Optical and Quantum Electronics **11** *(1979) 485–495*

Intensity fluctuations in a ring laser taking into consideration a spatial population-inversion grating

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Received 25 April 1979

The correlation functions of the intensity fluctuations are calculated by use of a linearization procedure for the equations of motion which include the coupling of the counter-rotating travelling waves of the ring laser in the cases of Doppler broadening, of homogeneous broadening with a self-induced population-inversion grating, and in intermediate cases. Depending on the strength of the mode competition various stable stationary solutions exist for the amplitudes. The transition between these stable states shows phenomena closely resembling phase transitions such as critical fluctuations, critical slowing down, etc. In particular, when the system passes from the state where both modes are above threshold to the state where one mode is below threshold, the negative cross-correlation of the fluctuations also becomes critical. In the case of the transient behaviour of the ring laser in the unstable region the time development of the amplitudes and the correlation functions are described in a short-time approximation.

1. Introduction

In contrast to a laser with a Fabry–Perot resonator the counter-running waves of a single-frequency ring laser are only coupled due to the active medium. The strength of this coupling depends on the spectral characteristic of the amplifying molecules. In the case of a spectrally inhomogeneously broadened line, e.g. in gas lasers, the coupling of the counter-running waves is determined by the common gain reservoir of both waves. In a gas laser with Doppler-broadened gain this common reservoir depends on the detuning of the laser frequency from the line centre. Far from the line centre each of the counter-running waves interacts with an ensemble of molecules of different velocity and the coupling is weak. Near the line centre the velocity groups of molecules interacting with the two waves overlap and a strong coupling between the counter-running waves results. In the case of a spectrally homogeneously broadened line both running waves are amplified by the same active molecules. If the molecules are at rest an additional coupling exists. The two oppositely directed running waves comprising the electric field add in the medium, yielding a standing wave, and burn spatial holes in the population inversion. The periodically modulated population-inversion grating acts like a Bragg grating to reflect one running wave back into the other [1]. For stationary atoms this grating is precisely out of phase with the standing wave pattern and destructive interference occurring between the reflected wave and the similarly directed running wave lead to an additional mode competition. This strong coupling significantly influences the mode spectrum of a ring laser as well as that of a Fabry–Perot laser [2–5].

Especially in a single-frequency ring laser unidirectional operation is caused by this strong coupling [1]. At the line centre in a gas laser also a spatial population-inversion grating occurs [9]. The contribution of Bragg scattering due to this grating to the mode competition depends on the ratio of the dipole decay constant γ determining the linewidth of the laser transition and the level decay constant Γ. In many gas lasers where $\Gamma \ll \gamma$ this contribution is negligibly small (e.g. in a He–Ne laser $\Gamma \simeq 15$ MHz,

0306–8919/79/060485–11$03.10/0

$\gamma \simeq 150$ MHz). The influence of the coupling between the counter-running waves in a gas ring laser on their fluctuations and correlations was treated in full detail by Tehrani and Mandel [6, 7]. They neglected the Bragg-scattering contribution at line centre and showed that the competition of the oppositely directed waves in a Doppler-broadened gain medium leads to negative correlations between their intensity fluctuations, whose magnitude depends on the detuning of the laser frequency from the line centre. At the line centre the relative intensity fluctuations do not die out in general as the pump parameter is increased, and the emitted light does not become fully coherent as in a Fabry–Perot laser. The aim of this paper is to take into account the influence of the strong coupling of the oppositely direct waves due to the population-inversion grating on the intensity fluctuations and their correlation. In Section 2, the basic equations are derived for a ring laser with homogeneously broadened gain. They take account of the fact that drift and diffusion of excitation energy can partially wash out the spatial holes burnt into the population inversion by the standing wave pattern [8]. The equations obtained there are similar to the ones which are the starting point in [6], but the coupling constant appearing there has a different physical content and range of values. However, this permits one to compare directly the effect of the two kinds of coupling as limiting cases and include intermediate cases. The basic equations are linearized with respect to the fluctuations and the solution of the linearized equations is given. Of course, this approximation restricts the size of the fluctuations which can be treated. On the other hand it is possible to obtain analytical results which show the influence of the mode competition on the fluctuations and their correlations and enables the behaviour of these quantities at the transition between various stable states of the ring laser to be considered. This is done from the point of view of co-operative effects [13, 14] in Section 3. Using a short-time approximation the time development of the fluctuations and their cross-correlation is considered in Section 4 for the transient behaviour of the ring laser from an unstable state.

2. Laser equations

A system of two level atoms which are in resonance with a cavity field consisting of two oppositely directed running waves of the same frequency ω is treated with a single transition frequency ω_0

$$A(z, t) = \sqrt{\left(\frac{\hbar}{2V\epsilon_0\omega}\right)}[b_1 e^{-i(\omega t-kz)} + b_1^+ e^{i(\omega t-kz)} + b_2 e^{-i(\omega t+kz)} + b_2^+ e^{+i(\omega t+kz)}]. \tag{1}$$

The field described by the vector potential $A(z, t)$ is assumed to be linearly polarized and directed along the z-axis. Applying the equations of motion for the light field and atoms [10] to this field we obtain the field equations

$$\dot{b}_n^+ = -\kappa_n b_n^+ + ig \sum_\mu a_\mu^+ e^{ikz_\mu} + F_{\kappa_n}^+;\ n = 1, 2 \tag{2}$$

and the matter equations

$$\dot{a}_\mu^+ = (i\omega_0 - \gamma)a_\mu^+ - ig^*[b_1^+ e^{i(\omega t-kz_\mu)} + b_2^+ e^{i(\omega t+kz_\mu)}]\, s_\mu + (F_\gamma^{(\mu)})^+ \tag{3}$$

$$\begin{aligned}\dot{s}_\mu = & -\Gamma(s_\mu - s_0) - 2i\{g a_\mu^+[b_1 e^{-i(\omega t-kz_\mu)} + b_2 e^{-i(\omega t+kz_\mu)}] - g^* a_\mu[b_1^+ e^{i(\omega t-kz_\mu)} \\ & + b_2^+ e^{i(\omega t+kz_\mu)}]\} + F_\Gamma^{(\mu)},\end{aligned} \tag{4}$$

where b_n ($n = 1, 2$) are the amplitudes of the counter-running waves varying slowly with time, κ_n is the cavity linewidth for the nth mode, k the wave number, z_μ the position of the μth atom, a_μ the operator of the transition of the μth atom, s_μ the inversion operator, s_0 the operator of inversion without a lasing field, V the normalization volume and g is the coupling coefficient of the interaction between field and atoms and is given in mks units by

$$g = -i(2V\hbar\omega\epsilon_0)^{-1}\,\omega_0 P \tag{5}$$

where P is the dipole matrix element of the transition. F_{κ_n}, $F_\gamma^{(\mu)}$ and $F_\Gamma^{(\mu)}$ are the fluctuating forces due to the cavity losses κ_n, the spontaneous emission described by the dipole decay constant γ, and the relaxation of the inversion described by the level decay constant Γ respectively. These fluctuating forces are assumed to be Markoffian and Gaussian. The random forces acting on different atoms are assumed to be independent. In order to eliminate the atomic variables in Equation 2 the following assumptions are made: (a) The relaxation times of the atomic system are short compared to all other times of the system. In this case the atomic variables, i.e. the dipole moment and the inversion, follow the motion of the field adiabatically. (b) Only the fluctuations due to the spontaneous emission are taken into account. F_{κ_n} and $F_\Gamma^{(\mu)}$ and the effect of $F_\gamma^{(\mu)}$ on the inversion are neglected. Assuming exact resonance $\omega = \omega_0$ and not too high photon numbers we obtain the population inversion from Equations 3 and 4:

$$s_\mu = s_0\left[1 - \frac{4|g|^2}{\Gamma\gamma}(b_1^+b_1 + b_2^+b_2 + b_2^+b_1 e^{i2kz_\mu} + b_1^+b_2 e^{-2ikz_\mu})\right]. \tag{6}$$

This formula describes the inversion saturation. The inversion s_0 caused by pumping and decay processes is lowered by the intensity of both modes. It is obvious that the saturated inversion is spatially modulated due to the standing wave pattern. At this point we want to include the fact that drift and diffusion of excitation energy can wash out this population-inversion grating. Therefore the terms in Equation 6 containing spatial dependence are multiplied by a factor which can vary from zero (i.e. the inversion grating is washed out completely) to one (i.e. the spatial grating has the highest possible amplitude). Then Equation 6 becomes

$$s_\mu = s_0\left\{1 - \frac{4|g|^2}{\gamma\Gamma}[b_1^+b_1 + b_2^+b_2 + (\epsilon - 1)(b_2^+b_1 e^{2ikz_\mu} + b_1^+b_2 e^{-2ikz_\mu})]\right\} \tag{7}$$

where $1 \leqslant \epsilon \leqslant 2$. By inserting Equation 7 into Equation 3 the atomic variables in Equation 2 can be eliminated by means of the adiabatic approximation in the usual way (see e.g. [10]). Replacing the operators by the corresponding classical quantities, e.g. the photon operators by the complex field amplitudes $b_n^+, b_n \to \beta_n^*, \beta_n$ Equation 2 takes the form

$$\dot{\beta}_1^* - [a_1 - d(|\beta_1|^2 + \epsilon|\beta_2|^2)]\beta_1^* = F_1^*(t) \tag{8}$$

$$\dot{\beta}_2^* - [a_2 - d(|\beta_2|^2 + \epsilon|\beta_1|^2)]\beta_2^* = F_2^*(t) \tag{9}$$

where $a_i = \kappa_i(N_0/N_{\text{thr}}^{(i)} - 1)$ is the pump parameter, $N_{\text{thr}}^{(i)} = \kappa_i\gamma/|g|^2$ the threshold inversion, $N_0 = Ns_0$ the unsaturated inversion, N the number of active atoms and $d = 4N_0|g|^4/\gamma^2\Gamma$. The fluctuating forces

$$F_j(t) = \frac{ig}{\gamma}\sum_\mu e^{\pm kz_\mu}F_\gamma^{(\mu)}(t)$$

(+ sign corresponds to $j = 1$ and $-$ to $j = 2$) have the property

$$\langle F_i(t)F_j^*(t')\rangle = 4q\delta_{ij}\delta(t-t') \tag{10}$$

where $q = |g|^2N/4\gamma$ is the noise strength. Equations 8 and 9 derived for an active medium with a homogeneously broadened laser line are the starting point for the following considerations. The parameter ϵ which is a measure of the amplitude of the spatial grating describes the coupling between the counter-running waves due to the Bragg scattering at the population-inversion grating. When deriving equations of motion of the complex field amplitudes for an active medium with a Doppler-broadened line one obtains equations similar to Equations 8 and 9 [11]. The coupling parameter occurring there depends on the detuning of the laser frequency from the atomic line centre and is a measure for overlapping of the holes burnt into the velocity distribution of the population inversion by the counter-running waves. It is given by

$$\epsilon_D = \left[1 + \left(\frac{\omega - \omega_0}{\gamma}\right)^2\right]^{-1} \tag{11}$$

and varies from zero to one. Equations 8 and 9 with Equation 11 are the starting point of [6, 7]. However, as mentioned above, in a gas laser tuned at line centre a spatial inversion grating also occurs [9]. If the influence of this grating must be taken into consideration, and this is the case if $\gamma \simeq \Gamma$, it may be expected that this can formally be done by choosing the following range for the coupling parameter

$$0 \leqslant \epsilon \leqslant \xi \tag{12}$$

where $1 < \xi \leqslant 2$. That means that Equation 8 and 9 formally describe a ring laser with an active medium of three possible spectral characteristics: (a) a Doppler-broadened laser line, i.e. $0 \leqslant \epsilon \leqslant 1$, (b) a homogeneous broadened line with a self-induced spatial grating of various amplitudes, i.e. $1 \leqslant \epsilon \leqslant 2$, and (c) the intermediate case of a Doppler-broadened line and a spatial grating at line centre, i.e. ϵ is given by Equation 12.

In order to solve Equations 8 and 9 approximately we use the linearization method [12]. A condition for applying this method is that the fluctuations are small enough to permit us to retain only linear terms involving deviations from the unperturbed quantities. This condition restricts the size of the fluctuations but the method enables analytical results to be obtained which give a good insight into the most essential effects. We use the decomposition

$$\beta_j(t) = [\alpha_j(t) + \delta_j(t)]\, e^{i\varphi_j(t)} \tag{13}$$

where $\alpha_j(t)$ are the smoothly varying averaged amplitudes, $\varphi_j(t)$ the phase which can still refer to the noise source [10] and $\delta_j(t)$ small deviations caused by the noise source. Inserting Equation 13 into Equations 8 and 9 leads to equations describing smooth variations of the averaged amplitudes

$$\dot{\alpha}_1 - [a_1 - d(\alpha_1^2 + \epsilon\alpha_2^2)]\,\alpha_1 = 0 \tag{14}$$

$$\dot{\alpha}_2 - [a_2 - d(\alpha_2^2 + \epsilon\alpha_1^2)]\,\alpha_2 = 0 \tag{15}$$

and equations linear in $\delta_i(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} + \begin{pmatrix}2d\alpha_1^2 - \Delta_1; & 2d\epsilon\alpha_1\alpha_2\\ 2d\epsilon\alpha_1\alpha_2; & 2d\alpha_2^2 - \Delta_2\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix} = \begin{pmatrix}N_1\\ N_2\end{pmatrix} \tag{16}$$

where $\Delta_{1,2} = a_{1,2} - d(\alpha_{1,2}^2 + \epsilon\alpha_{2,1}^2)$ and $N_i = \mathrm{Re}\,(e^{-i\varphi_i}F_i)$. Assuming that $e^{-i\varphi_i}$ can be absorbed into the fluctuation forces as a new phase factor [10] the random forces N_i have the property

$$\langle N_i(t)N_j(t')\rangle = 2q\delta_{ij}\delta(t-t'). \tag{17}$$

The equations for φ_i which are dropped here lead to the well-known phase diffusion [10].

3. Moments and correlation functions of the intensity fluctuations for stationary amplitudes

Equations 14 and 15 have two stationary solutions. (a) Both amplitudes are non-vanishing and are given by

$$\alpha_1^2 = \frac{1}{d}\frac{a_1 - \epsilon a_2}{1-\epsilon^2}; \quad \alpha_2^2 = \frac{1}{d}\frac{a_2 - \epsilon a_1}{1-\epsilon^2}. \tag{18}$$

(b) One of the amplitudes becomes zero:

$$\alpha_1^2 = \frac{a_1}{d}; \quad \alpha_2^2 = 0 \quad \text{or} \quad \alpha_1^2 = 0; \quad \alpha_2^2 = \frac{a_2}{d}. \tag{19}$$

Inserting first the stationary solution (Equation 18) into Equation 16 and taking into consideration that $\Delta_i = 0$ we obtain linear differential equations with constant coefficients yielding the following solution

$$\delta_i(t) = \sum_{j=1,2}\int_0^t \mathrm{d}s\,G_{ij}(t-s)N_j(s) \tag{20}$$

where

$$G_{ij}(t-s) = g_{ij}e^{\lambda_1(t-s)} + h_{ij}e^{\lambda_2(t-s)},$$

$$g_{ij} = \frac{d}{2\sqrt{D}}\begin{pmatrix} \alpha_2^2-\alpha_1^2+\sqrt{D}/d, & -2\epsilon\alpha_1\alpha_2 \\ -2\epsilon\alpha_1\alpha_2, & \alpha_1^2-\alpha_2^2+\sqrt{D}/d \end{pmatrix}$$

$$h_{ij} = \frac{d}{2\sqrt{D}}\begin{pmatrix} \alpha_1^2-\alpha_2^2+\sqrt{D}/d, & 2\epsilon\alpha_1\alpha_2 \\ 2\epsilon\alpha_1\alpha_2, & \alpha_2^2-\alpha_1^2+\sqrt{D}/d \end{pmatrix}$$

$$\lambda_{1,2} = -d(\alpha_1^2+\alpha_2^2) \pm \sqrt{D}$$

and

$$D = d^2[\alpha_1^4+\alpha_2^4+2\alpha_1^2\alpha_2^2(2\epsilon^2-1)]. \tag{21}$$

α_i is given by Equation 18 and $\delta_i(0)=0$ is assumed as the initial value. Using the result (Equation 20) with Equations 18 and 21 the normalized correlation and cross-correlation functions of the intensity fluctuations

$$\mu_{ij}(t,\tau) = \frac{\langle \Delta I_i(t)\Delta I_j(t+\tau)\rangle}{\langle I_i\rangle\langle I_j\rangle} = \frac{4\langle\delta_i(t)\delta_j(t+\tau)\rangle}{\alpha_i\alpha_j} \tag{22}$$

can be calculated. Here we have written $\langle I_i\rangle = \alpha_i^2$; $\Delta I_i(t) = I_i - \langle I_i\rangle \simeq 2\alpha_i\delta_i$. Taking into account the property of the fluctuation forces (Equation 17), Equation 22 with 20 takes the form

$$\begin{aligned}\mu_{11}(t,\tau) = {} & \frac{qd(1-\epsilon^2)^2}{2(a_1-\epsilon a_2)^2(a_2-\epsilon a_1)}\Bigg(\exp\left[\left(-\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)\tau\right]\left\{1-\exp\left[2\left(-\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)t\right]\right\} \\ & \times\left\{\sqrt{D}+\frac{1}{1-\epsilon^2}\left(-\frac{a_1^2-a_2^2}{\sqrt{D}}+2(a_2-\epsilon a_1)\right)\right\}+\exp\left[-\left(\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)\tau\right] \\ & \times\left\{1-\exp\left[-2\left(\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)\right]\right\}\left\{-\sqrt{D}+\frac{1}{1-\epsilon^2}\left[\frac{a_1^2-a_2^2}{\sqrt{D}}+2(a_2-\epsilon a_1)\right]\right\}\Bigg)\end{aligned} \tag{23}$$

$$\begin{aligned}\mu_{12}(t,\tau) = {} & -\frac{qd(1-\epsilon)\epsilon}{(a_1-\epsilon a_2)(a_2-\epsilon a_1)}\Bigg(\left(\frac{a_1+a_2}{\sqrt{D}}+\epsilon+1\right)\exp\left[-\left(\frac{a_1+a_2}{1+\epsilon}-\sqrt{D}\right)\tau\right] \\ & \times\left\{1-\exp\left[-2\left(\frac{a_1+a_2}{1+\epsilon}-\sqrt{D}\right)t\right]\right\}+\left(-\frac{a_1+a_2}{\sqrt{D}}+\epsilon+1\right)\exp\left[-\left(\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)\tau\right] \\ & \times\left\{1-\exp\left[-2\left(\frac{a_1+a_2}{1+\epsilon}+\sqrt{D}\right)t\right]\right\}\Bigg).\end{aligned} \tag{24}$$

$\mu_{22}(t,\tau)$ is obtained from Equation 23 by exchanging $1 \leftrightarrow 2$. For the case of equal pump rates $a_1 = a_2 = a$, Equations 23 and 24 take the form

$$\begin{aligned}\mu_{11}(t,\tau) = \mu_{22}(t,\tau) = {} & \frac{qd(1+\epsilon)}{a^2(1-\epsilon)}\Bigg((\epsilon+1)\exp\left[-\frac{2a(1-\epsilon)}{1+\epsilon}\tau\right]\left\{1-\exp\left[-\frac{4a(1-\epsilon)}{1+\epsilon}t\right]\right\} \\ & +(1-\epsilon)e^{-2a\tau}(1-e^{-4a\tau})\Bigg)\end{aligned} \tag{25}$$

$$\begin{aligned}\mu_{12}(t,\tau) = {} & -\frac{qd(1+\epsilon)}{a^2(1-\epsilon)}((1+\epsilon)\exp[-2a\tau(1-\epsilon)/(1+\epsilon)]\{1-\exp[-4at(1-\epsilon)/(1+\epsilon)]\} \\ & -(1-\epsilon)e^{-2a\tau}(1-e^{-4at})).\end{aligned} \tag{26}$$

Second the stationary solution (Equation 19) is inserted into Equation 16. Using again the initial condition $\delta_i(0)=0$ the solution of these equations is

$$\delta_1(t) = \int_0^t \mathrm{d}s \exp\left[-2a_1(t-s)\right] N_1(s) \tag{27}$$

$$\delta_2(t) = \int_0^t \mathrm{d}s \exp\left[(a_2 - \epsilon a_1)(t-s)\right] N_2(s). \tag{28}$$

The correlation functions can be calculated by means of the results in Equations 27 and 28. The normalized correlation function of the intensity fluctuations of the mode with non-vanishing intensity becomes

$$\mu_{11}(t,\tau) = \frac{2qd}{a_1^2} \mathrm{e}^{-2a_1\tau}(1 - \mathrm{e}^{-4a_1 t}). \tag{29}$$

The correlation function of the fluctuations of the mode below threshold is given by

$$\langle \delta_2(t)\delta_2(t+\tau)\rangle = \frac{q}{\epsilon a_1 - a_2} \exp\left[(a_2 - \epsilon a_1)\tau\right]\{1 - \exp\left[2(a_2 - \epsilon a_1)t\right]\}. \tag{30}$$

The cross-correlation of the two modes becomes zero. Now we want to discuss these results. The aim is to determine the range of the coupling parameter ϵ, where either the solution Equation 18 or the solution of Equation 19 is stable, and to consider the dependence of the fluctuations and their correlations on the coupling parameter ϵ. First we deal with the case of different pump rates of the counter-running waves and assume $a_1 > a_2$. A closer investigation of Equation 21 with Equation 18 reveals that $\sqrt{D} < (a_1 + a_2)/(1+\epsilon)$ for $\epsilon < a_2/a_1$ and the correlation functions Equations 23 and 24 approach a stationary value for $t \to \infty$. This also implies that the stationary solution for Equation 18 is stable in this range. For $\epsilon > a_2/a_1$ the correlation functions do not tend to a stationary value for $t \to \infty$ but diverge. This implies that the solution (Equation 18) is unstable and Equation 19 is now the stable solution. When ϵ reaches the critical value $\epsilon_c = a_2/a_1$ the solution of Equation 18 which is stable for $\epsilon < a_2/a_1$ approaches continuously the solution of Equation 19 which is stable for $\epsilon > a_2/a_1$ (soft transition) and the relaxation constant $2[(a_1 + a_2)/\epsilon - \sqrt{D}]$ tends to zero (critical slowing down [13, 14]). In other words, when ϵ passes from $\epsilon < \epsilon_c$ to $\epsilon > \epsilon_c$ the stable equilibrium positions are exchanged, i.e. we have the so-called exchange of stability. As is obvious from the variance $\mu_{ii} = \mu_{ii}(0,0)$ and the covariance μ_{12} of I_1, I_2

$$\mu_{11} = \frac{2qd(1-\epsilon^2)}{(a_1 - \epsilon a_2)^2} \tag{31}$$

$$\mu_{22} = \frac{2qd(1-\epsilon^2)}{(a_2 - \epsilon a_1)^2} \tag{32}$$

$$\mu_{12} = -\frac{2qd(1-\epsilon^2)\epsilon}{(a_1 - \epsilon a_2)(a_2 - \epsilon a_1)} \tag{33}$$

the fluctuations grow with increasing ϵ, i.e. when the laser frequency is tuned towards the line centre. Whereas the intensity fluctuations of the mode remaining above threshold for $\epsilon \uparrow a_2/a_1$ approach the value of $\mu_{11} = 2qd/(a_1^2 - a_2^2)$ the fluctuations of the mode which becomes zero tend to infinity† (critical fluctuations). The intensity fluctuations of the two counter-running modes become anti-correlated. The magnitude of the anti-correlation is proportional to the coupling constant ϵ. μ_{12} as well as $\langle \delta_1\delta_2 \rangle$ tends to infinity for $\epsilon \uparrow a_2/a_1$. As is also evident from Equations 23 and 24 the dependence of the correlation functions on τ is not always well represented by a single exponential as it is in the case of a conventional laser. The correlation time increases with increasing ϵ. Equations 31–33 agree with those obtained in [6] for a ring laser operating well above threshold. For $\epsilon > \epsilon_c$ the correlation functions given by Equations 29 and 30 approach the stationary value for $t \to \infty$. The correlation function of the intensity fluctuations of the mode above threshold corresponds to that of a conventional single-mode laser [10]. For $\epsilon \downarrow \epsilon_c$ the correlation function of the fluctuations of the mode below threshold also show the

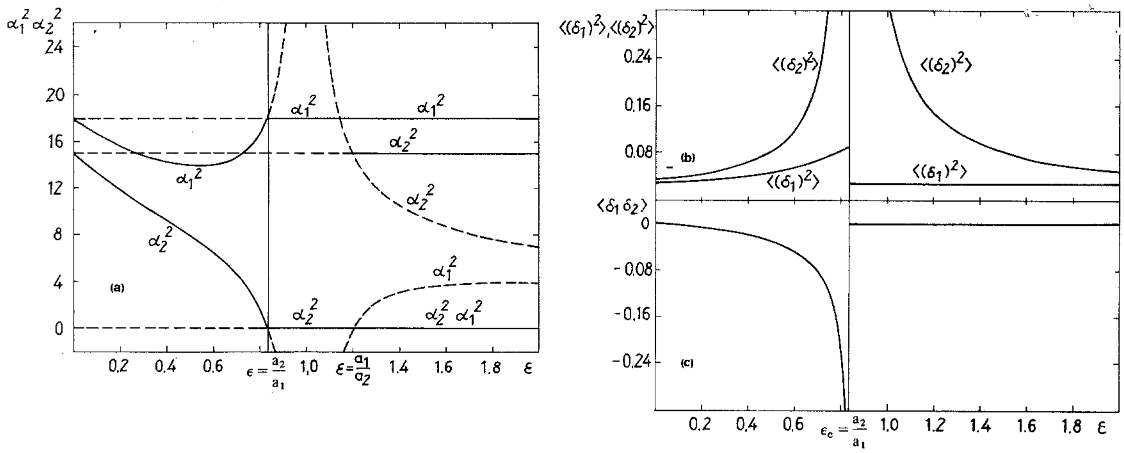

Figure 1 The variation of the intensities (a), the intensity fluctuations (b), and the cross-correlation (c) of the two ring laser modes having different pump parameters, $a_1 = 18$, $a_2 = 15$ with coupling parameter ϵ. In (a) the solid curves indicate the range of stable amplitudes, the dashed curves the range of unstable amplitudes. 1 and 2 label the course of the quantities belonging to the modes with amplitudes α_1 and α_2 respectively. The ϵ-dependence of $\langle \delta_i \delta_j \rangle$ in (b) and (c) are plotted for the case of exchange of stability at $\epsilon_c = a_2/a_1$. For all figures $q = d = 1$ is assumed.

phenomena of critical fluctuations and critical slowing down. So far we have seen that in the case of Doppler broadening for different pump rates there are two stable states. With the transition from $\epsilon < \epsilon_c$ to $\epsilon > \epsilon_c$ the system passes through an instability and shows typical features of a second-order phase transition [15]. This behaviour is illustrated in Fig. 1. It is also shown in Fig. 1a that in the range $a_1/a_2 \leqslant \epsilon \leqslant 2$ either the solution $\alpha_1^2 = a_1/d$, $\alpha_2^2 = 0$ or $\alpha_1^2 = 0$, $\alpha_2^2 = a_2/d$ is stable, i.e. the ring laser shows a bistable behaviour in this region. When ϵ passes from $\epsilon > a_1/a_2$ to $\epsilon < a_1/a_2$ the solution $\alpha_2^2 = a_2/d$ becomes unstable and the system jumps to the new stable state $\alpha_2^2 = 0$, $\alpha_1^2 = a_1/d$ (hard transition). Hysteresis occurs when the direction of changing ϵ is reversed. Both features characterize the analogy to a first-order phase transition [15]. At this transition the fluctuations of the mode α_1^2 become critical.

For equal pump rates the critical value of the coupling parameter is $\epsilon_c = 1$, which is the upper limit of Doppler broadening. Thus in the case of pure Doppler broadening the solution of Equation 18 is valid and in the case of a homogeneously broadened line with self-induced grating of variable amplitude the solution of Equation 19 is valid. However, if the spatial inversion grating occurring at the centre of a Doppler broadened line noticeably influences the behaviour of the ring laser (see Section 2), the critical point $\epsilon_c = 1$ lies within the range of values of ϵ given by Equation 12. For $\epsilon \uparrow 1$ the intensity fluctuations of the two modes grow and also the magnitude of their negative cross-correlation and becomes infinity for $\epsilon = 1$† (see Equations 25 and 26). In contrast to the case of different pump rates with the transition from $\epsilon < 1$ to $\epsilon > 1$ the solution of Equation 18 for $a_1 = a_2$ becomes transient and decays to the new stable solution given by Equation 19 (hard transition). Thus one can say, when exceeding a critical size the strong anti-correlation between the two counter-running modes forces the system to go into a new state where one of the two modes is below threshold and the cross-correlation vanishes. This state is bistable, i.e. either mode 1 or mode 2 is below threshold. Critical fluctuations and critical slowing down occur for both modes. This behaviour which also shows typical features of a first-order phase transition is illustrated in Fig. 2.

†It is an important point that the divergence of the correlation functions is caused by the linearization procedure. In the exact theory which avoids linearization it remains true, however, that at the critical points the fluctuations and their correlations become large (compare for the case of Doppler broadening [6]).

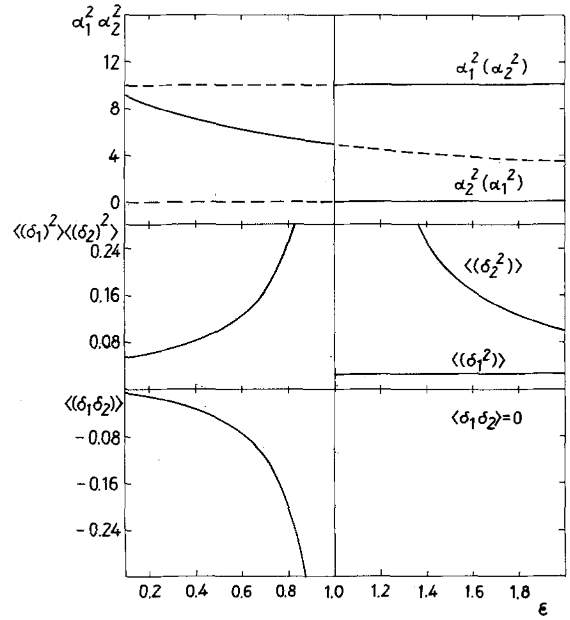

Figure 2 The same as for Fig. 1 for equal pump parameters $a_1 = a_2 = 10$.

4. Transient behaviour of the ring laser in the unstable region

In this section we are interested in determining the time development of the intensity fluctuations and their cross-correlation when the ring laser operation starts from an arbitrary initial state and tends to its stable state. Practically this can be realized by introducing spatially inhomogeneous losses into the ring resonator which force the laser to oscillate with counter-running waves of non-vanishing amplitudes [2] despite the Bragg grating occurring in the active medium. At the time $t = 0$ these spatially inhomogeneous losses are removed and the state becomes transient and decays to a new stable state. For arbitrary initial values of the amplitudes and the fluctuations the complete equations of motion (Equations 14–16) have to be solved. We will do this approximately by using a short-time approximation. We look for the solution of Equations 14 and 15 and the homogeneous part of Equation 16 in the form

$$\alpha_i(t) = \alpha_i(0) + \dot{\alpha}_i(0)t + 1/2\ddot{\alpha}_i(0)t^2 \tag{34}$$

$$\delta_i(t) = \delta_i(0) + \dot{\delta}_i(0)t + 1/2\ddot{\delta}_i(0)t^2 \tag{35}$$

taking into account only terms up to second order in t. Inserting Equation 34 into Equations 14 and 15 and comparing coefficients of the same power of t we obtain

$$\alpha_i(t) = \zeta_i[1 + \Delta_i t + \tfrac{1}{2}(\Delta_i^2 - 2d\zeta_i^2\Delta_i - 2\epsilon d\zeta_j^2\Delta_j)t^2]; \quad i \neq j \tag{36}$$

where

$$\zeta_i = \alpha_i(0); \quad \Delta_i = a_i - d(\zeta_i^2 + \epsilon\zeta_j^2); \quad i \neq j.$$

After putting Equation 36 into Equation 16 the homogeneous part of Equation 16 is solved in the approximation of Equation 35. Using this solution, the solution of the complete Equation 16 can be found in the form given by Equation 20. Thus we obtain the correlation function $\langle\delta_i(t)\delta_j(t)\rangle$ up to the second order in t

$$\begin{aligned}\langle[\delta_1(t)]^2\rangle &= 2d\epsilon\zeta_1\zeta_2[-2(t + A_1t^2)\langle q_1q_2\rangle + 2d\epsilon\zeta_1\zeta_2t^2\langle(q_2)^2\rangle] + \{1 + 2B_1t + 2[\Delta_1^2 - d\zeta_1^2(7\Delta_1\\ &\quad - 4d\zeta_1^2) - d\zeta_2^2(\epsilon\Delta_2 - 2d\epsilon^2\zeta_1^2)]t^2\}\langle(q_1)^2\rangle + 2q(t + B_1t^2)\end{aligned} \tag{37}$$

$$\langle\delta_1(t)\delta_2(t)\rangle = -2d\epsilon\zeta_1\zeta_2[(t + A_1t^2)\langle(q_1)^2\rangle + (t + A_2t^2)\langle(q_2)^2\rangle + 2qt^2]$$

$$+ (1 + (B_1 + B_2)t + \{1/2(\Delta_1 + \Delta_2)^2 - d\zeta_1^2[(\epsilon + 5)\Delta_1 + 2\Delta_2 - 2d\zeta_1^2 - 2d\zeta_2^2(2\epsilon^2 + 1)]$$
$$- d\zeta_2^2[(\epsilon + 5)\Delta_2 + 2\Delta_1 - 2d\zeta_2^2 - 2d\zeta_1^2(2\epsilon^2 + 1)]\}t^2)\langle q_1 q_2\rangle \qquad (38)$$

where $q_i = \delta_i(0)$ and $A_1 = 2\Delta_1 + \Delta_2 - 3d\zeta_1^2 - d\zeta_2^2$; $B_i = \Delta_i - 2d\zeta_i^2$. A_2 and $\langle[\delta_2(t)]^2\rangle$ can be obtained from A_1 and Equation 37 by substituting $1 \leftrightarrow 2$. Equations 37 and 38 describe the development of the fluctuations and their cross-correlation of the system having the initial values ζ_i, $\langle q_i q_j\rangle$ within a small time interval. Note that if as initial conditions the stationary solution (Equation 18) and vanishing fluctuations are chosen, Equations 37 and 38 change into Equations 23 and 24 approximated for small times. Assuming that at $t = 0$ the coupling between the modes is switched on, i.e. $\langle q_1 q_2\rangle = 0$ and $\epsilon > 1$ for $t \geqslant 0$, we see from Equation 38 that the fluctuations of the counter-running modes become anti-

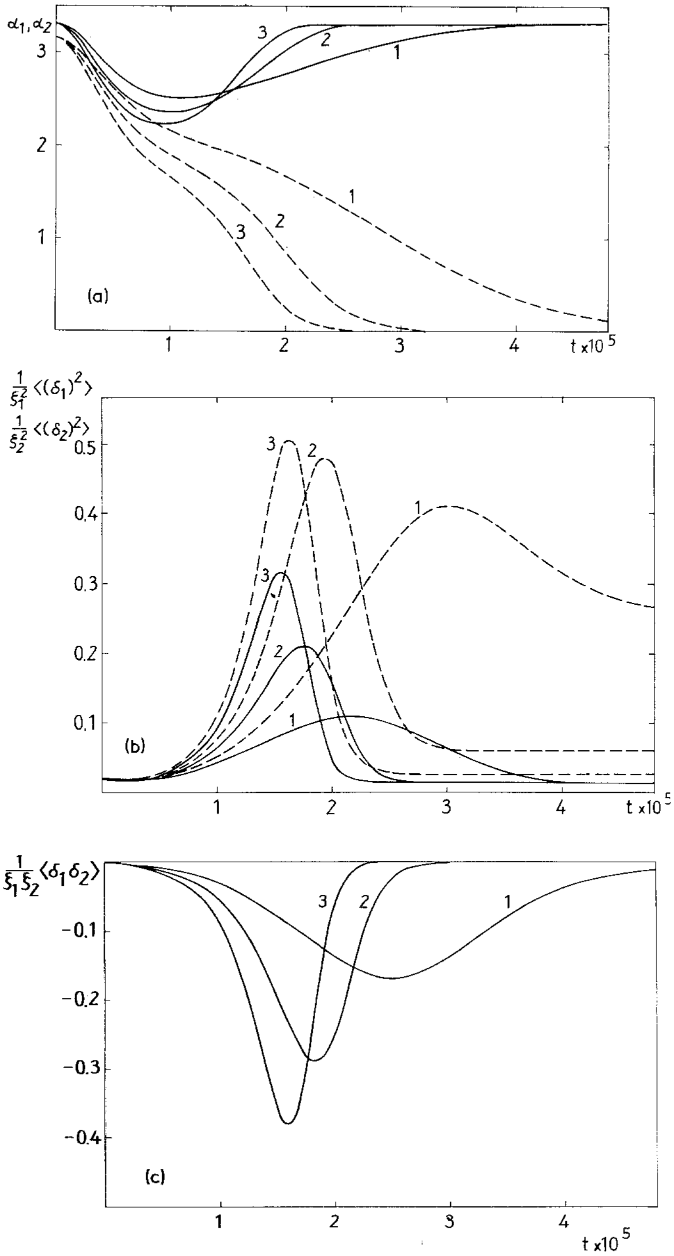

Figure 3 The evolution of the amplitudes (a), the intensity fluctuations normalized to the initial intensities (b), and the normalized cross-correlation (c) for values of the coupling parameter of $\epsilon = 1.05$ (1), $\epsilon = 1.5$ (2), $\epsilon = 2$ (3) in the case of a ring laser starting from the initial values $\zeta_i = a_i/d$, $\langle(q_i)^2\rangle = q/2a_i$, $\langle(q_1q_2)\rangle = 0$ for $q = d = 1$. The solid (dashed) curves show the time dependence of quantities belonging to the mode with the pump parameter $a_1 = 11$ ($a_2 = 10$).

correlated. The amount is proportional to the coupling parameter ϵ. Especially for the initial values $\zeta_1^2 = \zeta_2^2 = a/d, \langle (q_1)^2 \rangle = \langle (q_2)^2 \rangle = q/2a$, i.e. stationary amplitudes and fluctuations for $\epsilon = 0$, Equations 37 and 38 take the form

$$\langle [\delta_1(t)]^2 \rangle = \frac{q}{2a}(1 + 6\epsilon^2 a^2 t^2) \tag{39}$$

$$\langle \delta_1(t)\delta_2(t) \rangle = -2\epsilon q(t - 2at^2). \tag{40}$$

From Equation 39 it is evident that the fluctuations of a unidirectionally travelling wave $q/2a$ are increased due to the mode competition with second order in t. The magnitude of the anti-correlation is increased due to the coupling of the counter-running modes in the first order in t and diminished due to the pumping in the second order in t. Equations 36–38 can be used to calculate the evolution of the amplitudes and the correlation functions numerically when the system starts from an arbitrary initial state and tends to its stable state. For the situation that the coupling between the two modes is suddenly switched on this is illustrated in Fig. 3 for various values of the coupling parameter ϵ. These figures show that the increase and the magnitude of the intensity fluctuations, as well as of their negative cross-correlation, grow when ϵ is increased. It is also evident that the more slowly the initial state of the intensities decay, the closer ϵ approaches to the critical value (critical slowing down).

5. Conclusions

There are various mechanisms causing a competition of the counter-running modes of a single-frequency ring laser. In the case of a spectrally inhomogeneously broadened line, e.g. Doppler broadening, the strength of the competition is determined by the common gain reservoir of both modes; and in the case of homogeneous broadening and the active atoms being at rest, the coupling is enhanced by the self-induced population-inversion grating. Depending on the ratio of the lifetimes of the levels and the dipole moment of the active atoms, in the case of Doppler broadening an inversion grating occurring at line centre may influence the mode competition and intermediate cases are possible. Depending on the strength of the coupling there are different stable states: (a) both modes are above threshold and (b) only one mode is above threshold. In the first case the intensity fluctuations are anti-correlated. The magnitude of the anti-correlation increases with an increasing coupling parameter. In particular, in the case of equal pump rates and consequently of equal intensities of the oppositely directed waves one may conclude that when exceeding a critical size the strong anti-correlation between the counter-running modes forces the ring laser to go into a new state where one of the modes is below threshold and the fluctuations of the two modes are uncorrelated. The transition between the stable states shows phenomena closely resembling phase transitions such as critical fluctuations, critical slowing down, etc. For mode coupling enhanced by a population-inversion grating the time development of the amplitudes and the correlation functions during the transient behaviour of the ring laser is described in a short-time approximation. It is illustrated by means of special initial conditions that the strong coupling leads to negative cross-correlation to the first order in time and that the pumping has a tendency to decrease this anti-correlation in a second order in time.

There is a whole series of further physically interesting phenomena connected with a ring laser, the gain of which is spectrally homogeneously broadened and shows a population-inversion grating. Experiments show that if both directions are back-scattered by retro-reflecting elements the two modes begin to oscillate mutually with a frequency depending on the strength of back-scattering [16], or, in other words, the system pumped continuously shows order in the time domain. It would also be interesting to investigate the fluctuations and especially their cross-correlation of a homogeneously broadened ring laser, which is forced by spatially inhomogeneous (δ-shaped) losses to oscillate in both directions. This also offers the possibility of experimentally determining the influence of the mode competition enhanced by the population-inversion grating on the cross-correlation of the counter-running modes.

Acknowledgements
The authors wish to thank Dr V. Peřinova and Dr J. Peřina from the Palacky University of Olomouc, and Dr G. Jetschke from the University of Jena for helpful discussions.

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