

# A Multi-Scale Theory of Swelling Porous Media: I. Application to One-Dimensional Consolidation

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**Abstract.** A theory is developed which describes flow in multi-scale, saturated swelling media. To upscale information, both the hybrid theory of mixtures and the homogenization technique are employed. In particular, a model is formulated in which vicinal water (water adsorbed to the solid phase) is treated as a separate phase from bulk (non-vicinal) water. A new form of Darcy's law governing the flow of both vicinal and bulk water is derived which involves an interaction potential to account for the swelling nature of the system. The theory is applied to the classical one-dimensional consolidation problem of Terzaghi and to verify Low's empirical, exponential, swelling result for clay at the macroscale.

**Key words:** Swelling clay soil, multi-scale flow, hybrid mixture theory, homogenization, consolidation.

## 1. Introduction

The development of theories to simultaneously represent consolidation and the flow of water in saturated clay soils began with Terzaghi [48] and Biot [11]. Essentially, Biot and Terzaghi developed linear poroelastic models based upon a phenomenological approach at the macroscale. Their models are now well established and Biot's model can be recovered via a linearization procedure within a more general thermomechanical framework of modern mixture theory (e.g. Crochet and Naghdi [20], Green and Steel [31]) or by applying a homogenization technique to a local pore-scale problem where the solid is considered to be linearly elastic and the fluid is assumed to be Stokesian (Auriault [4]). Although limited to the linear elastic range, the Biot and Terzaghi models have been the basis for the development of more general approaches aimed at removing some of their restrictive assumptions. For one-dimensional consolidation, nonlinear extensions of the Terzaghi theory have been proposed by using nonlinear stress–strain relationships (e.g. Davis and

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Raymond [23], Mikasa [42]), by using the nonlinearized definition of strain for large deformations (e.g. Gibson *et al.* [28]), and by using a variable permeability coefficient (e.g. Cherubini [18], Low [39]). Similarly, rheological models based on a Kelvin–Voigt viscoelastic material have been proposed to extend the Biot theory into the range of secondary consolidation which is governed by creep phenomena and stress relaxation in viscous clay soils (e.g. Barden [8], Gibson and Lo [29]). Plasticity models have also been proposed which extend the Biot theory into the anelastic range by including features such as yield phenomena, irreversible deformations, hysteresis, and hardening and softening. Within this category, the Cam–Clay model and Modified Cam–Clay models (e.g. [45]) have been developed recently and are considered to be more appropriate for modeling nonlinear, path-dependent, hysteretic soil behavior. As in the Terzaghi and Biot models, the majority of models describing the consolidation of clay have been developed directly at the macroscale usually by taking measurements of a representative sample of clay and postulating constitutive equations which most accurately reflect the experimental data.

Despite the widespread use of generalized Terzaghi and Biot models, the authors are not aware of any models which are rigorously derived from the microstructure of clay. Saturated clays consist of an assemblage of clusters of hydrous aluminum and magnesium silicates plus adsorbed water (clay particles). The mixture often displays a tendency to swell when wetted and to shrink when dried. This swell/shrink phenomenon is associated with the extremely high specific surface area of the hydrophilic plate-like clay mineral. The hydrophilic nature of clay affects the properties of water near the clay minerals and hence this affected water, which is strongly attracted to clay minerals, has physical properties which are altered by the proximity of a solid (Low [39], [40], [41], Grim [32]). Such affected water is often called *vicinal water*, *adsorbed water*, or *interlayer water* to distinguish it from *bulk water* or free pore water (i.e. water which lies in porous openings and whose properties are not significantly affected by the presence of clay minerals). For example, the density of vicinal water has been reported to be between 0.97 and 1.7 times that of free water, and the vicinal water is more viscous than free water, with both properties varying with the distance from the surface and depending in a complex way on the structure and commensurability of the surfaces, the surface-liquid potential, the separation of the surfaces, the direction of shear, the rate of shear and the history of shear (Cushman [22]). In general, the vicinal water may behave as an inhomogeneous anisotropic fluid. The anomalous nature of vicinal fluids is born out both experimentally and computationally (e.g. Cushman [22]). The microscopic properties of vicinal water are revealed on the macroscale in such behavior as swelling, plasticity characteristics, and in deviations from such popularly used equations as Darcy's law. Darcy's law has given satisfactory results for granular porous media, where swelling is not predominant. However, it has been shown that the classical form of Darcy's law is not accurate in swelling media (e.g. Philips [44], Kiralis [37]). It is also well known that the classical effective stress

principle stated by Terzaghi has produced highly accurate results when applied to non-swelling media such as sand. However, an extension of Terzaghi's effective stress principle to swelling media is not straightforward (e.g. Sridharan and Rao [47], Kiralis [36], Chenmin and Hueckel [17]). Modifications of the principle of effective stress have also been proposed (Sridharan and Rao [47], Chenmin and Hueckel [17]) for clay soils in order to take into account features such as inter-particle contact stress, electro-chemical forces, adsorption of water by the clay minerals, etc.

Approaches which have been used to develop models of porous media include mixture theory and methods which propagate microscopic governing and constitutive equations to the larger scale (e.g. homogenization, volume averaging, etc.). Mixture theory is an approach in which the system is viewed as overlapping continua (e.g. Bowen, [13]). Additional terms are introduced in the conservation equations to account for interaction between constituents, and the entropy inequality is exploited using the Coleman and Noll method [19] to restrict the forms of the constitutive equations. This approach allows assumptions to be brought in only at the macroscale so that the form of the constitutive equations can be chosen to fit experimental data. One disadvantage of this technique is that microscopic information and features are lost. On the other hand, techniques such as homogenization (Bensoussan *et al.* [10], Sanchez-Palencia [46]) and volume averaging (e.g. Bachmat and Bear [7], Whitaker [49]) begin with microscopic conservation and constitutive equations and then propagate the information to the macroscale. Although this method generally gives useful closed-form relations for macroscopic material coefficients, it does not allow much flexibility in choosing the form of the constitutive equations at the macroscale. Both approaches are well-established and, after initial assumptions are made, have a firm mathematical basis. A combination of these two approaches designed to combine the advantages of each was first proposed by Hassanizadeh and Gray [33, 34]. We call this approach *hybrid mixture theory*. Essentially it consists of applying classical mixture theory to the macroscopically averaged balance laws for phases and interfaces. Thus some microscale information is passed onto the macroscale, yet constitutive equations are formulated on this latter scale.

In order to derive a mathematical model for flow in swelling clay soils we propose an approach significantly different from those generalizing Terzaghi and Biot models. The proposed model is developed from the microscale by taking advantage of the microstructure of clay. The model is multiscale as it includes three levels of observation: micro, meso and macro (see Figure 1). At the microscale the model has two phases, the disjoint clay minerals and the vicinal water. At the mesoscale the model includes the clay particles (vicinal water plus clay platelets) and the bulk water, and at the macroscale the medium is treated as a single continuum. Thus the vicinal water and the bulk water are treated as two distinct phases. To propagate information from the microscale to the mesoscale we adapt with some modifications the framework established by Achanta *et al.* [2]. In this work, to capture the

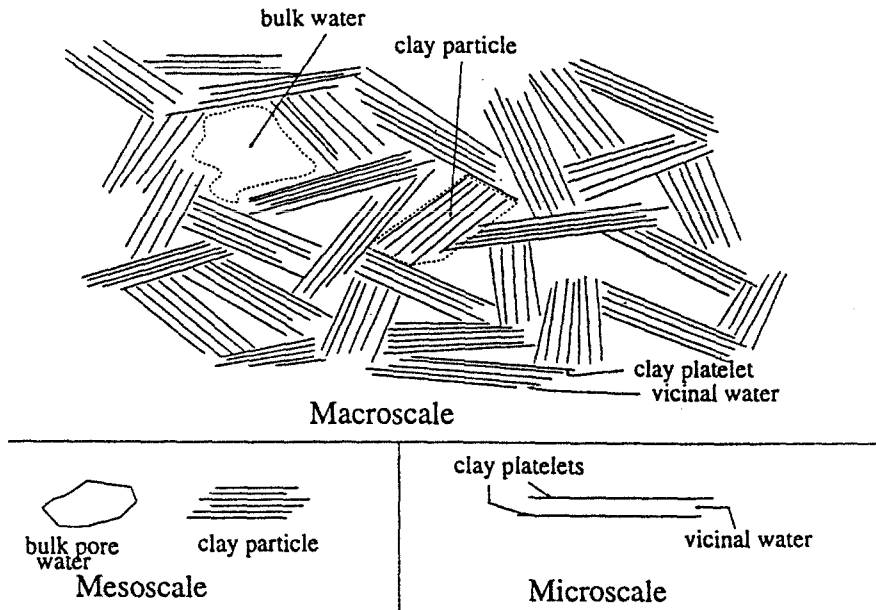


Fig. 1. Multi-scale model for clay.

effects of swelling, the upscaling approach used is hybrid mixture theory with the volume fraction included as one of the independent variables. It should be noted that using the volume fraction as an independent variable began within the context of mixture theory (Drumheller [25], Bowen [14, 15], Bedford and Drumheller [9]). Gray and Hassanizadeh [30, 35], followed by Achanta *et al.* [2], used the volume fraction as an independent variable within the framework of hybrid mixture theory and Achanta *et al.* [2] suggested that this is an appropriate method for modeling clay particles. One of the novel results derived using the hybrid mixture theory with the volume fraction as an independent variable is that flow of a single phase, single constituent fluid in swelling soils is driven by a gradient of chemical potential [2]. Contrary to classical belief (i.e. classical form of Darcy's Law), the gradient of the chemical potential is more general than a pressure gradient as it also involves an interaction potential to account for the swelling nature of the system.

Thus at the mesoscale we derive equations which govern the flow of vicinal water using the hybrid mixture theory. To model the Newtonian bulk water we assume flow is governed by the Stokes equations. Appropriate boundary conditions between the vicinal and bulk phases are proposed establishing a system of governing equations at the mesoscale. We then use the homogenization technique [10, 46] to propagate the mesoscopic information to the macroscale. This procedure results in a single system of equations. In using this upscaling technique the loss of information in reducing the number of degrees of freedom associated with the movement up the hierarchy manifests itself on the macroscale with the appearance of new constitutive variables (Cushman [21]). These new variables

appear, for example, in a modified Darcy's law which governs the total velocity field, including the flow of both the vicinal and bulk phases. The resultant system of homogenized equations is then linearized about an equilibrium state and a generalized solution is presented within the framework of the semigroup theory (Pazy [43], Sanchez-Palencia [46], Yosida [50]).

The homogenized model is applied to the classical one-dimensional consolidation problem described by Terzaghi [48] where the steady-state solution of the macroscopic governing equations reproduces Low's [41] swelling-pressure expression. In [41] Low reported on numerous experiments and empirically derived a relationship between the vicinal and bulk phase pressures at equilibrium (swelling pressure). Low's empirical result has recently been derived using the hybrid mixture theory model at the mesoscale by Achanta *et al.* [2]. Our model produces Low's result directly at the macroscale (including the bulk water phase) as well as the mesoscale.

A brief outline of the paper is as follows: In Section 2, we develop the model for the clay particles based on a combination of hybrid mixture theory and the technique of Lagrangian multipliers which is used to exploit the entropy inequality with internal constraints in the sense of Liu [38]. In Section 3, we propose appropriate boundary conditions between the vicinal and bulk phases and then establish the mesoscopic problem. In Section 4, we apply the homogenization procedure to derive macroscopic governing equations and present a generalized solution of the linearized problem. In Section 5, we apply the theory to the classical, one-dimensional consolidation problem described by Terzaghi [48].

## 2. Governing Equations for Clay Particles with Incompressible Constraints

In this section we present the balance equations, constitutive assumptions, and resulting constitutive relations governing the movement of the clay platelets and vicinal water. Our development will combine the approach presented by Achanta *et al.* [2] for compressible swelling media with the technique of Lagrangian multipliers (in the sense of Liu [38]) for exploiting the entropy inequality when the vicinal water and clay platelets are incompressible. In [2], the hybrid mixture theory is used to derive equations for the general case of a multiphase swelling media. Here, we consider a simpler case of a two-phase (solid and fluid), single constituent per phase, system which has negligible interfacial effects. The latter assumption is physically satisfied if the clay particles are assumed partially swollen and all clay platelets are thoroughly wetted.

In the usual framework of the mixture theory a typical clay particle consists of two liquid–solid coexisting continua which undergo independent motions  $\mathbf{x} = \mathbf{x}_\alpha(\mathbf{X}_\alpha, t)$ ,  $\alpha = l, s$  with respect to each reference configuration (here  $\mathbf{x}$  denotes the spatial position of the particle of the  $\alpha$ -phase at time  $t$  with respect to a reference position  $\mathbf{X}_\alpha$ ). We begin with the mesoscopic mass, momentum, energy

and entropy equations of Hassanizadeh and Gray [33] which are the microscopic conservation equations averaged over a representative elementary volume (REV). These equations are defined everywhere so that the microscopically distinct phases are now viewed as superimposed continua. One of the major advantages of the hybrid mixture theory is that each mesoscopic variable is precisely defined in terms of its microscopic counterparts.

The entropy inequality is postulated for the mixture as a whole. The local forms of the balance laws are presented for the liquid and solid phases (represented by subscripts  $l$  and  $s$  respectively) below for the case in which the entropy fluxes are solely due to heat fluxes, the only external source is gravity, and the solid and fluid are non-polar, non-reacting phases at local thermal equilibrium.

### *Conservation of Mass*

$$\frac{D_\alpha(\phi_\alpha\rho_\alpha)}{Dt} + \phi_\alpha\rho_\alpha \operatorname{div} \mathbf{v}_\alpha = 0, \quad \alpha = l, s,$$

where  $\phi_\alpha$ ,  $\rho_\alpha$  and  $\mathbf{v}_\alpha$  are respectively the volume fraction, intrinsic volume-average density, and mass-average velocity of phase  $\alpha$ .  $D_\alpha/Dt$  is the material time derivative following the  $\alpha$ -phase

$$\frac{D_\alpha}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_\alpha \cdot \nabla.$$

### *Conservation of Momentum*

$$\phi_\alpha\rho_\alpha \frac{D_\alpha\mathbf{v}_\alpha}{Dt} - \operatorname{div}(\phi_\alpha\mathbf{t}_\alpha) - \phi_\alpha\rho_\alpha\mathbf{g} = \hat{\mathbf{T}}_\alpha, \quad \alpha = l, s, \quad (2.1)$$

where  $\mathbf{t}_\alpha$  denotes the average symmetric stress tensor for phase  $\alpha$ ,  $\hat{\mathbf{T}}_\alpha$  the exchange of momentum to phase  $\alpha$  from the other phase and  $\mathbf{g}$  is the body force (gravity).

### *Conservation of Energy*

$$\phi_\alpha\rho_\alpha \frac{D_\alpha E_\alpha}{Dt} - \phi_\alpha \operatorname{tr}(\mathbf{t}_\alpha\mathbf{d}_\alpha) - \operatorname{div}(\phi_\alpha\mathbf{h}_\alpha) = \hat{Q}_\alpha, \quad \alpha = l, s, \quad (2.2)$$

where  $E_\alpha$  is the average internal energy of phase  $\alpha$  per unit mass,  $\mathbf{h}_\alpha$  is the heat flux,  $\mathbf{d}_\alpha$  is the symmetric part of  $\nabla\mathbf{v}_\alpha$  and  $\hat{Q}_\alpha$  is the net exchange of energy between phases due to mechanical interactions.

### *Entropy Inequality*

$$\Lambda = \sum_{\alpha=l,s} \left[ \phi_\alpha\rho_\alpha \frac{D_\alpha\eta_\alpha}{Dt} - \operatorname{div} \left( \frac{\phi_\alpha\mathbf{h}_\alpha}{T} \right) \right] \geq 0, \quad (2.3)$$

where  $\eta_\alpha$  is the entropy of phase  $\alpha$  per unit mass of the mixture,  $T$  is the temperature and  $\Lambda$  is the rate of net entropy production. In addition, the conservation of momentum and energy for the clay particle (vicinal fluid plus clay platelets) requires

$$\sum_{\alpha=l,s} \hat{\mathbf{T}}_\alpha = 0,$$

$$\sum_{\alpha=l,s} [\hat{Q}_\alpha + \hat{\mathbf{T}}_\alpha \cdot \mathbf{v}_\alpha] = 0,$$

or

$$\sum_{\alpha=l,s} \hat{Q}_\alpha = -\hat{\mathbf{T}}_l \cdot \mathbf{v}_{l,s}, \quad (2.4)$$

where  $\mathbf{v}_{l,s} = \mathbf{v}_l - \mathbf{v}_s$  is the mass-average relative velocity (i.e. the objective part of the vicinal fluid velocity).

To obtain a constitutive theory formulated in terms of the temperature rather than entropy we choose to replace  $E_\alpha$  by its Legendre transform

$$A_\alpha = E_\alpha - T\eta_\alpha,$$

where  $A_\alpha$  is the Helmholtz free energy density of the  $\alpha$ -phase. Replacing  $E_\alpha$  in (2.2) by  $A_\alpha$ , combining with (2.3) and using (2.4) gives

$$\begin{aligned} \Lambda = \sum_{\alpha=l,s} \left[ -\frac{\phi_\alpha \rho_\alpha}{T} \left( \frac{D_\alpha A_\alpha}{Dt} + \eta_\alpha \frac{D_\alpha T}{Dt} \right) + \frac{\phi_\alpha}{T} \text{tr}(\mathbf{t}_\alpha \mathbf{d}_\alpha) + \frac{\phi_\alpha}{T^2} \mathbf{h}_\alpha \cdot \nabla T \right] - \\ - \frac{1}{T} \mathbf{v}_{l,s} \cdot \hat{\mathbf{T}}_l \geq 0. \end{aligned}$$

The choice of independent variables is based on the experimental constraints and internal state behaviors of the system (the latter is related to the swelling properties of the clay particles). Assuming that on the mesoscale, the solid and fluid phases are incompressible and not heat conducting and that the vicinal water is non-viscous, the behavior of the system is dictated by the following independent variables:

$$T, \mathbf{E}_s, \phi_l, \nabla \phi_l, \mathbf{v}_{l,s}, \quad (2.5)$$

where  $\mathbf{E}_s$  is the averaged strain tensor of the skeleton formed by the solid phase and is given by

$$\mathbf{E}_s = \frac{1}{2}(\mathbf{F}_s^T \mathbf{F}_s - \mathbf{I}),$$

where  $\mathbf{F}_s = \text{grad } \mathbf{x}_s$  denotes the deformation gradient (with grad denoting the differentiation with respect to a material particle on the averaged state).

We include  $\nabla \phi_l$  in the set (2.5) to provide a potential for vicinal water flow in swelling systems. This is analogous to including  $\nabla T$  as an independent variable in a heat-conducting medium (see e.g. [27]). Our dependent variables, i.e. variables which are functions of the set of independent variables (2.5), are thus

$$A_\alpha, \eta_\alpha, \frac{D_s \phi_l}{Dt}, \mathbf{t}_\alpha, \mathbf{h}_\alpha, \hat{\mathbf{T}}_l, \quad \alpha = l, s. \quad (2.6)$$

In Hassanizadeh and Gray [34]  $\mathbf{E}_s$  and  $\phi_l$  are not considered to be independent variables, since knowledge of  $\mathbf{E}_s$  determines  $\phi_l$  through the continuity equation for the incompressible solid phase. On the other hand it is pointed out by Achanta *et al.* [2] that the inclusion of the volume fraction as an independent internal state variable is crucial for a proper description of swelling. The inclusion of the volume fraction is necessary to recover the well known empirical exponential swelling relation between the pressures of the vicinal and bulk phase waters determined by Low [41]. The disparity between the two approaches can be mollified by viewing the continuity equations for the solid and fluid phases as constraints (Liu [38]) rather than balance laws. These constraints are enforced by adding to the entropy inequality the mass balances for incompressible media premultiplied by Lagrangian multipliers. The theorem proposed by Liu gives an equivalence between the entropy inequality with an imposed restriction and a modified entropy inequality using the Lagrange multiplier formulation.

The closure issue, which is discussed extensively by Bouré [12], must be addressed. For compressible materials and/or multiphase flows,  $\rho_\alpha$  and  $\phi_\alpha$  may be included in the list of independent variables [15, 35] and [2] (the latter as an internal variable). To have the same number of equations as unknowns an additional closure law is provided by postulating that  $D_s \phi_l / Dt$  (the material derivative following the solid phase) is a dependent variable with dependence given in terms of a volume fraction topological law in the sense of Bouré [12] (or rate law for  $\phi_l$ ). As noted by Bouré, this law is distinct from the other constitutive equations. It contains features specific to multiphase flows such as interfacial structure and geometry, that must be restored to enable closure. Because both vicinal fluid and the clay platelets are assumed incompressible, we diverge slightly from our predecessors' approaches. In [2] incompressibility is not considered and in [34]  $D_s \phi_l / Dt$  is eliminated using the continuity equation. As pointed out in [2], the rate law for  $\phi$  contains important information related to swelling pressures in clay colloidal systems. We wish to retain the additional information obtained when considering  $D_s \phi_l / Dt$  as a dependent variable and still enforce the incompressibility of the two phases. This is accomplished by imposing the continuity equation for incompressible media

$$\frac{D_\alpha \phi_\alpha}{Dt} + \phi_\alpha \text{div } \mathbf{v}_\alpha = 0, \quad \alpha = l, s \quad (2.7)$$



using two Lagrange multipliers,  $\lambda_l$  and  $\lambda_s$  in the entropy inequality

$$\begin{aligned} \Lambda = & \sum_{\alpha=l,s} \left[ -\frac{\phi_\alpha \rho_\alpha}{T} \left( \frac{D_\alpha A_\alpha}{Dt} + \eta_\alpha \frac{D_\alpha T}{Dt} \right) + \frac{\phi_\alpha}{T} \text{tr}(\mathbf{t}_\alpha \mathbf{d}_\alpha) + \frac{\phi_\alpha}{T^2} \mathbf{h}_\alpha \cdot \nabla T \right] - \\ & - \frac{1}{T} \mathbf{v}_{l,s} \cdot \hat{\mathbf{T}}_l + \sum_{\alpha=l,s} \frac{\lambda_\alpha}{T} \left( \frac{D_\alpha \phi_\alpha}{Dt} + \phi_\alpha \text{div } \mathbf{v}_\alpha \right) \geq 0. \end{aligned}$$

According to Liu [38], the problem now can be treated as a free variation problem with the Lagrange multipliers,  $\lambda_l$  and  $\lambda_s$  as unknowns which are neither independent nor dependent variables in the sense of (2.5) and (2.6), i.e. they are functions of  $\mathbf{x}$  and  $t$ .

Since in our formulation the continuity equation was imposed as a constraint rather than a balance law,  $D_s \phi_l / Dt$  was not eliminated from (2.7) and therefore the swelling information available in the topological law for  $D_s \phi_l / Dt$  is maintained.

As in [2] the dependence of the Helmholtz free energies on the independent variables is postulated to be

$$A_l = A_l(T, \phi_l), \quad A_s = A_s(T, \mathbf{E}_s).$$

The retention of the volume fraction in the constitutive dependence of  $A_l$  is motivated by Low [41], who verified experimentally that the free energy of the vicinal water is strongly dependent on the separation of the clay platelets.

The Coleman and Noll method [19] is now used to exploit the restrictions placed by the entropy inequality on the constitutive theory. By substituting the above functional forms into the entropy inequality, expanding the material time derivatives and using the relation  $D_s \mathbf{E}_s / Dt = \mathbf{F}_s^T \mathbf{d}_s \mathbf{F}_s$  [27], we obtain

$$\begin{aligned} \Lambda = & \sum_{\alpha=l,s} -\frac{\phi_\alpha \rho_\alpha}{T} \left( \frac{\partial A_\alpha}{\partial T} + \eta_\alpha \right) \frac{D_s T}{Dt} + \frac{\phi_l}{T} \text{tr}((\mathbf{t}_l + \lambda_l \mathbf{I}) \mathbf{d}_l) + \\ & + \frac{\phi_s}{T} \text{tr}((\mathbf{t}_s + \lambda_s \mathbf{I} - \mathbf{t}_s^e) \mathbf{d}_s) + \frac{1}{T^2} \nabla T \cdot \\ & \times \left( \sum_{\alpha=l,s} \phi_\alpha \mathbf{h}_\alpha - \phi_l \rho_l T \mathbf{v}_{l,s} \left( \frac{\partial A_l}{\partial T} + \eta_l \right) \right) - \\ & - \frac{1}{T} \mathbf{v}_{l,s} \cdot \left( \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} \nabla \phi_l - \lambda_l \nabla \phi_l + \hat{\mathbf{T}}_l \right) - \\ & - \frac{1}{T} \frac{D_s \phi_l}{Dt} \left( \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} - \lambda_l + \lambda_s \right) \\ & \geq 0, \end{aligned}$$

where

$$\mathbf{t}_s^e = \rho_s \mathbf{F}_s \frac{\partial A_s}{\partial \mathbf{E}_s} \mathbf{F}_s^T$$

is the classical effective stress tensor of the solid phase introduced by Terzaghi [48].

Because  $D_s T / Dt$ ,  $\mathbf{d}_l$ ,  $\mathbf{d}_s$  and  $\nabla T$  are neither dependent (constitutive) nor independent variables,  $\Lambda$  is a linear function in these variables. Hence, for the inequality to hold for all possible processes, the coefficients of these variables must be identically zero. This gives the following relations which must always hold

$$\begin{aligned} \sum_{\alpha=l,s} \phi_\alpha \rho_\alpha \left( \frac{\partial A_\alpha}{\partial T} + \eta_\alpha \right) &= 0, \\ \sum_{\alpha=l,s} \phi_\alpha \mathbf{h}_\alpha - \phi_l \rho_l T \mathbf{v}_{l,s} \left( \frac{\partial A_l}{\partial T} + \eta_l \right) &= 0, \\ \mathbf{t}_l &= -\lambda_l \mathbf{I}, \\ \mathbf{t}_s &= -\lambda_s \mathbf{I} + \mathbf{t}_s^e. \end{aligned}$$

We are left with the residual entropy inequality

$$\begin{aligned} \Lambda &= -\frac{1}{T} \mathbf{v}_{l,s} \cdot \left( \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} \nabla \phi_l - \lambda_l \nabla \phi_l + \hat{\mathbf{T}}_l \right) - \\ &\quad - \frac{1}{T} \frac{D_s \phi_l}{Dt} \left( \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} - \lambda_l + \lambda_s \right) \geq 0. \end{aligned} \quad (2.8)$$

## 2.1. EQUILIBRIUM RESTRICTIONS

The state of thermodynamic equilibrium is defined to be the state in which the variables  $\{\nabla T, \mathbf{v}_{l,s}, D_s \phi_l / Dt\}$  vanish. At equilibrium, entropy is maximum and entropy generation attains its minimum value and therefore we must have  $(\partial \Lambda / \partial z_a)_e = 0$  and  $(\partial^2 \Lambda / \partial z_a \partial z_b)_e$  positive definite. Here  $(\cdot)_e$  denotes the evaluation of the functions at equilibrium and  $z_a$  and  $z_b$  are any of the above set of variables. Application of these requirements to the residual entropy inequality yields at equilibrium

$$\begin{aligned} (\hat{\mathbf{T}}_l)_e &= \left( \left( \lambda_l - \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} \right) \nabla \phi_l \right)_e, \\ (\lambda_l - \lambda_s)_e &= \left( \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} \right)_e. \end{aligned} \quad (2.9)$$

By comparing the above equations with those of [2], where no incompressibility assumption is made, we note that the Lagrange multipliers  $\lambda_\alpha$  represent the pressures  $p_\alpha$  of the phase  $\alpha$ . For compressible media these have been previously defined as (e.g. [34])

$$p_\alpha = (\rho_\alpha)^2 \frac{\partial A_\alpha}{\partial \rho_\alpha}, \quad \alpha = l, s.$$

In classical literature on interfacial thermodynamics, the difference  $\lambda_l - \lambda_s$  is commonly referred to as ‘interfacial pressure’ (Zangwill [51]). When moisture content is high it is reasonable to assume that the majority of the overburden pressure is carried by the vicinal water. This assumption implies that the interfacial pressure is approximately equal to the pressure in the vicinal water and henceforth we may assume

$$\lambda_l - \lambda_s = p_l - p_s \approx p_l. \tag{2.10}$$

Under the above hypothesis, application of (2.9) to a clay-swelling experiment [2] leads to the exponential swelling pressure relationship between the pressures of the vicinal and bulk water postulated by Low.

### 2.2. NEAR-EQUILIBRIUM THEORY

To obtain results which hold near equilibrium, the non-equilibrium restrictions can be expanded in a Taylor series about equilibrium and a linear approximation obtained by neglecting quadratic and higher order terms in the expansion. In the subsequent discussion we linearize the residual entropy inequality about variables which are zero at equilibrium (i.e.  $D_s \phi_l / Dt, \mathbf{v}_{l,s}$ ) and ensure positive quadratic forms appear after the linearization so that (2.8) is always satisfied. We have using the assumption (2.10),

$$\phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} \nabla \phi_l - p_l \nabla \phi_l + \hat{\mathbf{T}}_l = -(\phi_l)^2 [\mathbf{K}_l]^{-1} \mathbf{v}_{l,s} \tag{2.11}$$

and

$$p_l = \phi_l \rho_l \frac{\partial A_l}{\partial \phi_l} + \mu_* \frac{D_s \phi_l}{Dt}, \tag{2.12}$$

where  $\mathbf{K}_l$  and  $\mu_*$  are material coefficients (the latter has dimensions of viscosity) which depend on the set of independent variables and which are constrained to be positive by the residual entropy inequality. We assume isotropic behavior of the clay particles at the mesoscale so that  $\mathbf{K}_l = K_l \mathbf{I}$  where  $K_l$  is the permeability of the clay particles. Based on previous experimental results (Low [39]), the conductivity of clay soils is assumed to be of the form  $K_l = K_l(\phi_l)$ . Also through the remainder of the paper we assume  $\mu_* = \mu_*(\phi_l)$ .

Equation (2.12) introduces a novel definition of swelling pressure for non-equilibrium processes in colloidal systems (Achanta and Cushman [1]). When  $D_s\phi_l/Dt = 0$  it reduces to the traditional equilibrium definition.

A generalized form of Darcy's law can now be derived by combining (2.11) with the momentum equation (2.1) for the liquid phase to eliminate  $\hat{\mathbf{T}}_l$ . Neglecting inertial terms and noting that  $\mathbf{t}_l$  has only a pressure component we have

$$\phi_l \mathbf{v}_{l,s} = -K_l(\phi_l) \left( \nabla p_l + \rho_l \frac{\partial A_l}{\partial \phi_l} \nabla \phi_l - \rho_l \mathbf{g} \right). \quad (2.13)$$

The above form of Darcy's law differs from the standard one used to model saturated flows in non-swelling systems. The important difference is the appearance of an additional 'interaction potential' which accounts for swelling. We remark that a similar additional term namely the 'wettability potential', was derived in [30, 35] for unsaturated flows.

### 3. Mesoscopic Problem for Clay Particles and Bulk Water

In this section we formulate the mesoscopic governing equations for the clay particles and bulk water along with the appropriate boundary and initial conditions. We begin by re-writing Equations (2.7), (2.12) and (2.13) for the case in which the convective terms in the continuity equations are negligible for the clay particles, i.e.  $\mathbf{v}_\alpha \cdot \nabla \phi_\alpha \ll 1$ . We have

$$\begin{aligned} \frac{\partial \phi_l}{\partial t} + (1 - \phi_l) \operatorname{div} \mathbf{q}_l &= 0, \\ \mathbf{q}_l &= -K_l(\phi_l) (\nabla p_l + p_* \nabla \phi_l - \rho_l \mathbf{g}), \\ p_l &= \phi_l p_* + \mu_* \frac{\partial \phi_l}{\partial t}, \end{aligned}$$

where

$$p_* = \rho_l \frac{\partial A_l}{\partial \phi_l} \quad \text{and} \quad \mathbf{q}_l = \phi_l \mathbf{v}_{l,s},$$

with  $p_*$  having the same units as pressure. The above system is supplemented by experimentally obtained coefficients  $K_l = K_l(\phi_l)$ ,  $p_* = p_*(\phi_l)$  and  $\mu_* = \mu_*(\phi_l)$  completing the mesoscopic description of the clay particles. In addition, it is postulated that the bulk phase flow is governed by the classical Stokes problem

$$\begin{aligned} \operatorname{div} \mathbf{t}_f + \rho_f \mathbf{g} &= 0, \\ \mathbf{t}_f &= -p_f \mathbf{I} + 2\mu^f \mathbf{d}_f, \\ \operatorname{div} \mathbf{v}_f &= 0, \end{aligned} \quad (3.1)$$

where the subscript  $f$  denotes the bulk phase and  $\mu^f$  the viscosity coefficient. A complete mesoscopic description of the entire system requires additional boundary conditions on the particle-bulk water interface  $\Gamma$  and initial conditions. Through the remainder of the paper we shall assume  $\rho_l = \rho_f = \rho$  (which is satisfied if there is enough vicinal fluid) and hence continuity of the mass on the interface requires

$$\mathbf{q}_l \cdot \mathbf{n} = \mathbf{v}_{f,s} \cdot \mathbf{n} \quad \text{on } \Gamma,$$

where  $\mathbf{n}$  is the unit normal exterior to  $\Omega_f$ . We introduce the chemical potential  $\mu_\alpha$  of phase  $\alpha$  which, for a single component system is defined as (Callen [16])

$$\mu_\alpha = G_\alpha = A_\alpha + \rho^{-1} p_\alpha, \quad \alpha = l, f, \quad (3.2)$$

where  $G_\alpha$  is the Gibbs free energy of phase  $\alpha$ . Applying the above definition to the vicinal water we have by the chain rule

$$\rho \nabla \mu_l = p_* \nabla \phi_l + \nabla p_l,$$

which when combined with Darcy's law yields

$$\mathbf{q}_l = -\rho K_l (\nabla \mu_l - \mathbf{g}).$$

The above equation emphasizes a well-known classical result which states that the gradient of the chemical potential provides the generalized force for flow of matter, i.e. matter tends to flow from regions of high chemical potential to regions of low chemical potential. Thus in our form of Darcy's law,  $\nabla \mu_l$  is more general than  $\nabla p_l$  since it involves a coupling term  $p_* \nabla \phi_l$  accounting for swelling. This result can be exploited to derive the second boundary condition on  $\Gamma$ . To this end we shall pursue a generalization of the framework of Douglas and Arbogast [24] for single-phase flow in non-swelling fractured media, where continuity of the pressure in the fluid on the interface between the porous blocks and fracture domains is postulated. Since our driving force is the chemical potential, the approach of Douglas and Arbogast can be generalized to swelling media by postulating continuity of the chemical potential on the boundary

$$\mu_l = \mu_f \quad \text{on } \Gamma \quad (3.3)$$

or using (3.2)

$$\rho A_l + p_l = \rho A_f + p_f \quad \text{on } \Gamma. \quad (3.4)$$

Note that this assumption is consistent, but more general than that of Douglas and Arbogast, since it reduces to the continuity of the pressure when swelling is absent, i.e. when  $A_l$  does not depend on  $\phi_l$  so that  $A_l = A_f$ .

The above equation imposes a boundary condition involving the trace of the bulk-phase stress tensor. It should be recalled that we have assumed the vicinal water supports no shear. Thus in addition to (3.4) we assume the following condition for the deviatoric part of  $\mathbf{t}_f$ :

$$\mathbf{d}_f \mathbf{n} \approx \mathbf{d}_s \mathbf{n} \quad \text{on } \Gamma. \quad (3.5)$$

The Stokes problem can be posed in terms of the bulk-fluid velocity relative to the solid phase,  $\mathbf{v}_{f,s}$ . Since  $\mathbf{v}_s$  is zero outside its domain of definition we can simply replace  $\mathbf{v}_f$  by  $\mathbf{v}_{f,s}$  and  $\mathbf{d}_f$  by  $\mathbf{d}_{f,s}$  in (3.1) where  $\mathbf{d}_{f,s}$  denotes the symmetric part of  $\nabla \mathbf{v}_{f,s}$ . In addition, if we define  $\mathbf{t}_{f,s} = -p_f \mathbf{I} + 2\mu^f \mathbf{d}_{f,s}$ , then (3.5) implies that the normal component of the deviatoric part of  $\mathbf{t}_{f,s}$  vanishes on the boundary. Together with (3.4) this gives

$$(\rho A_l + p_l) \mathbf{n} = (\rho A_f \mathbf{I} - \mathbf{t}_{f,s}) \mathbf{n} \quad \text{on } \Gamma, \quad (3.6)$$

which represents continuity of the normal component of Bowen's [13] chemical potential tensor,  $\mu_\beta$ ,

$$\mu_\beta = A_\beta \mathbf{I} - \rho^{-1} \mathbf{t}_\beta,$$

with  $\beta = l$  and  $f, s$ . Because  $A_l = A_f$  when  $\phi_l = 1$ , we may rewrite (3.6) as

$$(\mathbf{t}_{f,s} + p_l \mathbf{I}) \mathbf{n} = \left( \int_\phi^1 p_*(s) \mathbf{I} ds \right) \mathbf{n} \quad \text{on } \Gamma.$$

Together with the above Neumann boundary condition, we shall henceforth consider the Stokes problem posed in terms of  $\{\mathbf{t}_{f,s}, p_f, \mathbf{v}_{f,s}\}$ . To simplify notation we redefine  $\mathbf{t}_f$  and  $\mathbf{v}_f$  as relative to the solid phase and then pose our mesoscopic problem in terms of these variables.

Let  $\Omega_l$  and  $\Omega_f$  denote the vicinal and bulk fluid domains, respectively. Then our mesoscale problem is given by the following system of equations for the five unknowns  $\phi_l, \mathbf{q}_l, \mathbf{v}_f, p_l$  and  $p_f$

$$\operatorname{div} \mathbf{v}_f = 0 \quad \text{in } \Omega_f,$$

$$-\mu^f \Delta \mathbf{v}_f + \nabla p_f = \rho \mathbf{g} \quad \text{in } \Omega_f,$$

$$\frac{\partial \phi_l}{\partial t} + (1 - \phi_l) \operatorname{div} \mathbf{q}_l = 0 \quad \text{in } \Omega_l,$$

$$\mathbf{q}_l = -K_l(\phi_l) [\nabla p_l + p_*(\phi_l) \nabla \phi_l - \rho \mathbf{g}] \quad \text{in } \Omega_l,$$

$$p_l = \phi_l p_*(\phi_l) + \mu_*(\phi_l) \frac{\partial \phi_l}{\partial t} \quad \text{in } \Omega_l,$$

$$\mathbf{q}_l \cdot \mathbf{n} = \mathbf{v}_f \cdot \mathbf{n} \quad \text{on } \Gamma,$$

$$(\mathbf{t}_f + p_l \mathbf{I}) \mathbf{n} = \left( \int_{\phi_l}^1 p_*(s) \mathbf{I} ds \right) \mathbf{n} \quad \text{on } \Gamma,$$

$$\phi_l = \phi_0 \quad \text{in } \Omega_l, t = 0.$$

When the driving force for the vicinal water flow is the gradient of the pressure, our mesoscopic problem is similar to that discussed by Ene and Vernescu [26] for the slow motion of a Newtonian viscous fluid in a non-swelling fissured medium.

#### 4. Two-Scale Asymptotic Expansions

##### 4.1. MACROSCOPIC EQUATIONS FOR A SWELLING MEDIUM WITH A BULK PHASE

By upscaling the mesoscopic problems posited in the previous section we derive governing equations at the macroscale (Fig. 1). We begin by postulating the existence of a periodic reference cell,  $\Omega_0$ , consisting of the union of the vicinal and bulk water domains. In order to determine a macroscopic equivalent description, the asymptotic behavior of the periodic solution of the mesoscopic equations is sought as the period tends to zero. Following the general framework of the homogenization procedure [10, 46] we introduce two lengths characterizing the pore size ( $l$ ) and the observation size ( $L$ ), and their ratio  $\varepsilon = l/L (\varepsilon \ll 1)$ . The transition from the mesoscopic (in which the heterogeneities can be distinguished) to the macroscopic description (in which the dual porosity of the structure is not distinguishable) is carried out by letting  $\varepsilon \rightarrow 0$ . This process is accomplished by considering every property to be of the form  $f(\mathbf{x}, \mathbf{y})$  (where  $\mathbf{x}$  and  $\mathbf{y}$  denote the macroscopic and mesoscopic coordinate respectively with  $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$ ) and then postulating two scale asymptotic expansions for the set  $\mathbf{u}^\varepsilon$  of primary unknowns  $\{\mathbf{q}_l, \mathbf{v}_f, p_l, p_f, \phi_l\}$  in terms of the perturbation parameter  $\varepsilon$

$$\mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \dots$$

with the coefficients  $\mathbf{u}^i$ ,  $\Omega_0$ -periodic in  $\mathbf{y}$ . A difficulty inherent to the swelling of the clay particles is the treatment of the free interface,  $\Gamma$ . One way of overcoming this difficulty, as discussed in [3, 5] when homogenizing immiscible flows, is to assume that the movement of the interfacial boundary is small relative to the movement of the fluid phase. Formally the free boundary problem is treated as a perturbation of order  $\varepsilon$  of a zero order problem posed in fixed domains  $\Omega_f^0$  and  $\Omega_c^0$  separated by a zero order interface  $\Gamma^0$ .

We assume that clay particles and bulk phase are of dimension  $\varepsilon$ , and that all dimensionless quantities associated with the mesoscopic problem are of order  $\varepsilon^0$  except (Auriault *et al.* [6])

$$Q_L = \frac{|\nabla p_f|}{\mu^f |\Delta \mathbf{v}_f|} = \frac{p_f L}{\mu^f v_f} = O(\varepsilon^{-2}).$$

Using this estimate the viscosity term in the Stokes problem is rescaled by  $\varepsilon^2$ . Inserting the above developments into the set of governing mesoscopic equations with the differential operator  $\partial/\partial_x$  replaced by  $\partial/\partial_x + \varepsilon^{-1}\partial/\partial_y$  we obtain, after a

formal matching of the powers of  $\varepsilon$ , successive cell problems. For the bulk water we have

$$\nabla_y p_f^0 = 0 \quad \text{in } \Omega_f, \quad (4.1)$$

$$\mu^f \Delta_y \mathbf{v}_f^0 - \nabla_x p_f^0 - \nabla_y p_f^1 + \rho \mathbf{g} = 0 \quad \text{in } \Omega_f, \quad (4.2)$$

$$\operatorname{div}_y \mathbf{v}_f^0 = 0 \quad \text{in } \Omega_f, \quad (4.3)$$

$$\operatorname{div}_x \mathbf{v}_f^0 + \operatorname{div}_y \mathbf{v}_f^1 = 0 \quad \text{in } \Omega_f, \quad (4.4)$$

and for the vicinal water, after expanding the set of phenomenological coefficients  $\Theta = \{K_l, p_*, \mu_*\}$  about  $\phi_l^0$ , we obtain

$$\nabla_y p_l^0 + p_*^0 \nabla_y \phi_l^0 = 0 \quad \text{in } \Omega_l, \quad (4.5)$$

$$\mathbf{q}_l^0 = -K_l^0 [\nabla_x p_l^0 + \nabla_y p_l^1 + p_*^0 (\nabla_x \phi_l^0 + \nabla_y \phi_l^1) - \rho \mathbf{g}] \quad \text{in } \Omega_l, \quad (4.6)$$

$$p_l^0 = p_*^0 \phi_l^0 + \mu_*^0 \frac{\partial \phi_l^0}{\partial t} \quad \text{in } \Omega_l, \quad (4.7)$$

$$\operatorname{div}_y \mathbf{q}_l^0 = 0 \quad \text{in } \Omega_l, \quad (4.8)$$

$$\frac{\partial \phi_l^0}{\partial t} + (1 - \phi_l^0) (\operatorname{div}_x \mathbf{q}_l^0 + \operatorname{div}_y \mathbf{q}_l^1) = 0 \quad \text{in } \Omega_l, \quad (4.9)$$

along with boundary and initial conditions

$$(\mathbf{q}_l^1 - \mathbf{v}_f^1) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \quad (4.10)$$

$$(\mathbf{t}_f^0 + p_l^0 \mathbf{I}) \mathbf{n} = \left( \int_{\phi_l^0}^1 p_*^0(s) \, ds \right) \mathbf{n} \quad \text{on } \Gamma, \quad (4.11)$$

$$(\mathbf{t}_f^1 + p_l^1 \mathbf{I}) \mathbf{n} = \left( \frac{d}{d\phi_l^0} \int_{\phi_l^0}^1 p_*^0(s) \, ds \right) \phi_l^1 \mathbf{n} = -p_*^0 \phi_l^1 \mathbf{n} \quad \text{on } \Gamma, \quad (4.12)$$

$$\phi_l^0 = \phi_0(\mathbf{x}) \quad \text{in } \Omega_l, t = 0, \quad (4.13)$$

where  $\Theta^0 = \{K_l^0, p_*^0, \mu_*^0\}$  satisfies

$$\Theta = \Theta^0 + \varepsilon \phi_l^1 \frac{\partial \Theta}{\partial \phi_l} (\phi_l^0) + \mathcal{O}(\varepsilon^2).$$



At  $O(\varepsilon^0)$ ,  $\mathbf{t}_f$  has only a pressure component and using (4.1)  $\mathbf{t}_f^0 = -p_f^0(\mathbf{x}, t)\mathbf{I}$ . Combining (4.5) and (4.7) we obtain

$$c^0 \nabla_y \phi_l^0 + \mu_*^0 \nabla_y \frac{\partial \phi_l^0}{\partial t} = 0, \tag{4.14}$$

with the coefficient  $c^0(\phi_l^0, \partial \phi_l^0 / \partial t)$  given by

$$c^0 = \phi_l^0 \frac{dp_*^0}{d\phi_l^0} + 2p_*^0 + \frac{d\mu_*^0}{d\phi_l^0} \frac{\partial \phi_l^0}{\partial t}.$$

Integration of (4.14) leads to

$$\nabla_y \phi_l^0(\cdot, t) = \nabla_y \phi_l^0(\cdot, 0) \exp \left[ - \int_0^t c^0(s) (\mu_*^0(s))^{-1} ds \right]$$

which when combined with assumption (4.13) on the initial data leads to  $\phi_l^0 = \phi_l^0(\mathbf{x}, t)$ . Together with (4.7) this also implies  $p_l^0 = p_l^0(\mathbf{x}, t)$  and therefore from (4.7) and (4.11) we have

$$p_l^0 - p_f^0 = \int_{\phi_l^0}^1 p_*^0(s) ds, \quad p_l^0 = p_*^0 \phi_l^0 + \mu_*^* \frac{\partial \phi_l^0}{\partial t}.$$

We now derive the macroscopic Darcy’s law governing the flow of both the vicinal and bulk water. To this end we analyze the local problems  $\{(4.2), (4.3)\}$  and  $\{(4.6), (4.8)\}$ . Let  $L^2(\Omega_0)$  be the usual Hilbert space of square integrable scalar valued  $\Omega_0$ -periodic functions defined on  $\Omega_i, (i = 0, l, f)$ , equipped with the usual inner product

$$(f, g)_i = \int_{\Omega_i} fg \, d\Omega, \quad i = 0, l, f.$$

Let  $H^1(\Omega_0)$  denote the Sobolev subspace of  $L^2(\Omega_0)$  consisting of  $\Omega_0$ -periodic functions with the derivative  $\partial f$  in the distributional sense satisfying  $\partial f \in L^2(\Omega_0)$  and equipped with the inner product

$$(f, g)_1 = \sum_{|\alpha| \leq 1} \int_{\Omega_0} \partial^\alpha f \partial^\alpha g \, d\Omega.$$

Let the space  $W(\Omega_0)$  be defined by

$$W(\Omega_0) = \{\mathbf{w} \in L^2(\Omega_0), \quad \text{div}_y \mathbf{w} = 0\}.$$

Since the solution,  $\mathbf{v}_f^0$ , of the local problem  $\{(4.2), (4.3)\}$  with Neumann conditions is determined up to an additive constant we also need the quotient space  $W_f(\Omega_f)$

$$W_f(\Omega_f) = \{\mathbf{w}_f \in H^1(\Omega_f), \quad \text{div}_y \mathbf{w}_f = 0, \quad \tilde{\mathbf{w}}_f = 0\}.$$

The weak formulation of the local problems is given by

$$\begin{aligned} & \mu^f(\nabla_y \mathbf{v}_f^0, \nabla_y \mathbf{w}_f)_f \\ &= (\rho \mathbf{g} - \nabla_x p_f^0, \mathbf{w}_f)_f + \int_{\Gamma} \mathbf{t}_f^1 \mathbf{n} \cdot \mathbf{w}_f \, d\Gamma \quad \forall \mathbf{w} \in W_f, \end{aligned} \quad (4.15)$$

$$\begin{aligned} & (K_l^0)^{-1}(\mathbf{q}_l^0, \mathbf{w})_l \\ &= (\rho \mathbf{g} - \nabla_x p_l^0, \mathbf{w})_l - p_*^0(\nabla_x \phi_l^0, \mathbf{w})_l + \\ &+ \int_{\Gamma} (p_l^1 + p_*^0 \phi_l^1) \mathbf{n} \cdot \mathbf{w} \, d\Gamma \quad \forall \mathbf{w} \in W. \end{aligned} \quad (4.16)$$

Let

$$U(\Omega_0, \Omega_f) = \{\mathbf{w} \in W(\Omega_0), \mathbf{w}_f \in W_f(\Omega_f)\}$$

and define the total velocity

$$\mathbf{V} = \begin{cases} \mathbf{v}_f & \text{in } \Omega_f \\ \mathbf{q}_l & \text{in } \Omega_l. \end{cases}$$

By adding (4.15) and (4.16) and using the boundary condition (4.12),  $\mathbf{V}^0$  satisfies the variational problem: Find  $\mathbf{V}^0 \in U$  such that  $\forall \mathbf{w} \in U$

$$\begin{aligned} & (K_l^0)^{-1}(\mathbf{V}^0, \mathbf{w})_l + \mu^f(\nabla_y \mathbf{V}^0, \nabla_y \mathbf{w})_f \\ &= (\rho \mathbf{g}, \mathbf{w})_0 - (\nabla_x p_l^0, \mathbf{w})_l - p_*^0(\nabla_x \phi_l^0, \mathbf{w})_l - (\nabla_x p_f^0, \mathbf{w})_f \end{aligned} \quad (4.17)$$

The left-hand side defines a symmetric coercive bilinear form and thus the existence of a unique solution is an immediate consequence of the Lax–Milgram lemma.

From (3.2) the chemical potential at  $\mathbf{O}(\varepsilon^0)$ ,  $\mu_i^0(\mathbf{x}, t)$ , ( $i = l, f$ ) of both the vicinal and bulk water phases is given by

$$\rho \mu_i^0 = \rho A_i^0 + p_i^0, \quad i = l, f \quad (4.18)$$

where  $A_f^0 = A_f$  is constant and  $A_l^0 = A_l(\phi_l^0)$ . Application of the boundary condition (3.3) at  $\mathbf{O}(\varepsilon^0)$ , implies  $\mu_f^0 = \mu_l^0 = \mu^0$ . Moreover we have from (4.18)

$$\rho \nabla_x \mu^0 = \begin{cases} \nabla_x p_l^0 + p_*^0 \nabla_x \phi_l^0 & \text{in } \Omega_l \\ \nabla_x p_f^0 & \text{in } \Omega_f \end{cases}$$

and therefore (4.17) can be rewritten as

$$\begin{aligned}
 (K_l^0)^{-1}(\mathbf{V}^0, \mathbf{w})_l + \mu^f(\nabla_y \mathbf{V}^0, \nabla_y \mathbf{w})_f \\
 = \rho(\mathbf{g} - \nabla_x \mu^0, \mathbf{w})_0, \quad \forall \mathbf{w} \in U.
 \end{aligned} \tag{4.19}$$

If we define  $\kappa(\mathbf{y})$  as the particular solution of (4.19) corresponding to  $\rho(\mathbf{g} - \nabla_x \mu^0) = \mathbf{I}$  and introduce the mean value operator

$$\tilde{\kappa} = |\Omega_0|^{-1} \int_{\Omega_0} \cdot d\Omega_i(\mathbf{y}), \quad i = l, f,$$

then by linearity we have after averaging

$$\tilde{\mathbf{V}}^0 = -\rho \mathbf{K}(\nabla_x \mu^0 - \mathbf{g}), \quad \text{with } \mathbf{K} = \tilde{\kappa}, \tag{4.20}$$

where

$$\rho \nabla_x \mu^0 = \nabla_x p_l^0 + p_*^0 \nabla_x \phi_l^0 + \nabla_x p_f^0.$$

Equation (4.20) is the macroscale Darcy law which accounts for the flow of both the vicinal and bulk water. The coefficient  $\mathbf{K}$  is a symmetric positive definite tensor which depends on the mesoscale permeability of the clay particles,  $K_l$ , the viscosity of the bulk water,  $\mu^f$ , and the geometry of the cell. This coefficient is identified with the macroscale conductivity of the entire system. It can be determined by averaging the solution of the cell problem (4.19) with a unit source. We remark that this definition differs from the classical one, but is similar to that of Ene and Vernescu [26] for non-swelling fissured media.

Finally, we derive the macroscopic mass balance. To this end we introduce the zero-order volume fraction of the clay particles relative to the cell,  $n^0 = |\Omega_l^0|/|\Omega|$  and formally treated it as a constant within the context of the perturbation analysis of the free boundary problem. Applying the mean value operator to (4.4), coupled with the boundary condition (4.10) and the periodicity assumption yields

$$\begin{aligned}
 \widetilde{\text{div}}_x \mathbf{v}_f^0 &= -|\Omega_0|^{-1} \int_{\Omega} \text{div}_y \mathbf{v}_f^1 d\Omega \\
 &= -|\Omega_0|^{-1} \int_{\Gamma} \mathbf{v}_f^1 \cdot \mathbf{n} d\Gamma = -|\Omega_0|^{-1} \int_{\Gamma} \mathbf{q}_l^1 \cdot \mathbf{n} d\Gamma = \widetilde{\text{div}}_y \mathbf{q}_l^1.
 \end{aligned} \tag{4.21}$$

Averaging (4.9) and using (4.21) gives

$$n^0 \frac{\partial \phi_l^0}{\partial t} + (1 - \phi_l^0)(\text{div}_x \tilde{\mathbf{q}}_l^0 + \text{div}_x \tilde{\mathbf{v}}_f^0) = 0$$

or in terms of the total velocity

$$n^0 \frac{\partial \phi_l^0}{\partial t} + (1 - \phi_l^0) \operatorname{div}_x \widetilde{\mathbf{V}}^0 = 0.$$

After dropping superscripts and subscripts, for a given set of coefficients  $\Theta = \{K_l, p_*, \mu_*\}$ , the system of homogenized equations is

$$n \frac{\partial \phi_l}{\partial t} + (1 - \phi_l) \operatorname{div} \mathbf{V} = 0,$$

$$\mathbf{V} = -K(\nabla p_l + p_* \nabla \phi_l + \nabla p_f - \rho \mathbf{g}),$$

$$p_l - p_f = \int_{\phi}^1 p_*(s) \, ds,$$

$$p_l = p_*(\phi_l) \phi_l + \mu_*(\phi_l) \frac{\partial \phi_l}{\partial t}, \quad (4.22)$$

which can be rewritten as

$$n \frac{\partial \phi_l}{\partial t} + (1 - \phi_l) \operatorname{div} \mathbf{V} = 0,$$

$$\mathbf{V} = -A \nabla \phi_l - B \nabla \frac{\partial \phi_l}{\partial t} + K \rho \mathbf{g}, \quad (4.23)$$

with the coefficients  $A$  and  $B$  given by

$$A = 2K \left( \phi_l \frac{dp_*}{d\phi_l} + 2p_* + \frac{d\mu_*}{d\phi_l} \frac{\partial \phi_l}{\partial t} \right), \quad B = 2K \mu_*.$$

If the gravity term is negligible, the macroscopic form of Darcy's law in (4.23) is analogous to the viscoelastic constitutive equations for the effective stress tensor in rheological models for secondary consolidation and creep of clay (e.g. Barden [8], Gibson and Lo [29]). In the viscoelastic model the constitutive equation for the effective stress tensor is given by a combination of linear or non-linear 'springs' with linear or non-linear 'dashpots' with short time memory. After homogenizing, the mesoscopic viscoelastic form of Darcy's law (2.13) for the vicinal water also governs the flow of vicinal and bulk water at the macroscale, i.e. the macroscopic flow governed by (4.22) and follows the gradient of the chemical potential for the entire system including the vicinal and bulk phase contributions.

#### 4.2. SOLUTION OF THE LINEARIZED MACROSCALE MODEL

We assume that both the clay particles and the bulk water are initially at equilibrium with a constant volume fraction  $\overline{\phi}_l$ . Volume fraction changes, together with their

space and time derivatives are assumed small. Linear terms in the governing equations and quadratic terms in the Helmholtz free energy of the vicinal water are retained. We have

$$\rho A_l = \rho A_0 + \alpha(\phi_l - \bar{\phi}_l) + \frac{\beta}{2}(\phi_l - \bar{\phi}_l)^2,$$

where  $A_0$ ,  $\alpha$  and  $\beta$  are constants depending on  $\bar{\phi}_l$ . Retaining only linear terms, denoting  $\{\bar{K}, \bar{\mu}_*\} = \{K(\bar{\phi}_l), \mu_*(\bar{\phi}_l)\}$  and noting that  $p_*$  is given by

$$p_* = \alpha + \beta(\phi_l - \bar{\phi}_l),$$

our system of linearized homogenized equations is

$$n \frac{\partial \phi}{\partial t} + (1 - \bar{\phi}_l) \operatorname{div} \mathbf{V} = 0,$$

$$\mathbf{V} = -\bar{K}(\nabla p_l + \alpha \nabla \phi_l + \nabla p_f - \rho \mathbf{g}),$$

$$p_l - p_f = \int_{\phi_l}^1 p_*(s) \, ds - \alpha(\phi_l - \bar{\phi}_l),$$

$$p_l = \alpha \bar{\phi}_l + (\alpha + \beta \bar{\phi}_l)(\phi_l - \bar{\phi}_l) + \bar{\mu}_* \frac{\partial \phi_l}{\partial t}. \quad (4.24)$$

We now present a generalized solution of the above linearized system. Consider the macroscopic clay soil domain  $\Omega$  and without loss of generality, assume homogeneous Dirichlet boundary conditions on  $\partial\Omega$ . Then (4.24) can be written as

$$\bar{a} \Delta \phi_l + \bar{b} \Delta \frac{\partial \phi_l}{\partial t} = n \frac{\partial \phi_l}{\partial t} \quad \text{in } \Omega,$$

$$\phi_l = 0 \quad \text{on } \partial\Omega,$$

$$\phi_l = \phi_0 \quad \text{in } \Omega, t = 0, \quad (4.25)$$

where

$$\bar{a} = 2\bar{K}(1 - \bar{\phi}_l)(2\alpha + \beta\bar{\phi}_l), \quad \bar{b} = 2\bar{K}(1 - \bar{\phi}_l)\bar{\mu}_*.$$

The above system is the linear Kelvin–Voigt viscoelasticity problem with an additional external damping term in the right-hand side. The existence of a generalized solution of linear viscoelastic problems is usually obtained within the framework of the semigroup theory (Pazy [43], Sanchez-Palencia [46], Yosida [50]). Define

$$V = H_0^1(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \partial\Omega\}$$

and the symmetric bilinear forms

$$a(u, v) = \bar{a}(\nabla u, \nabla v), \quad b(u, v) = \bar{b}(\nabla u, \nabla v) + n(u, v), \quad \forall \{u, v\} \in V^2$$

where  $(\cdot)$  denoting the inner product in  $L^2(\Omega)$ . The weak form of (4.25) is: Find  $\phi_l \in V$  such that

$$a(\phi_l, v) + b\left(\frac{\partial \phi_l}{\partial t}, v\right) = 0, \quad \forall v \in V. \quad (4.26)$$

Clearly the forms  $a$  and  $b$  are continuous and coercive. Using the Riesz representation theorem define the linear self-adjoint, bounded and coercive operator  $A: V \rightarrow V$  such that

$$b(A\phi_l, v) = -a(\phi_l, v), \quad \forall v \in V,$$

so that (4.26) is given by

$$b\left(\frac{\partial \phi_l}{\partial t} - A\phi_l, v\right) = 0, \quad \forall v \in V.$$

Together with the initial condition this can be rewritten as an evolution problem of the form

$$\frac{\partial \phi_l}{\partial t} = A\phi_l, \quad \phi_l(\cdot, 0) = \phi_0. \quad (4.27)$$

Since  $A$  is a bounded operator in  $V$ ,  $A$  is the generator of the uniformly continuous semigroup,  $T(t) = e^{tA}$  (Pazy [43]). Moreover,  $T(t)\phi_0$  defines a strongly continuous function on  $V$  which is the solution of (4.27).

## 5. Application to One-Dimensional Consolidation of Clay

In this section we apply the macroscopic model of Section 4 to the classical one-dimensional consolidation problem described by Terzaghi [48]. The problem consists of a porous column of length  $L$  bounded by rigid, impermeable walls, except on its top where it is loaded by a pressure  $p_0$  and is free to drain. The clay particles are assumed partially swollen so that the overburden pressure equals that in the vicinal water. In one-dimensional consolidation the displacement of the porous medium can be calculated using the mass balance of the solid phase. Denoting atmospheric pressure by  $p_{\text{atm}}$  and for simplicity dropping the gravity term, the one-dimensional simplification of (4.22) is

$$n \frac{\partial \phi_l}{\partial t} + (1 - \phi_l) \frac{\partial V}{\partial x} = 0,$$

$$V = -K \left( \frac{\partial p_l}{\partial x} + p_* \frac{\partial \phi}{\partial x} + \frac{\partial p_f}{\partial x} \right),$$

$$p_l = p_* \phi_l + \mu_* \frac{\partial \phi_l}{\partial t},$$

$$p_l - p_f = \int_{\phi}^1 p_*(s) \, ds,$$

with boundary conditions

$$p_l = p_0 \quad \text{on } x = 0,$$

$$p_f = p_{\text{atm}} \quad \text{on } x = 0, \quad (5.1)$$

$$V = 0 \quad \text{on } x = l,$$

and initial condition

$$\phi_l = \phi_0 \quad \text{in } (0, L), t = 0. \quad (5.2)$$

### 5.1. EQUILIBRIUM SOLUTION

We show that the equilibrium solution of the above system at steady state reproduces Low's [41] exponential swelling relation. Setting  $\partial \phi_l / \partial t = 0$ , we have

$$\mathbf{V} = 0,$$

$$\frac{\partial p_l}{\partial x} + p_* \frac{\partial \phi_l}{\partial x} + \frac{\partial p_f}{\partial x} = 0, \quad (5.3)$$

$$p_l = p_* \phi_l, \quad (5.4)$$

$$p_l - p_f = \int_{\phi_l}^1 p_*(s) \, ds. \quad (5.5)$$

From (5.5)

$$\frac{\partial p_f}{\partial x} = \frac{\partial p_l}{\partial x} + p_* \frac{\partial \phi_l}{\partial x},$$

which when combined with (5.3) leads to

$$\frac{\partial p_l}{\partial x} + p_* \frac{\partial \phi_l}{\partial x} = 0.$$

Hence, using (5.4) we have

$$\phi_l \frac{\partial p_l}{\partial x} + p_l \frac{\partial \phi_l}{\partial x} = \frac{\partial(p_l \phi_l)}{\partial x} = 0,$$

which after integration yields

$$p_l \phi_l = p_*(\phi_l)^2 = C,$$

where  $C$  is a constant. Hence,  $\phi_l, p_l$  and  $p_f$  are constants with  $p_l = p_0$  and  $p_f = p_{\text{atm}}$ . To solve for  $\phi_l$  we have, from (5.4) and (5.5),

$$\frac{dp_l}{d\phi_l} = -p_* = -\frac{p_l}{\phi_l} \quad (5.6)$$

which after integration and using the condition  $p_l = p_f = p_{\text{atm}}$  at  $\phi_l = 1$  yields

$$p_l = \frac{p_{\text{atm}}}{\phi_l}. \quad (5.7)$$

When the clay consists of flat platelets, the swelling of the system is dictated by the separation of the platelets,  $\lambda$ . If we denote the effective thickness of a clay platelet by  $\lambda_s$ , the volume fraction is given by

$$\phi_l = \frac{\lambda}{\lambda + \lambda_s}.$$

On rewriting (5.6) in terms of the separation between platelets, we find

$$\left( \frac{\lambda^2}{\lambda_s} + \lambda \right) \frac{\partial p_l}{\partial \lambda} = -p_l.$$

At high moisture contents,  $\lambda \ll \lambda^2/\lambda_s$  and, hence, after integrating we obtain

$$p_l = p_f \exp(\lambda_s/\lambda).$$

The swelling pressure  $\Pi$  is given by

$$\Pi \equiv p_l - p_f = p_f(\exp(\lambda_s/\lambda) - 1).$$

The above relation for the swelling pressure is identical to that obtained empirically by Low [41] at equilibrium. We also remark that Achanta *et al.* [2] obtained the identical result at the mesoscale by applying the hybrid mixture theory to the clay particles and assuming compressibility. Here we generalize the approach of Achanta *et al.*, where we show that Low's result also obtains from the macroscopic problem with incompressibility assumptions. To the authors' knowledge it is



the first time a model for one-dimensional consolidation reproduces the classical swelling pressure result at equilibrium.

## 5.2. SOLUTION OF THE LINEARIZED EVOLUTION PROBLEM

The transient solution of the one-dimensional consolidation example is now derived by considering the problem governed by the following simplified form of the linearized system (4.24)

$$n \frac{\partial \phi_l}{\partial t} + (1 - \bar{\phi}_l) \frac{\partial V}{\partial x} = 0,$$

$$V = -\bar{K} \left( \frac{\partial p_l}{\partial x} + \alpha \frac{\partial \phi_l}{\partial x} + \frac{\partial p_f}{\partial x} \right),$$

$$p_l - p_f = \int_{\bar{\phi}_l}^1 \alpha(s) ds - \alpha(\phi_l - \bar{\phi}_l),$$

$$p_l = \alpha \bar{\phi}_l + (\alpha + \beta \bar{\phi}_l)(\phi_l - \bar{\phi}_l) + \bar{\mu}_* \frac{\partial \phi_l}{\partial t},$$

together with the boundary and initial conditions (5.1) and (5.2). Setting  $\bar{\phi}_l = \phi_l^\infty$  where the superscript  $\infty$  denotes the steady state solution, from (5.4) and (5.7),  $p_0 = \alpha \phi_l^\infty$  and  $\phi_l^\infty = p_{\text{atm}}/p_0$ . Introducing the dimensionless quantities

$$x^* = \frac{x}{L}, \quad S = \frac{2\bar{K}\bar{\mu}_*(1 - \bar{\phi}_l)}{nL^2}, \quad t^* = Rt,$$

where

$$R = \frac{2\bar{K}(1 - \bar{\phi}_l)(2\alpha + \beta \bar{\phi}_l)}{nL^2},$$

the above problem can be rewritten as

$$\frac{\partial \phi_l}{\partial t^*} = \frac{\partial^2 \phi_l}{\partial x^{*2}} + S \frac{\partial^3 \phi_l}{\partial x^{*2} \partial t^*}.$$

The solution admits the following Fourier series expansion

$$\phi_l = \frac{p_{\text{atm}}}{p_0} + \sum_{n=0}^{\infty} \frac{2\phi_0}{M} \sin(Mx^*) \exp\left(-\frac{M^2 t^*}{1 + SM^2}\right),$$

$$p_l = p_0 + \sum_{n=0}^{\infty} \frac{2\phi_0}{M} \left( \bar{f} - \frac{\bar{\mu}_* M^2 R}{1 + SM^2} \right) \sin(Mx^*) \exp\left(-\frac{M^2 t^*}{1 + SM^2}\right),$$

$$p_f = p_{\text{atm}} + \alpha \sum_{n=0}^{\infty} \frac{2\phi_0}{M} \left( \bar{g} - \frac{\bar{\mu}_* M^2 R}{1 + SM^2} \right) \sin(Mx^*) \exp\left(-\frac{M^2 t^*}{1 + SM^2}\right),$$

$$V = -4\bar{K} \sum_{n=0}^{\infty} \phi_0 \left( \bar{g} - \frac{\bar{\mu}_* M^2 R}{1 + SM^2} \right) \cos(Mx^*) \exp\left(-\frac{M^2 t^*}{1 + SM^2}\right),$$

where

$$M = \frac{\Pi(2n+1)}{2}, \quad \bar{f} = \alpha + \frac{\beta p_0}{\alpha}, \quad \bar{g} = 2\alpha + \frac{\beta p_0}{\alpha}.$$

The displacement  $U_s$  of the solid can be calculated in a post-processing scheme using the linearized macroscopic continuity equation for the solid phase,

$$n \frac{\partial \phi_s}{\partial t} + \bar{\phi}_s \frac{\partial^2 U_s}{\partial x \partial t} = 0,$$

with  $\bar{\phi}_s = 1 - \bar{\phi}_l$ . Hence, we have

$$n \frac{\partial \phi_l}{\partial t} = (1 - \bar{\phi}_l) \frac{\partial^2 U_s}{\partial t \partial x},$$

which can be expressed in terms of the dimensionless quantities and the corresponding dimensionless displacement  $U_s^* = U_s(p_0 - p_{\text{atm}})/nLp_0$  as

$$\frac{\partial^2 U_s^*}{\partial t^* \partial x^*} = \frac{\partial \phi_l}{\partial t^*}.$$

Together with the condition  $U_s^*(x, 0) = 0$  this yields the solution

$$U_s^* = \sum_{n=0}^{\infty} \frac{2\phi_0}{M^2} \cos(Mx^*) \left[ 1 - \exp\left(-\frac{M^2 t^*}{1 + SM^2}\right) \right].$$

Figure 2 shows different evolutions in time of the normalized displacement  $U^n(0, t)$  with respect to the steady state solution ( $U^n(0, t) = U_s^*(0, t)/U_s^*(0, \infty)$ ) parametrized by the number  $S$ . Terzaghi's solution of the parabolic problem is recovered from the viscoelastic problem when  $S \rightarrow 0$ . Due to the viscous relaxation term, as  $S$  increases,  $U_s^*$  approaches  $U_\infty$  less rapidly than would be expected from the Terzaghi theory. This phenomenon is extensively discussed in the literature on secondary consolidation and has been modeled using different rheological models (e.g. Gibson and Lo [29]).

## 6. Remarks on Evaluating the Phenomenological Coefficients

In our model there are three phenomenological coefficients which must be experimentally determined,  $\mu_*$ ,  $K_l$ , and  $p_*$ . Of these coefficients  $K_l$  and  $\mu_*$  arise when

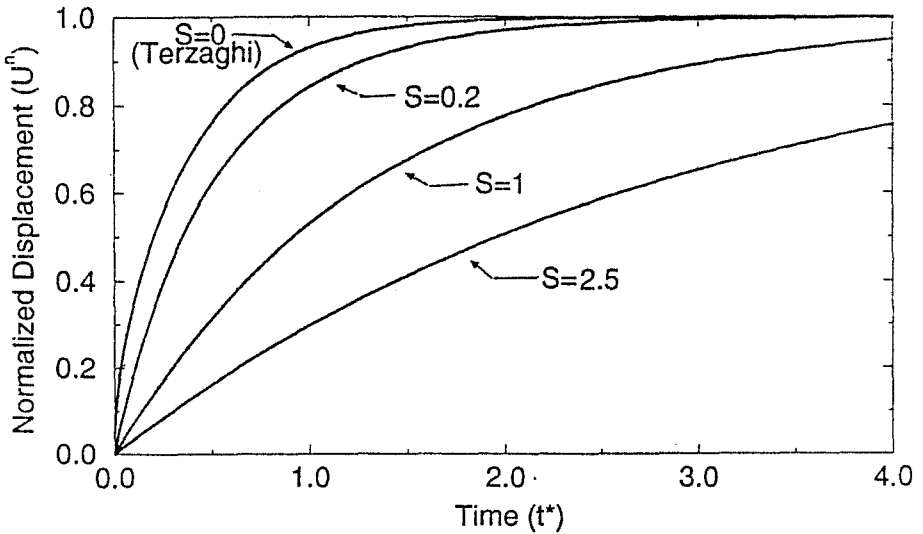


Fig. 2. Evolution in time of the normalized displacement.

linearizing the residual entropy inequality (2.8) about equilibrium. They can be evaluated from Darcy’s law and the topological law (2.11) and (2.12) for  $\mathbf{q}_l$  and  $D_s \phi_l / Dt$  respectively. The latter coefficient can be measured in a non-equilibrium swelling pressure experiment. Both of these phenomenological coefficients require knowledge of  $p_*$ . Thus one must first obtain  $p_*$ , which is given by  $\rho_l \partial A_l / \partial \phi$  where  $A_l = A_l(\phi_l)$  for the incompressible vicinal fluid. Since  $A_l$  may be multivalued, determining the constitutive form of  $A_l$  is not as simple as it may seem. Because we linearized about equilibrium, which is an unknown state in this problem,  $p_*$  is dependent on the choice of the equilibrium reference configuration (for the consolidation problem equilibrium was taken as the steady-state configuration). As a first approximation  $p_*$  can be evaluated using the equilibrium swelling pressure relationship. In this case  $p_* = p_{atm} / \phi_l^2$ .

**7. Conclusions**

We have presented a multi-scale theory for flow in swelling media. Vicinal and bulk waters were treated as distinct phases. Swelling at the mesoscale was studied within the framework of hybrid mixture theory. Upscaling the mesoscopic equations for the particles and bulk phase within the framework of homogenization led to Darcy’s law at the macroscale. Darcy’s law states that the flow of vicinal and bulk phases is driven by a gradient of the chemical potential for the entire system including both vicinal and bulk-phases contributions.

A simplified theory was applied to the classical one-dimensional consolidation problem. It was shown that the steady-state solution of this problem reproduces

Low's swelling-pressure relationship between the vicinal and bulk-phase pressures at equilibrium. Such a result is novel for consolidation problems.

Extensions of the theory to multi-dimensional consolidation problems which include particle shearing can be obtained by combining the present approach with a more general description of the solid matrix. This can be accomplished by postulating more general sets of independent variables in the hybrid-mixture theory.

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