}+\frac{1}{2}(3l+4b-15)Ar^{(13-3l-4b)/2}}{c_1'+c_2'r^{-4b}}+\right.$$
$$\left.+\frac{4bc_2'r^{-4b}\left[c_1''+c_2''r^{-4b}-Ar^{(13-3l-4b)/2}\right]}{(c_1'+c_2'r^{-4b})^2}\right]. \tag{37}$$

We observe that the central pressure and central density are infinite but their ratio is finite. We obtain

$$\frac{1}{c^2}\left(\frac{p}{\rho}\right)_{r=0} = \frac{1}{c^2}\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_{r=0} = -\frac{l^2+4b(l-1)-7}{l^2-2l+4b(l-3)-5}$$

provided $l \neq 4$. We observe that the fluid obeys the causality principle at the centre for

$$3 \leq l < 4.$$

Applying the junction conditions over the boundary $r = r_b$ the constants are evaluated as follows:

$$c_1 = \frac{(1-2u)^{-1/2}}{8br_b^{(l-1+4b)/2}}\left[1-l+4b+2u(l-4b)\right],$$

$$c_2 = \frac{(1-2u)^{-1/2}}{8br_b^{(l-1-4b)/2}}\left[-1+l+4b-2u(l+4b)\right],$$

$$A = \left[c_1''+c_2''r_b^{-4b}+(1-2u)(c_1'+c_2'r_b^{-4b})\right]r_b^{(3l-13+4b)/2}.$$

In the following sections we shall discuss physical properties of stellar models based on some particular members of the two classes and the corresponding equations of state.

5. Solution Common to Class I and Class II

In this section we shall present a detailed study of the particular solution common to each of the Classes I and II. The solution is obtained by assuming $m = 1$ in (16), (19), or by letting $l = 19/6$ in (31) and (32):

$$\mathrm{e}^{\nu} = \left(\tilde{c}_1x^{1/2}+\tilde{c}_2x^{5/3}\right)^2,$$

$$e^{-\lambda} = \frac{6}{7}\frac{6\tilde{c}_1 + 12\tilde{c}_2 x^{7/6} + 7\tilde{A}x^{7/3}}{9\tilde{c}_1 + 16\tilde{c}_2 x^{7/6}},$$

$$\frac{8\pi G}{c^4} r_b^2 p = \frac{1}{x^2}\left[\frac{2}{7}\frac{(6\tilde{c}_1 + 13\tilde{c}_2 x^{7/6})(6\tilde{c}_1 + 12\tilde{c}_2 x^{7/6} + 7\tilde{A}x^{7/3})}{(\tilde{c}_1 + \tilde{c}_2 x^{7/6})(9\tilde{c}_1 + 16\tilde{c}_2 x^{7/6})} - 1\right],$$

$$\frac{8\pi G}{c^2} r_b^2 \rho = \frac{1}{x^2}\left[1 - \frac{2}{7}\frac{18\tilde{c}_1 + 78\tilde{c}_2 x^{7/6} + 70\tilde{A}x^{7/3}}{9\tilde{c}_1 + 16\tilde{c}_2 x^{7/6}} + \right.$$
$$\left. + \frac{16\tilde{c}_2 x^{7/6}(6\tilde{c}_1 + 12\tilde{c}_2 x^{7/6} + 7\tilde{A}x^{7/3})}{(9\tilde{c}_1 + 16\tilde{c}_2 x^{7/6})^2}\right],$$

where

$$\tilde{c}_1 \equiv \frac{2}{7}(5 - 13u)(1 - 2u)^{-1/2},$$

$$\tilde{c}_2 \equiv \frac{3}{7}(4u - 1)(1 - 2u)^{-1/2},$$

$$\tilde{A} \equiv \frac{1}{49}(25 - 135u + 98u^2)(1 - 2u)^{-1/2}$$

and

$$x \equiv \frac{r}{r_b}.$$

In Table II the march of pressure, density, pressure–density ratio and the square of the adiabatic sound speed $\mathrm{d}p/\mathrm{d}\rho$ is given for $u = 0.20$.

We observe that these fluid parameters decrease monotonically with the increase in the radial coordinate throughout within the causal fluid ball. It is to be noted that the monotonic decrease in p and ρ is essential for a realistic fluid ball. A similar behaviour of p/ρ and $\mathrm{d}p/\mathrm{d}\rho$ is the characteristic feature of a polytrope.

We note that the behaviour of fluid parameters within the ball depends upon u the mass-radius ratio. One may construct models where p/ρ and $\mathrm{d}p/\mathrm{d}\rho$ be non-monotonic or even monotonically increasing.

6. A Particular Member of Class I for $m = \frac{6}{5}$

The solution is

$$e^{\nu} = (\hat{c}_1 x^{2/5} + \hat{c}_2 x^{9/5})^{5/3},$$

$$e^{-\lambda} = \frac{(\hat{c}_1 + \hat{c}_2 x^{7/5})^2}{(\frac{8}{5}\hat{c}_1 + 3\hat{c}_2 x^{7/5})}\left[\frac{\hat{A}x^{7/3}}{(\hat{c}_1 + \hat{c}_2 x^{7/5})^{5/3}} + \frac{36}{35\hat{c}_1}\right],$$

TABLE II

March of pressure, density, pressure-density ratio and square of the adiabatic sound speed within the fluid ball ($0 \le x \le 1$) corresponding to the common solution to Class I and Class II ($m = 1$ or $l = 19/6$) with $u = 0.20$

x	$\frac{8\pi G}{c^4} p r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$\frac{1}{c^2}\frac{p}{\rho}$	$\frac{1}{c^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}$
0	∞	∞	1/3	1/3
0.1	12.958621	43.037187	0.3011028	0.3234033
0.2	2.844013	10.775176	0.2639412	0.2978245
0.3	1.0821266	4.7724689	0.2267435	0.282644
0.4	0.5068652	2.6578049	0.1907082	0.2593828
0.5	0.2609897	1.6701588	0.1562664	0.2522642
0.6	0.1391623	1.1269624	0.1234844	0.210349
0.7	0.0731653	0.7863437	0.0930449	0.1839726
0.8	0.0354187	0.4953364	0.0715043	0.1774149
0.9	0.0131874	0.4163359	0.0316749	0.1448071
1.0	0	0.3	0	0

$$\frac{8\pi G}{c^4} r_b^2 p = \frac{1}{x^2}\left[\frac{(\frac{5}{3}\hat{c}_1 + 4\hat{c}_2 x^{7/5})[5\hat{A}x^{7/3} + \frac{36}{7\hat{c}_1}(\hat{c}_1 + \hat{c}_2 x^{7/5})^{5/3}]}{(\hat{c}_1 + \hat{c}_2 x^{7/5})^{2/3}(8\hat{c}_1 + 15\hat{c}_2 x^{7/5})} - 1\right]$$

$$\begin{aligned}\frac{8\pi G}{c^2} r_b^2 \rho = \frac{1}{x^2}\Bigg[& 1 + (\hat{c}_1 + \hat{c}_2 x^{7/5})^{-2/3}(8\hat{c}_1 + 15\hat{c}_2 x^{7/5})^{-2} \times \\ & \times\Big\{ - \hat{A}x^{7/3}(\frac{400}{3}\hat{c}_1^2 + 297\hat{c}_1\hat{c}_2 x^{7/5} + 180\hat{c}_2^2 x^{14/5}) - \\ & - \frac{36}{35\hat{c}_1}(40\hat{c}_1^2 + 122\hat{c}_1\hat{c}_2 x^{7/5} + 180\hat{c}_2^2 x^{14/5})(\hat{c}_1 + \hat{c}_2 x^{7/5})^{5/3}\Big\}\Bigg],\end{aligned}$$

where

$$\hat{c}_1 \equiv \frac{3}{7}(3 - 8u)(1 - 2u)^{-2/5},$$

$$\hat{c}_2 \equiv \frac{2}{7}(5u - 1)(1 - 2u)^{-2/5},$$

$$\hat{A} \equiv \frac{6}{5}\frac{1 - 7u + 8u^2}{3 - 8u}(1 - 2u)^{2/5}.$$

Table III shows the march of p, ρ, $p/(c^2\rho)$ and $(1/c^2)(\mathrm{d}p/\mathrm{d}\rho)$, within the fluid sphere whose mass to radius ratio is given by $u \equiv GM/(c^2 r_b) = 0.18$.

TABLE III

March of pressure, density, pressure-density ratio and square of the adiabatic sound speed within the fluid ball ($0 \le x \le 1$) corresponding to the common solution to Class I and Class II ($m = 6/5$) with $u = 0.18$

x	$\frac{8\pi G}{c^4} p r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$\frac{1}{c^2}\frac{p}{\rho}$	$\frac{1}{c^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}$
0	∞	∞	0.2	0.2
0.1	6.8644356	35.747112	0.1920277	0.1950493
0.2	1.6017015	8.9499976	0.1789611	0.1902851
0.3	0.6490889	3.9847439	0.1628935	0.1863598
0.4	1.3245283	2.245673	0.1445127	0.1841447
0.5	0.17885	1.4441397	0.1238453	0.1846894
0.6	0.1023952	1.0019634	0.1021945	0.1910861
0.7	0.0580192	0.7374889	0.0786712	0.2038191
0.8	0.0303991	0.5655912	0.0537474	0.2346442
0.9	0.0123099	0.4475398	0.0275057	0.2885637
1.0	0	0.3629265	0	0.5517486

7. A Particular Member of Class II for $l = 10/3$

For $l = 10/3$, the class of solutions (31) and (32) gives rise to the following solution:

$$\mathrm{e}^{\nu} = (\bar{c}x^{1/3} + \bar{c}_2 x^2)^2,$$

$$\mathrm{e}^{-\lambda} = \frac{3}{7}\frac{6\bar{c}_1 + 21\bar{c}_2 x^{5/3} + 7\bar{A}x^{7/3}}{4\bar{c}_1 + 9\bar{c}_2 x^{5/3}},$$

$$\frac{8\pi G}{c^4} r_b^2 p = \frac{1}{x^2}\left[-1 + \frac{5}{7}\frac{(\bar{c}_1 + 3\bar{c}_2 x^{5/3})(6\bar{c}_1 + 21\bar{c}_2 x^{5/3} + 7\bar{A}x^{7/3})}{(\bar{c}_1 + \bar{c}_2 x^{5/3})(4\bar{c}_1 + 9\bar{c}_2 x^{5/3})}\right],$$

$$\frac{8\pi G}{c^2} r_b^2 \rho = \frac{1}{7x^2}\frac{20\bar{c}_1(2\bar{c}_1 - 3\bar{c}_2 x^{5/3}) - 35\bar{A}x^{7/3}(8\bar{c}_1 + 9\bar{c}_2 x^{5/3})}{(4\bar{c}_1 + 9\bar{c}_2 x^{5/3})^2},$$

where

$$\bar{c}_1 \equiv \frac{3}{5}(2 - 5u)(1 - 2u)^{-1/2},$$

$$\bar{c}_2 \equiv \frac{1}{5}(5u - 1)(1 - 2u)^{-1/2},$$

TABLE IV

March of pressure, density, pressure-density ratio and square of the adiabatic sound speed within the fluid ball ($0 \leq x \leq 1$) corresponding to the member of Class II ($l = 10/3$) with $u = 0.18$

x	$\frac{8\pi G}{c^4} p r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$\frac{1}{c^2}\frac{p}{\rho}$	$\frac{1}{c^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}$
0	∞	∞	0.2	0.2
0.1	6.932199	35.82089	0.1935239	0.1969473
0.2	1.6261602	8.9978856	0.1807269	0.1926609
0.3	0.6597907	4.017648	0.1642231	0.1854423
0.4	0.3292216	2.2679643	0.1451617	0.1764199
0.5	0.1806186	1.5102877	0.1195921	0.1601132
0.6	0.1027301	1.0091015	0.1018035	0.1540776
0.7	0.0579647	0.7387083	0.0784676	0.1415633
0.8	0.0299569	0.5615387	0.0533478	0.1277493
0.9	0.0119994	0.4386787	0.0273535	0.1137353
1.0	0	0.3495936	0	0.099499

$$\overline{A} \equiv \frac{1}{7}(14u^2 - 24u + 4)(1 - 2u)^{-1/2}.$$

For $u = 0.18$, we have calculated p, ρ, $(1/c^2)(p/\rho)$ and $(1/c^2)(\mathrm{d}p/\mathrm{d}\rho)$ for a monotonic sequence of the dimensionless variable x within the fluid sphere ($0 \leq x \leq 1$). We observe that these fluid parameters fall monotonically from their maximum central values (Table IV).

8. Conclusion

A method to integrate Einstein's field equations for equilibrium stellar interiors has been devised by which to obtain a variety of classes of physically meaningful solutions. We have seen as to how by the transformation (5) along with the assumption (8) one is able to derive new classes and that some of a few known classes are rediscovered in the process. To make our point home we have presented two new classes of solutions. We have shown that members of these classes provide stellar models with physically significant fluid properties.

The assumption (8) can further be generalized for an exhaustive exploration of the existence of classes of physically meaningful solutions. In this way one hopes to rediscover Kuchowicz's class (Kuchowicz, 1968) which is a generalization of uniform density interior solution due to Schwarzschild, deriving new classes not covered by the method discussed in the present paper.

References

Adler, R.J.: 1974, *J. Math. Phys.* **15**, 727.
Durgapal, M.C.: 1982, *J. Phys. Math.* **15**, 2637.
Goldman, S.P.: 1978, *Astrophys. J.* **226**, 1079.
Kuchowicz, B.: 1968, *Acta Phys. Polon.* **33**, 541.
Kuchowicz, B.: 1970, *Acta Phys. Polon.* **B1**, 437.
Leibovitz, C.: 1969, *Phys. Rev.* **185**, 1664.
Matese, J.J. and Whitman, P.G.: 1980, *Phys. Rev. D* **22**, 1270.
Pant, D.N. and Sah, A.: 1982, *Phys. Rev. D* **26**, 1254.
Pant, D.N. and Sah, A.: 1985, *Phys. Rev. D* **32**, 1358.
Pant, D.N. and Pant, N.: 1993, *J. Math. Phys.* **34**, 2440.
Pant, D.N. and Pant, N.: 1993a, *Class. Quant. Grav.*, (submitted).
Pant, D.N. and Pant, N.: 1993b, *Ind. J. Pure Appl. Math.*, (submitted)
Tolman, R.C.: 1939, *Phys. Rev.* **55**, 367.
Whitman, P.G.: 1983, *Phys. Rev. D* **27**, 1722.
Wyman, M.: 1949, *Phys. Rev.* **75**, 1930.
Zel'dovich, Ya.B. and Novikov, I.D. : 1971, *Relativistic Astrophysics*, Vol. 1, University of Chicago Press, Chicago, 88.