EXPLODING RADIATING VISCOUS SPHERES IN GENERAL RELATIVITY

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Abstract. The influence of viscosity on the gravitational collapse in radiating fluid spheres is investigated. The interior solution is matched with the Vaidya metric at the boundary of the fluid distribution. Prescribing an equation of state to take into account the degree of induced anisotropy by the viscosity and using the Herrera, Jiménez and Ruggeri method, we obtain an explicit Tolman VI-like exploding model. The sphere explodes more violently when the anisotropy due to the viscosity is smaller. The shear viscosity diminishes with the expansion of the distribution of matter.

1. Introduction

Dissipative processes are frequently excluded in general relativistic calculations (Carter, 1988; Israel and Stewart, 1980). However, the viscosity may be important (Arnett, 1977; Brown, 1982; Bhete, 1982) in the neutrino trapping during gravitational collapse, which is expected to occur when the central density is of the order of 10^{11}–10^{12} g cm^{-3}. Although the mean free path of the neutrinos is much greater than other particles, the radiative Reynolds number of the trapped neutrinos is nevertheless small at high density (Mihalas and Mihalas, 1984), rendering the core fluid viscous (Kazanas, 1978; Kazanas and Schramm, 1979). Recently a method was reported by Herrera *et al.* (1989) for radiating viscous spheres. This procedure may be considered as a first-order iterative method, in the sense that the radial dependence of the one relevant physical variable is extracted from the classical approximation and introduced into fully relativistic equations. More recently, was reported a proposition (Barreto and Rojas, 1992) for studying dissipative fluids (viscosity + heat flow) in a general relativistic approach which consists of a prescription of an equation of state to obtain the tangential stress induced by the shear viscosity.

In this paper we shall use the Herrera *et al.* (1980) method for studying radiating viscous spheres in the streaming out approximation (Herrera *et al.*, 1989). But now prescribing an equation of state (through the sphere), relating the tangential induced pressure to the other dynamical variables (Cosenza *et al.*, 1982).

The paper is organized as follows. In Section 2 are sketched the conventions, the Einstein field equations, an equation of state for the tangential pressure induced by the viscosity and the junction conditions at the boundary of the sphere. Section 3 contains the description of the model worked out. The discussion of the results is presented in Section 4.

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2. The Field Equations, an Equation of State and the Junction Conditions

Let us consider a spherically symmetric nonstatic distribution of matter which consists of viscous fluid and unpolarized radiation traveling in the radial direction. in radiation coordinates (Bondi, 1964) the metric takes the form

$$ds^2 = e^{2\beta}[(V/r)\,du^2 + 2\,du\,dr] - r^2[d\vartheta^2 + \sin^2\vartheta\,d\varphi^2]\,, \tag{1}$$

where β and V are functions of u and r. Here $u \equiv x^0$ is a time-like coordinate (in flat space-time u is just the retarded time, so that surfaces of u = constant represent null cones open to the future), $r \equiv x^1$ is a null coordinate ($g_{rr} = 0$) such that surfaces r = constant, u = constant are spheres, and $\vartheta, \varphi \equiv x^2, x^3$ are the usual angle coordinates. For the matter distribution considered here, the energy-momentum tensor has the form

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - Pg_{\mu\nu} + \varepsilon k_\mu k_\nu + \tau_{\mu\nu}\,, \tag{2}$$

where U_μ and k_μ denote, respectively, the four-velocity of the fluid and a null-vector pointing in the direction of the outgoing radiation. The tensor $\tau_{\mu\nu}$ is given by

$$\tau_{\mu\nu} = \eta(U_{\mu;\nu} + U_{\nu;\mu} - \dot{U}_\mu U_\nu - \dot{U}_\nu U_\mu) + \left(\zeta - \frac{2}{3}\eta\right)\theta\,P_{\mu\nu}\,, \tag{3}$$

where η and ζ are the shear and the bulk viscosity, respectively, and as usual

$$\theta = U^\mu_{;\mu} \tag{4}$$

is the expansion,

$$P_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu$$

is the projection tensor, and

$$\dot{U}_\nu = U^\mu U_{\nu;\mu}$$

is the four-acceleration. We can write the expression for the energy momentum tensor in the more convenient form

$$T_{\mu\nu} = (\rho + P - \zeta\theta)U_\mu U_\nu - (P - \zeta\theta)g_{\mu\nu} + \varepsilon k_\mu k_\nu + 2\eta\sigma_{\mu\nu}\,, \tag{5}$$

where the shear tensor $\sigma_{\mu\nu}$, which is given by

$$\sigma_{\mu\nu} = U_{(\mu;\nu)} - U_{(\mu}\dot{U}_{\nu)} - \frac{1}{3}\theta P_{\mu\nu} \tag{6}$$

satisfies the conditions

$$\sigma_{\mu\nu}U^\nu = \sigma_{\mu\nu}g^{\mu\nu} = 0. \tag{7}$$

Also, we can introduce the scalar σ by means of

$$\sigma^2 = \frac{1}{2}\sigma^{\mu\nu}\sigma_{\mu\nu}. \tag{8}$$

Now in order to provide a physical meaning to the components of the energy-momentum tensor, as given by (2), let us (following Bondi (1964)) introduce purely locally Minkowskian coordinates (t, x, y, z) by

$$\begin{aligned} \mathrm{d}t &= e^{2\beta}[(V/r)^{1/2}\,\mathrm{d}u + (r/V)^{1/2}\,\mathrm{d}r] \\ \mathrm{d}x &= e^{\beta}(r/V)^{1/2}\,\mathrm{d}r;\ \ \mathrm{d}y = r\,\mathrm{d}\vartheta;\ \ \mathrm{d}z = r\sin\vartheta\,\mathrm{d}\varphi. \end{aligned} \tag{9}$$

Denoting Minkowskian components of the energy-momentum tensor by a caret we have

$$\begin{aligned} T_{00} &= \hat{T}_{00}e^{2\beta}\,V/r, \\ T_{01} &= (\hat{T}_{00} + \hat{T}_{01})\,e^{2\beta}, \\ T_{11} &= (\hat{T}_{00} + \hat{T}_{11} + 2\hat{T}_{01})\,e^{2\beta}(r/V), \\ T_2^2 &= T_3^3 = \hat{T}_2^2 + \hat{T}_3^3. \end{aligned} \tag{10}$$

Next, one assumes that for an observer moving with velocity ω relative to these coordinates in the radial direction the space contains:

(a) a viscous fluid of density $\hat{\rho}$ and pressure $\hat{P}$
(b) isotropic radiation of energy density $3\hat{\lambda}$
(c) unpolarized radiation of energy density $\hat{\varepsilon}$ traveling in the radial direction.

For this specific observer the covariant energy tensor is

$$\begin{pmatrix} \hat{\rho} + 3\hat{\lambda} + \hat{\varepsilon} & -\hat{\varepsilon} & 0 & 0 \\ -\hat{\varepsilon} & \begin{array}{c}\hat{P} + \hat{\lambda} - \zeta\theta - \\ -4\eta\sigma/\sqrt{3} + \hat{\varepsilon}\end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c}\hat{P} + \hat{\lambda} - \zeta\theta + \\ +2\eta\sigma/\sqrt{3}\end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c}\hat{P} + \hat{\lambda} - \zeta\theta + \\ +2\eta\sigma/\sqrt{3}\end{array} \end{pmatrix},$$

then the Lorentz transformation readily shows that

$$T_{00} = e^{4\beta}(1 - 2\frac{\tilde{m}}{r})\left(\frac{\rho + P\omega^2}{1-\omega^2} + \varepsilon\right), \tag{11}$$

$$T_{01} = e^{2\beta}\frac{\rho - P\omega}{1-\omega}, \tag{12}$$

$$T_{11} = \frac{(\rho + P)(1-\omega)}{(1+\omega)(1-2\tilde{m}/r)}, \tag{13}$$

$$T_2^2 = T_3^3 = -P_t; \tag{14}$$

where

$$\rho = \hat{\rho} + 3\hat{\lambda}; \qquad P = \hat{P} + \hat{\lambda} - \zeta\theta - 4\eta\sigma/\sqrt{3};$$

$$P_t = P + 2\sqrt{3}\eta\sigma; \qquad \varepsilon = \hat{\varepsilon}\left(\frac{1+\omega}{1-\omega}\right);$$

$$V = e^{2\beta}(r - 2\tilde{m}(u,r))$$

can be shown by Eq. (4) that

$$\begin{aligned}\theta = {} & \frac{2\omega(1-2\tilde{m}/r)^{1/2}}{(1-\omega^2)^{1/2}}(\beta_1 + 1/r) + \\ & + e^{-2\beta}\left[\left(\frac{1-\omega}{1+\omega}\right)^{1/2}(1-2\tilde{m}/r)^{-3/2}\tilde{m}_0/r - \frac{\omega_0(1-2\tilde{m}/r)^{-1/2}}{(1+\omega)(1-\omega^2)^{1/2}}\right] + \\ & + \frac{\omega}{(1-\omega^2)^{1/2}}(1-2\tilde{m}/r)^{-1/2}\left(\frac{\tilde{m}}{r^2} - \frac{\tilde{m}_1}{r}\right) + \\ & + (1-2\tilde{m}/r)^{1/2}\frac{\omega_1}{(1-\omega^2)^{3/2}};\end{aligned} \tag{15}$$

and, by Equation (8)

$$\sigma = \frac{3\omega}{r^2}\left(\frac{r - 2\tilde{m}(u,r)}{1-\omega^2}\right)^{1/2} + \frac{\theta}{\sqrt{3}}. \tag{16}$$

Thus, it can be shown (Herrera *et al.*, 1989) that the Einstein fields equations may be written in radiative (null) coordinates as

$$\frac{\rho + P\omega^2}{1-\omega^2} + \varepsilon = \frac{1}{4\pi r(r-2\tilde{m})}\left(-\tilde{m}_0 e^{-2\beta} + \frac{(r-2\tilde{m})}{r}\tilde{m}_1\right), \tag{17}$$

$$\frac{\rho - \omega P}{1+\omega} = \frac{\tilde{m}_1}{4\pi r^2}, \tag{18}$$

$$\frac{1-\omega}{1+\omega}(\rho + P) = \frac{\beta_1(r-2\tilde{m})}{2\pi r^2}, \tag{19}$$

$$\begin{aligned}P_t = {} & -\frac{\beta_{01}e^{-2\beta}}{4\pi} + \frac{1}{8\pi}(1-2\tilde{m}/r)(2\beta_{11} + 4\beta_1^2 - \beta_1/r) + \\ & + \frac{3\beta_1(1-2\tilde{m}_1) - \tilde{m}_{11}}{8\pi r}.\end{aligned} \tag{20}$$

where differentiation with respect to u and r is denoted by subscripts 0 and 1, respectively. Observe that from a purely formal point of view, the system (17)–(20) is the same as for a radiating anisotropic fluid (without viscosity), with radial pressure P and tangential pressure P_t.

Now, the method consists in assuming that the r dependence auxiliary functions

$$\tilde{\rho} = \frac{\rho - P\omega}{1+\omega}, \tag{21}$$

$$\tilde{P} = \frac{P - \rho\omega}{1+\omega}, \tag{22}$$

are the same as that of the energy density and pressure corresponding to the static "seed" model (Herrera *et al.*, 1980). With this r dependence of $\tilde{P}$ and $\tilde{\rho}$, we can integrate the field Equations (18) and (19) to obtain the geometrical variables $\tilde{m}$ and β, up to some functions of u, which will be specified for the model worked out.

It remains for us to give a prescription to obtain the tangential stress induced by the viscosity. For this purpose we shall use an Equation of State (throughout the sphere) relating the tangential induced pressure to the other dynamical variables (Cosenza *et al.*, 1982; Barreto and Rojas, 1992), namely

$$P_t - P = C\frac{(\tilde{\rho} + \tilde{P})}{(r - 2\tilde{m})}(4\pi r^3 \tilde{P} + \tilde{m}), \tag{23}$$

where C is a constant throughout the sphere. Equation (23) specify the degree of anisotropy induced by the viscosity, $P_t - P = 2\sqrt{3}\eta\sigma$. Therefore, this prescription permits us to avoid a first-order iterative methods (Herrera *et al.*, 1989).

Before finishing this section, it is worth noticing that, in order to make the method outlined above completely consistent, it is necessary to match the interior solution with the Vaidya metric at the boundary of the fluid distribution (Darmois or Lichnerowicz conditions). It is easy to check that these conditions are equivalent to the continuity of the functions β_a and $\tilde{m}$ across the boundary of the sphere (Herrera and Jiménez, 1983), and to the equation

$$-\beta_{0a} + (1 - 2\tilde{m}_a/a)\beta_{1a} - \tilde{m}_{1a}/2a = 0, \tag{24}$$

where the subscript a indicates that the quantity is evaluated at the surface $r = a(u)$. Using the continuity of β and $\beta = 0$ for the Vaidya metric, we may expand it near the boundary

$$\beta_{0a} + \dot{a}\beta_{1a} = 0, \tag{25}$$

where now the differentiation with respect to u also is denoted by a dot. Substituting this last expression back into (24) and using the field Equations (18) and (19), we obtain

$$\dot{a} = (1 - 2\tilde{m}_a/a)\left\{\frac{(\omega_a\rho_a - P_a)/(1-\omega_a)}{\rho_a + P_a}\right\}. \tag{26}$$

In radiation coordinates, the velocity of matter is given by

$$\frac{\mathrm{d}r}{\mathrm{d}u} = e^{2\beta}(1 - 2\tilde{m}/r)\frac{\omega}{1-\omega}. \tag{27}$$

Therefore, it follows that

$$\dot{a} = (1 - 2\tilde{m}_a/a)\frac{\omega_a}{1-\omega_a}. \tag{28}$$

Comparing (26) and (28) we get

$$\overline{P}_a = \frac{4}{\sqrt{3}}\eta_a\sigma_a, \tag{29}$$

where

$$\overline{P}_a = \hat{P}_a + \hat{\lambda}_a - \zeta_a\theta_a,$$

Thus the radial pressure at the boundary of the viscous sphere does not vanish. This situation, discontinuity of the radial pressure at the boundary, also occurs in the case of thermal conduction (Herrera *et al.*, 1987; Santos, 1985), where the radial pressure at the boundary is proportional to the heat flux evaluated at the surface.

3. An Exploding Model

We shall illustrate the method with a simple but illustrative model inspired in the Tolman VI solution (Tolman, 1939) The equation of state of the static Tolman VI solution approaches the one of a highly relativistic Fermi gas. Thus, following the method, let us choose (Herrera *et al.*, 1980 ; Cosenza *et al.* 1982)

$$\tilde{\rho} = \frac{3g(u)}{r^2} \tag{30}$$

where g is an arbitrary function, and

$$\tilde{P} = \frac{\tilde{\rho}}{3h}\left[\frac{I - K\alpha(r/a)^{\sqrt{(4-3h)}}}{I - \alpha(r/a)^{\sqrt{(4-3h)}}}\right] \tag{31}$$

with

$$h = 1 - 2C,$$

$$I = 8 - 3h - 4\sqrt{4-3h},$$

$$K = 8 - 3h + 4\sqrt{4-3h},$$

$$\alpha = \frac{I + 3h\omega_a}{K + 3h\omega_a}.$$

It is worth noticing that with the choice of the above auxiliary variables, the Tolman-Oppenheimer-Volkov-like equation for these variables is not satisfied, except in the static case (Barreto *et al.*, 1991).

Next, the field Equations (18) and (19) are integrated to give

$$\tilde{m} = m(r/a) \tag{32}$$

and

$$\beta = \frac{2m}{3ah(1-2m/a)} \ln\left\{ \left(\frac{1-\alpha(r/a)^{\sqrt{(4-3h)}}}{1-\alpha} \right)^2 (r/a)^{2-\sqrt{(4-3h)}} \right\}. \tag{33}$$

where $m \equiv \tilde{m}(u, a)$. For the functions of u appearing in (32) and (33) (m, a, ω_a) we have the following three equations (surface equations) which are obtained from (17), (27) and the conservation law $T^{1\mu}_{;\mu} = 0$, evaluated at the surface (modulo the junction equations),

$$\dot{A} = F(\Omega - 1) \tag{34}$$

$$\dot{F} = \frac{2FE + \dot{A}(1-F)}{A}, \tag{35}$$

and

$$\dot{\Omega} = \Omega \left\{ \frac{2\Omega \dot{A}}{A} + \frac{h[3F\Omega - 4(1-F)]}{8A\Omega} - \frac{\dot{F}}{F(1-F)} \right\}, \tag{36}$$

where

$$m(u=0) = m(0); \qquad A = a/m(0);$$

$$M = m/m(0); \qquad u/m(0) \to u;$$

$$F = 1 - 2M/A; \qquad \Omega = 1/(1-\omega_a);$$

$$E = 4\pi a^2 \varepsilon_a.$$

The system (34)–(36) may be integrated, provided one function of u is given. As usual we assume that the total luminosity is a specific function of u, such that a given portion of the initial mass is carried away. In this paper we assume that

$$FE = \frac{m_{\text{rad}}}{\gamma\sqrt{2\pi}} \exp\left[-\frac{(u-u_0)^2}{2\gamma^2} \right], \tag{37}$$

where the constants m_{rad} and γ have been chosen such that 1/100 of the initial mass is carried away by the pulse. Then the result of the integration of (34)–(36) (with

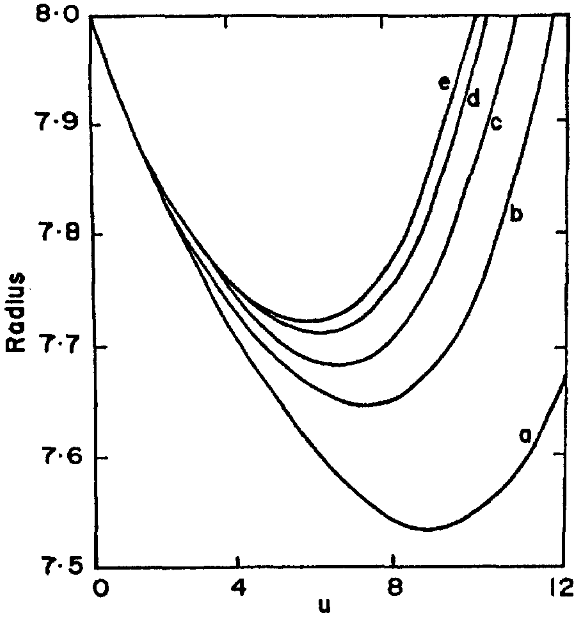

Fig. 1. Evolution of the radius for different values of h. The curves labelled *a*–*e* are for $h = 0.7334$, 0.8667, 0.9334, 0.9866 and 1.0134, respectively.

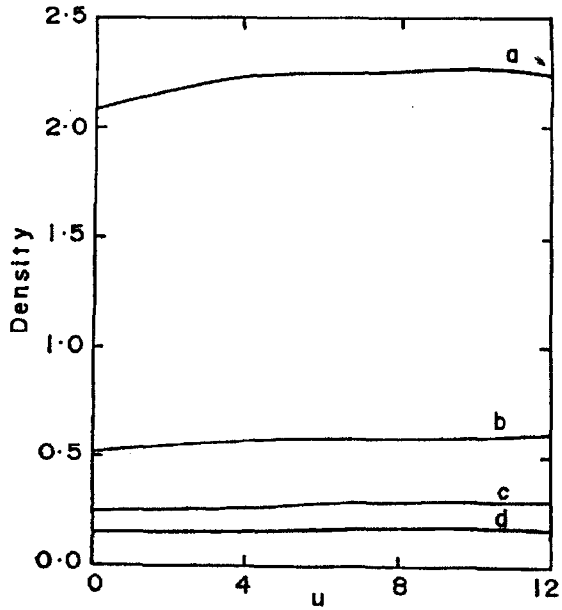

Fig. 2. Evolution of the dimensionless density $\overline{\rho} = \rho m(0)^2$ for $h = 0.8667$, monitored at different regions. The curves labelled *a*–*d* are for $r/a(0) = 0.25$, 0.5, 0.75 and 1, respectively. The density function is shown multiplied by 10^3.

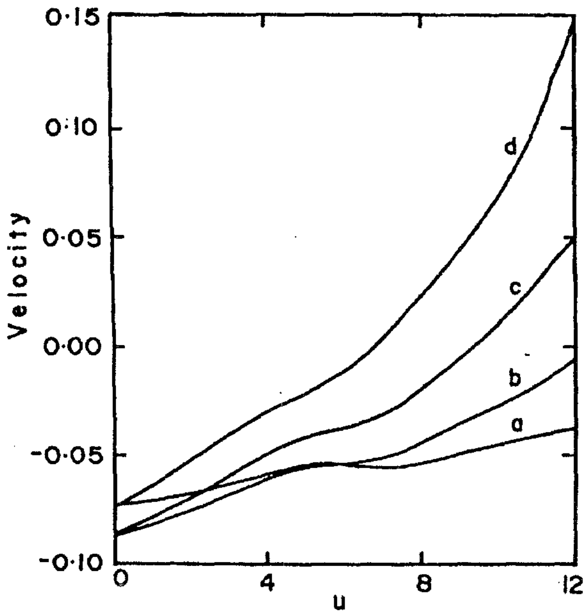

Fig. 3. Evolution of matter velocity (dr/du) for h = 0.8667, monitored at different regions: curves a–d are for $r/a(0)$= 0.25, 0.5, 0.75 and 1, respectively.

(37)) is fed back into (32) and (33), fully determining functions $\tilde{m}$ and β. The field Equations (17)–(20) and (23) remain to be used algebraically for determining the physical variables (ρ, P, ω, ε and η) for any piece of material. As the degree of anisotropy induced by the viscosity was specified (Equation (23)) some additional numerical work was necessary to calculate the shear viscosity. It is clear from Equations (15) and (16) that σ depend explicitly of ω_0 and ω_1.

The following set of initial data has been considered

$$A(0) = 8; \quad F(0) = 0.75; \quad \Omega(0) = 0.9.$$

This initial data and the running time of the numerical integration was suggested by the behavior of the matter variables themselves. A discussion of the results is deferred to the next section.

4. Conclusions

Figure 1 displays the evolution of the radius for different values of h. It is worth noticing that the sphere explodes more violently when the degree of anisotropy induced by the viscosity is the smaller. Nevertheless, when $P_t < P$ (curve e) the profiles of the shear viscosity are negative. Likewise, when $P_t > P$ but the difference is too large (curve a), the resultant shear viscosity is not physically acceptable ($\eta < 0$). Figs. (2) – (4) show the evolution of the density, matter velocity

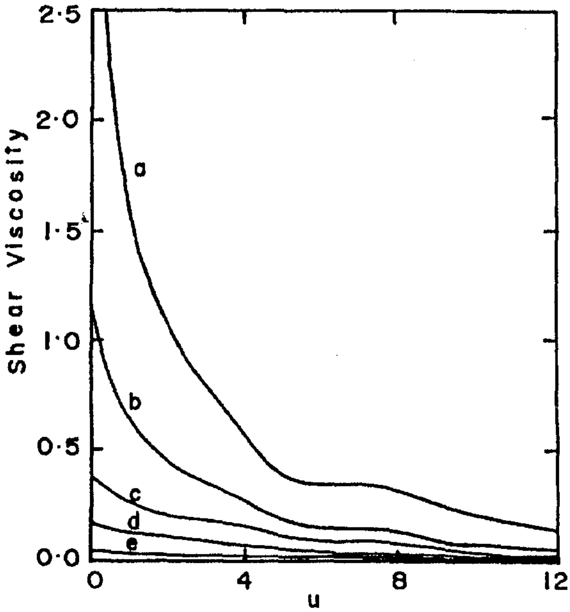

Fig. 4. Evolution of the shear viscosity for h monitored at different regions. The curves a–e are for $r/a(0)$= 0.2, 0.3, 0.4, 0.5 and 0.9, respectively. The shear viscosity function is shown multiplied by 10^3.

and shear viscosity, respectively. The shear viscosity is high in the regions with high density, as expected, and diminishes with the expansion of the sphere. The bouncing of the sphere should not be related to the appearance of a shock wave within the sphere. Rather it is connected with the simple, but not extremely unphysical equation of state.

We would like to conclude with the following comment: Although the Eckart and Landau-Lifshitz (Eckart, 1940; Landau and Lifshitz, 1959) methods of including viscosity in general relativity have been widely used in the past, it is well known that these theories present serious difficulties. However, they led to the Navier-Stokes equations in the classical approximation.

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