BIANCHI TYPE-II, VIII, AND IX COSMOLOGICAL MODELS WITH MATTER AND ELECTROMAGNETIC FIELDS

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(Received 13 May, 1992)

Abstract. We present analytic solutions of the Einstein–Maxwell equations for cosmological models of LRS Bianchi type-II, VIII, and IX. The solutions represent anisotropic universes with source-free electromagnetic fields and perfect fluids matter satisfying the equation of state that is a function of the cosmic-time. Some physical properties of the models have been discussed.

1. Introduction

Modern cosmology is concerned with nothing less than a thorough understanding and explanation of the past history, the present state, and the future evolution of the Universe. Astronomical observations of the large-scale distribution of galaxies in the Universe show that the distribution of matter can be satisfactorily described by a perfect fluid. The Friedmann–Robertson–Walker models are only globally acceptable perfect fluid space-times which are spatially-homogeneous and isotropic. The adequancy of the isotropic models for describing the present state of the Universe is no basis for expecting that they are equally suitable for its early stages of evolution. After spatially-homogeneous and isotropic models, the simplest cosmologies are spatially-homogeneous but anisotropic models called the Bianchi models I–IX (Ryan and Shepley, 1975). A great deal of theoretical work has been done to build analytical models of the Universe by solving Einstein's field equations for the unknown metric tensor associated to perfect fluid distribution for various Bianchi spaces by choosing the equation of state linking the pressure and matter energy density (Kramer *et al.*, 1980).

In connection with the hypothesis of a primordial magnetic field, a systematic investigation of the solutions of Einstein's field equations with source-free electromagnetic fields in generic cosmological models with Bianchi symmetries has made by Hughston and Jacobs (1970), Jacobs (1977), Lorentz (1980, 1981, 1982), and many more. Davidson (1962) and Coley and Tupper (1986) used an alternative approach of solving Einstein's field equations for perfect fluids without assuming the equations of state linking pressure and energy density. Following this approach, Hajj-Boutrös (1989) constructed an LRS Bianchi type-II perfect fluid cosmological model with an equation of state that is a function of the cosmic-time t.

In this paper, we follow Hajj-Boutrös (1989) to investigate perfect fluid solutions of the Einstein's field equations for metrics of Bianchi type-II, VIII, and Ix space-times in the presence of source-free electromagnetic fields. The equation of state is a function of the cosmic-time t. The solutions represent anisotropic cosmological models

Astrophysics and Space Science **201**: 29–33, 1993.

which would give essentially an empty universe for large time. The physical behaviour of the models near and far from the initial point singularity $t = 0$ are discussed. The solution of Hajj-Boutrös (1989) is shown to be a particular case.

2. Field Equations and Solutions

The LRS metric for the Bianchi type-II ($\delta = 0$), type-VIII ($\delta = -1$), and type-IX ($\delta = 1$) is given by

$$s^2 = dt^2 - (S\,dx - Sh\,dz)^2 - (R\,dy)^2 - (Rf\,dz)^2\,, \tag{1}$$

where R and S are functions of the cosmic-time t and

$$f(y) = \begin{bmatrix} \sin y \\ \sinh y \end{bmatrix}, \qquad h(y) = \begin{bmatrix} \cos y \\ -\cosh y \end{bmatrix} \quad \text{for} \quad \delta = \begin{bmatrix} +1 \\ -1 \end{bmatrix}. \tag{2}$$

For the metric (1) the Einstein–Maxwell's equations with perfect fluid matter and source-free electromagnetic fields are

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} = -(p + \varepsilon)\,, \tag{3}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\delta}{r^2} - \frac{3S^2}{4R^4} = -(p - \varepsilon)\,, \tag{4}$$

$$\frac{2\dot{R}\dot{S}}{RS} + \frac{\dot{R}^2}{R^2} + \frac{\delta}{r^2} - \frac{S^2}{4R^4} = (\rho + \varepsilon)\,, \tag{5}$$

where p is the pressure and ρ the energy density of matter, $\varepsilon = a^2/R^4$ is the enrgy density of the electromagnetic field (a is constant). An overdot denotes differentiation with respect to t. For details of the field equations see Venkateswarlu and Reddy (1991).

From Equations (3) and (4), we have

$$\frac{\ddot{S} - \ddot{R}}{S\ \ R} + \frac{\dot{R}\dot{S}}{RS} - \frac{\dot{R}^2}{R^2} - \frac{\delta}{R^2} + \frac{1}{R^4}\,(S^2 + 2a^2) = 0\,. \tag{6}$$

Using the scale transformation (7)

$$dT = S\,dt$$

in (6), we get

$$\frac{S''}{S} + \frac{S'^2}{S^2} - \frac{R''}{R} - \frac{R'^2}{R^2} + \frac{1}{R^4} - \frac{\delta}{R^2S^2} + \frac{2a^2}{R^4S^2} = 0\,, \tag{8}$$

where the prime denotes differentiation with respect to T. Setting

$$r = R^2\,, \qquad s = S^2\,, \tag{9}$$

Equation (8) becomes

$$\frac{r''}{r} - \frac{s''}{s} - \frac{2}{r^2} + \frac{2\delta}{rs} - \frac{4a^2}{sr^2} = 0 . \tag{10}$$

If we insert the *ad hoc* relation

$$\frac{s''}{s} = + \left(-\frac{2}{r^2} + \frac{2\delta}{rs} - \frac{4a^2}{sr^2} \right) \tag{11}$$

into (10), we obtain

$$\frac{r''}{r} = 0 ; \tag{12}$$

which on integration yields

$$r(T) = R^2 = cT + c_1 , \tag{13}$$

where c and c_1 are constants.

Again, making the scale transformation

$$\tau = cT + c_1 \tag{14}$$

in Equation (11), we obtain

$$\tau^2 \frac{d^2 s}{d\tau^2} + \frac{2s}{c^2} = \frac{2\delta\tau}{c^2} - \frac{4a^2}{c^2} . \tag{15}$$

The general solution of Equation (15) is

$$s(\tau) = S^2(\tau) = A\tau^{\alpha_1} + B\tau^{\alpha_2} + \delta\tau - 2a^2 , \tag{16}$$

where A, B are integration constants and α_1, α_2 are roots of the equation

$$c^2\alpha^2 - c^2\alpha + 2 = 0 . \tag{17}$$

For $c^2 \geq 8$, α_1 and α_2 are real. If we take

$$\alpha_1 = \frac{1}{2} + \frac{(c^2 - 8)^{1/2}}{2c} , \tag{18}$$

$$\alpha_2 = \frac{1}{2} - \frac{(c^2 - 8)^{1/2}}{2c} , \tag{19}$$

then

$$\tfrac{1}{2} \leq \alpha_1 \leq 1 , \qquad 0 \leq \alpha_2 \leq \tfrac{1}{2} \tag{20}$$

(Hajj-Boutrös, 1989).

The pressure and energy density, as calculated from Equations (3) and (5), are given by

$$p = \tfrac{1}{4}[(c^2 - 3 - 2\alpha_1 c^2)A\tau^{\alpha_1 - 2} + (c^2 + 3 - 2\alpha_2 c^2)B\tau^{\alpha_2 - 2} - \\ - \delta(c^2 + 1)\tau^{-1} - 2(c^2 + 1)a^2\tau^{-2}] , \tag{21}$$

$$\rho = \tfrac{1}{4}[(c^2 - 1 + 2\alpha_1 c^2)A\tau^{\alpha_1 - 2} + (c^2 - 1 + 2\alpha_2 c^2)B\tau^{\alpha_2 - 2} + $$
$$+ 3\delta(c^2 + 1)\tau^{-1} - 2(c^2 + 1)a^2\tau^{-2}] . \tag{22}$$

The energy density of the electromagnetic field is given by

$$\varepsilon = a^2/\tau^2 . \tag{23}$$

3. Bianchi Type-II Solutions

We first consider the case $\delta = 0$. The Bianchi type-II perfect fluid solution with source-free electromagnetic field is given by the metric functions

$$R^2 = \tau , \quad S^2 = A\tau^{\alpha_1} + B\tau^{\alpha_2} - 2a^2 . \tag{24}$$

In the absence of the electromagnetic field ($a = 0$), it reduces to the solution given by Hajj-Boutrös (1989). If $a = 0$ and $A = 0$ (or $B = 0$) this solutions reduces to that of Dunn and Tupper (1980) with the equation of state $\rho = np$, $n > 1$. The pressure, energy density of matter, and the charge density are infinite at $\tau = 0$. The spatial volume is zero there which verifies the singularity of the metric at the Big-Bang event $\tau = 0$. The pressure, matter energy density, and charge density tend to zero and spatial volume becomes infinite as $\tau \to \infty$. Thus the model gives essentially an empty universe for large τ.

The fluid motion is irrotational and acceleration-free. The expansion scalar θ has the value

$$\theta = \frac{1}{\tau} + \frac{1}{2}\left[\frac{A\alpha_1\tau^{\alpha_1 - 1} + B\alpha_2\tau^{\alpha_2 - 1}}{A\tau^{\alpha_1} + B\tau^{\alpha_2} - 2a^2}\right], \tag{25}$$

which is infinite at $\tau = 0$ but monotonic decreasing for $\tau > 0$. In fact, $\theta \to 0$ as $\tau \to \infty$. On the other hand, there is shear; and the shear scalar σ has the value of

$$\sigma = \frac{1}{2\sqrt{3}}\left[\frac{1}{\tau} - \frac{A\alpha_1\tau^{\alpha_1 - 1} + B\alpha_2\tau^{\alpha_2 - 1}}{A\tau^{\alpha_1} + B\tau^{\alpha_2} - 2a^2}\right], \tag{26}$$

which is infinite at $\tau = 0$ and tends to zero as $\tau \to \infty$. The ratio σ/θ does not tend to zero as $\tau \to \infty$ which shows that the shear does not tend to zero faster than the expansion.

For $c^2 > 8$, we have

$$\frac{p}{\rho} \to \frac{c^2 + 3 - 2\alpha_2 c^2}{c^2 - 1 + 2\alpha_2 c^2} \quad \text{as} \quad \tau \to 0 \tag{27}$$

and

$$\frac{p}{\rho} \to \frac{c^2 + 3 - 2\alpha_1 c^2}{c^2 - 1 + 2\alpha_1 c^2} \quad \text{as} \quad \tau \to \infty , \tag{28}$$

in view of the inequalities (20). The behaviour of the model depends essentially on the values of c. As shown by Hajj-Boutrös (1989), it behaves as a radiation-filled universe with source-free electromagnetic field for small τ and as a dust model for large τ.

4. Bianchi type-VIII and IX Solutions

For $\delta = -1$ and $\delta = 1$, the Bianchi type-VIII and IX perfct fluid solutions with source-free electromagnetic field are given by the functions

$$R^2 = \tau\,, \qquad S^2 A\tau^{\alpha_1} + B\tau^{\alpha_2} + \delta\tau - 2a^2\,. \tag{29}$$

The physical behaviour of the models is similar to that of the Bianchi type-II models discussed in Section 3. The ratio

$$\frac{p}{\rho} = \frac{(c^2 + 3 - 2\alpha_1 c^2)A\tau^{\alpha_1} + (c^2 + 3 - 2\alpha_2 c^2)B\tau^{\alpha_2} - (c^2 + 1)\delta\tau - 2a^2(c^2 + 1)}{(c^2 - 1 + 2\alpha_1 c^2)A\tau^{\alpha_1} + (c^2 - 1 + 2\alpha_2 c^2)B\tau^{\alpha_2} + 3\delta(c^2 + 1)\tau - 2a^2(c^2 + 1)}\,, \tag{30}$$

tends to 1 as $\tau \to 0$. Thus the models behave like stiff-matter dominated universes with source-free electromagnetic field for small τ. For $\tau \to \infty$, the ratio p/ρ tends to $-\frac{1}{3}$ which means that the models behave like de Sitter universes for large τ.

In the absence of electromagnetic field ($a = 0$), the Bianchi type-VIII and IX models filled with perfect fluid recently obtained by the present authors (Shri Ram and Singh, 1991).

5. Conclusions

We have presented analytical solutions to Einstein's field equations for Bianchi type-II, VIII, and IX universes of perfect fluid and source-free electromagnetic field that expand irrotationally from the initial time singularity $\tau = 0$ with shear. The equation of state is a function of τ. As far as we know these solutions are new, exhibiting such type of behaviour in the case of Bianchi type-II, VIII, and IX cosmologies.

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