SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS FOR CHARGED STATIC FLUID SPHERES

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Abstract. In this paper the authors have extended Bayin's (1978) work to the case of charged fluid spheres. These solutions are matched at the boundary with the Reissner–Nordström solution.

1. Introduction

The problem of charged matter distributions in general relativity has received considerable attention. A number of authors (Papapetrou, 1947; Majumdar, 1947; Bonnor, 1960, 1965, 1980; De and Raychaudhuri, 1968; Tikekar, 1985; Ibrahim and Nutku, 1976) have studied the charged fluid distribution in equilibrium. Interior solutions for charged fluid spheres have been investigated by Efinger (1965), Kyle and Martin (1967), Florides (1962, 1977, 1983), Chakravarti and De (1979) under different conditions. Shah (1968) has considered the generalization of Nordström's solution corresponding to the external field of a radiating charged particle. Wilson (1969) has presented an exact solution for the interior of static charged spheres. Solutions for charged fluid spheres have also been obtained by Bonnor and Wickramasuriya (1972), Bailyn and Eimerl (1972), Omote (1973), Krori and Barua (1975), Mehra (1980), Nduka (1977), Singh and Yadav (1978), and Koppar *et al.* (1991).

Due to nonlinearity of Einstein–Maxwell equations exact solutions are difficult to obtain. In order to solve this system of equations, it is necessary to specify, in some manner, the unknowns or to introduce extra relation between them. In this paper we have obtained exact solutions for the spherically-symmetric charged fluid distribution by the method of quadratures making a specific choice of metric functions and total charge. Our work extends the method of Bayin (1978) to the case of charged fluids. Finally, the solutions are matched with exterior Reissner–Nordström solution.

2. Field Equations

The Einstein–Maxwell field equations for perfect matter fluid are given by

$$G_{ij} = -8\pi(T_{ij} + E_{ij}), \tag{2.1}$$

where the energy momentum tensor of perfect fluid distribution is

$$T_{ij} = (\rho + p)u_{ikj} - pg_{ij}, \tag{2.2}$$

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and the energy momentum tensor of the electromagnetic field is

$$E_{ij} = \frac{1}{4\pi}\,[g^{kl}F_{ik}F_{jl} - \tfrac{1}{4}g_{ij}F_{kl}F^{kl}]\,. \tag{2.3}$$

The electromagnetic field equations are given by

$$[(-g)^{1/2}F^{ij}]_{,j} = 4\pi J^i(-g)^{1/2} \tag{2.4}$$

and

$$F_{[ij,k]} = 0\,, \tag{2.5}$$

where u^i is the 4-velocity of a fluid element, F^{ij} is the electromagnetic field tensor and J^i is the 4-current. The units are so chosen that $C = G = 1$.

We consider a static spherically-symmetric system for which the line element is

$$\mathrm{d}s^2 = -B^2\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 + A^2\,\mathrm{d}t^2\,, \tag{2.6}$$

In co-moving coordinates, the field equations may be written as

$$8\pi p = \frac{1}{B^2}\left(\frac{2A'}{rA} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \frac{2Q^2}{r^4}\,, \tag{2.7}$$

$$8\pi p = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] + \frac{Q^2}{r^2}\,, \tag{2.8}$$

$$8\pi\rho = \frac{1}{B^2}\left(\frac{2B'}{rB} - \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{Q^2}{r^4}\,; \tag{2.9}$$

where

$$Q(r) = 4\pi\int_0^r J^4\chi^2 AB\,\mathrm{d}\chi \tag{2.10}$$

is the total charge within a sphere of radius r. The corresponding electric field is given by

$$F^{41} = \frac{Q(r)}{ABr^2}\,; \tag{2.11}$$

where the prime denotes differentiation with respect to radial coordinate r.

If we eliminate p from Equations (2.7) and (2.8) we write the result in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1-B^2}{B^2r^2}\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{A'}{B^2Ar}\right) + \frac{1}{B^2A^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{AA'}{r}\right) + \frac{4Q^2}{r^5} = 0\,. \tag{2.12}$$

Assuming $A'/Ar = C(r)$ Equation (2.12) can be written as

$$\left(\frac{1}{r^2B^3} + \frac{C}{B^3}\right)\mathrm{d}B - \left(\frac{1}{B^2}\right)\mathrm{d}C - \left(\frac{B^2-1}{r^3B^2} + \frac{C^2r}{B^2} + \frac{Q^2}{r^5}\right)\mathrm{d}r = 0\,. \tag{2.13}$$

3. Solutions of the Field Equations

Now, we first try to solve Equation (2.13) for $B(r)$ if $C(r)$ is known or suitably chosen. For this purpose we write Equation (2.13) in the form

$$\frac{dB}{dr} = \frac{1 + Q^2}{r^3\left(\frac{1}{r^2} + C\right)} B^3 + \frac{-\frac{1}{r^3} + C^2 r + \frac{dC}{dr}}{\frac{1}{r^2} + C} B . \tag{3.1}$$

Choosing $1/B^2 = F$ Equation (3.1) can be written as

$$\frac{dF}{dr} + 2 \frac{-\frac{1}{r^3} + C^2 r + \frac{dC}{dr}}{\frac{1}{r^2} + C} F = - \frac{2(1 + Q^2)}{\left[r^2\left(\frac{1}{r^2} + C\right)\right]} . \tag{3.2}$$

Thus if $C(r)$, i.e., $A(r)$ and $Q(r)$ are known, the first-order linear differential equation (3.2) gives $B(r)$. Hence, we can try a number of interesting cases.

In order to find solution of Equation (3.2) we choose

$$\frac{-\frac{1}{r^3} + C^2 r + \frac{dC}{dr}}{\frac{1}{r^2} + C} = -\frac{1}{r} . \tag{3.3}$$

Then

$$C(r) = \frac{1}{(C_0 r + r^2)} \tag{3.4a}$$

and

$$A^2 = (a_0 + a_1 r)^2 . \tag{3.4b}$$

Equation (3.2) yields the solution

$$F(r) = B^{-2} = 1 + B_0 r^2 - (2/C_0) r + \left(\frac{4r^2}{C_0^2}\right) \log[(C_0 + 2r)/r] - \\ - \frac{2K}{(n-2)} r^n + 2Kr^2 \int \frac{r^{n-2}}{(C_0 + 2r)} dr , \tag{3.5}$$

where we have taken

$$Q(r) = Kr^n ; \tag{3.6}$$

B_0 and K being constants.

If we substitute the values of A, B, and Q in Equation (2.7), we obtain

$$p = \frac{1}{8\pi}\left[2Kr^{n-4} - \frac{1}{r^2} + \frac{(a_0 + 3a_1 r)}{r^2(a_0 + a_1 r)}\left\{1 + B_0 r^2 - (2/C_0)r + \right.\right.$$

$$\left.\left. + (4r^2/C_0^2)\log[(C_0 + 2r)/r] - \frac{2K}{(n-2)} r^n + 2Kr^2 \int \frac{r^{n-2}}{(C_0 + 2r)}\,\mathrm{d}r\right\}\right]. \tag{3.7}$$

Similarly, Equation (2.9) gives

$$\rho = \frac{1}{8\pi}\left[K^2 r^{2n-4} + \frac{4}{C_0 r} + \frac{2(n+1)Kr^{n-2}}{(n-2)} - \frac{12}{C_0^2}\log\left(\frac{C_0 + 2r}{r}\right) + \right.$$

$$\left. + \frac{4}{C_0(C_0 + 2r)} - \frac{2Kr^{n-1}}{(C_0 + 2r)} - 6K \int \frac{r^{n-2}}{(C_0 + 2r)}\,\mathrm{d}r - 3B_0\right]. \tag{3.8}$$

Next we consider the second possibility, that is, to solve $C(r)$ for given $B(r)$. Equation (2.13) reduces to

$$\frac{\mathrm{d}C}{\mathrm{d}r} = \left(\frac{1}{Br^2}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{B^2 - 1}{r^3}\right) + \left(\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)C - C^2 r + \frac{B^2 Q^2}{r^5}\,. \tag{3.9}$$

For known $Q(r)$ given by (3.6), Equation (3.9) is a Riccati equation for $C(r)$ and quite difficult to solve, in general. Therefore, we consider some cases of physical interest.

Case II

If we take $B = W_0 C(r)$ where W_0 is a constant, then one can find the following solution

$$C = \left[C_1 r^2 + W_0^2 - r^4 + \frac{W_0^2 K^2 r^{2n-2}}{(n-2)}\right]^{-1/2}, \tag{3.10}$$

where $n \neq 2$.

Sub-case (i) $n = 1$

$$\log A = \tfrac{1}{2}\sin^{-1}\left[\frac{(-C_1 + 2r^2)}{\{C_1^2 + 4W_0^2(1 - K^2)\}^{1/2}}\right] + C_2\,, \tag{3.11}$$

$$B = W_0[C_1 r^2 + W_0^2 - r^4 - W_0^2 K^2]^{-1/2}. \tag{3.12}$$

By use of Equations (3.11) and (3.12) Equations (2.7) and (2.9) give, respectively,

$$p = \frac{1}{8\pi W_0^2}\,[2(W_0^2 + C_1 r^2 - r^4 - W_0^2 K^2)^{1/2} + C_1 - r^2] + \frac{K^2}{8\pi r^2}\,, \tag{3.13}$$

$$\rho = \frac{1}{8\pi W_0^2}\,[5r^2 - 3C_1] + \frac{K^2}{4\pi r^2}\,. \tag{3.14}$$

Sub-case (*ii*) $n = 3$

In this case, we have

$$\log A = \frac{1}{2(1 - K^2 W_0^2)^{1/2}} \sin^{-1}\left[\frac{2r^2(1 - K^2 W_0^2) - C_1}{C_1^2 - 4W_0^2(1 - K^2 W_0^2)}\right] + C_3 , \tag{3.15}$$

$$B = W_0[W_0^2 + C_1 r^2 - (1 - W_0^2 K^2) r^4]^{-1/2} . \tag{3.16}$$

With the help of Equations (3.15) and (3.16), from Equation (2.7), we obtain

$$p = \frac{1}{8\pi W_0^2} [2\{W_0^2 + C_1 r^2 - (1 - W_0^2 K^2 r^4\}^{1/2} + C_1 - r^2] + \frac{3K^2 r^2}{8\pi} . \tag{3.17}$$

Similarly from Equation (2.9), one finds that

$$\rho = \frac{1}{8\pi W_0^2} (5r^2 - 3C_1) - \frac{K^2 r^2}{2\pi} . \tag{3.18}$$

Case III

If we consider $B(r) = W(r)C(r)$ then Equation (3.9) reduces to

$$\frac{\mathrm{d}C}{\mathrm{d}r} = -\left(\frac{W'}{W} + \frac{1}{r}\right) C - \frac{W'}{W} r^2 C^2 + C^3 \left(\frac{W^2}{r} + r^3 - W^2 K^2 r^{2n-3}\right). \tag{3.19}$$

If we assume

$$\frac{W^2}{r} + r^3 - W^3 K^2 r^{2n-3} = 0 ,$$

we get

$$W^2 = \frac{r^4}{(K^2 r^{2n-2} - 1)} . \tag{3.20}$$

When $n = 1$, we have

$$C = \frac{1}{r^2(C_3 r - 2)} , \tag{3.21}$$

$$A = C_4 \left(C_3 - \frac{2}{r}\right)^{1/2} , \tag{3.22}$$

$$B^2 = \frac{1}{(K^2 - 1)(C_3 r - 2)^2} . \tag{3.23}$$

If we substitute the values of A and B in (2.7) and (2.9), we get

$$p = \frac{1}{8\pi r^2} [1 + (K^2 - 1)\{2(C_3 r - 1) + (C_3 r - 2)^2\}], \tag{3.24}$$

$$\rho = \frac{1}{8\pi r^2} [1 + K^2 + (K^2 - 1)(C_3 r - 2)(2 - 3C_3 r)]. \tag{3.25}$$

Similarly, we can obtain the solution for $n = 3$.

Case IV

We can now try the substitution

$$\frac{1}{r^2 B}\frac{dB}{dr} - \frac{B^2 - 1}{r^3} + \frac{B^2 K^2 r^{2n}}{r^5} + \frac{C}{B}\frac{dB}{dr} = \theta(r)C(r). \tag{3.26}$$

If we assume $\theta(r) = r^{-1}$, one can easily obtain

$$C(r) = \frac{3r}{(3C_5 + r^3)}, \tag{3.27}$$

$$A(r) = C_6(3C_5 + r^3), \tag{3.28}$$

$$B^2 = (3C_5 + 4r^3)\left[3C_5 + (C_7 - 2r)r^2 + \frac{3K^2 C_5 r^{2n-2}}{(n-2)} + \frac{2K^2 r^{2n+1}}{(2n-1)}\right]^{-1}. \tag{3.29}$$

With the help of Equations (3.32) and (3.33) one can easily find the pressure p which is given by

$$p = \frac{(3C_5 + 7r^3)}{8\pi r^2(3C_5 + 4r^3)(3C_5 + r^3)}\left[(3C_5 + C_7 r^2 - 2r^3) + \right.$$

$$\left. + \left\{\frac{3C_5}{(n-2)} + \frac{2r}{(2n-1)}\right\} r^{2n-2}\right] - \frac{1}{r^2}(1 - 2K^2 r^{2n-6}). \tag{3.30}$$

Similarly we can find the density ρ (to avoid lengthy and cumbersome expression we omit it).

4. Matching with Exterior Solution

Now we consider the matching of exterior Reissner–Nordström metric with different interior solutions which have been obtained. The exterior solution is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 +$$

$$+ \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2, \tag{4.1}$$

where

$$F_{14} = \frac{q}{r^2} . \tag{4.2}$$

We take components g_{11}, g_{44}, and $\partial g_{44}/\partial r$ continuous across the boundary of the sphere $r = r_0$. Also

$$Q(r_0) = q . \tag{4.3}$$

Therefore,

$$Kr_0^n = q . \tag{4.4}$$

Case I

$$(a_0 + a_1 r_0)^2 = \left(1 - \frac{2M}{r_0} + \frac{q^2}{r_0^2}\right), \tag{4.5}$$

$$B_0 r_0^2 - 2r_0^3 - \frac{2Kr_0^{n+3}}{(n+1)} + 2r_0 \int_0^{r_0} \frac{r + Kr^{n+1}}{C_0 + 2r}\, \mathrm{d}r = \left(1 - \frac{2M}{r_0} + \frac{q^2}{r_0^2}\right), \tag{4.6}$$

$$a_1(a_0 + a_1 r_0) = \frac{M}{r_0^2} - \frac{q^2}{r_0^3} . \tag{4.7}$$

Subcase (*i*) $n = 1$

In this case the boundary conditions are given by

$$(a_0 + a_1 r_0)^2 = \left(1 - \frac{2M}{r_0} + \frac{q^2}{r_0^2}\right), \tag{4.8}$$

$$\left(B_0 + 1 - \frac{KC_0}{2}\right) r_0^2 - \left(2 - \frac{K}{2}\right) r_0^3 - Kr_0^4 + \tfrac{1}{4}(KC_0^2 - 2C_0) r_0 \log(C_0 + 2r) =$$

$$= \left(1 - \frac{2M}{r_0} + \frac{q^2}{r_0^2}\right), \tag{4.9}$$

$$a_1(a_0 + a_1 r_0) = \frac{M}{r_0^2} - \frac{q^2}{r_0^3} . \tag{4.10}$$

If we solve (4.4), (4.8), (4.9), and (4.10) we get the constants K, a_0, a_1, B_0, and C_0 in terms of M, q, and r_0 where r_0 is obtained by equating p to zero.

Subcase (*ii*) $n = 3$

$$(a_0 + a_1 r_0)^2 = \left(1 - \frac{2M}{r_0} + K^2 r_0^4\right), \tag{4.11}$$

$$(KC_0^3 - 8)\,\frac{C_0 r_0}{16}\,\log(C_0 + 2r_0) + \left(B_0 + 1 - \frac{KC_0^3}{8}\right) r_0^2 + $$

$$+ \left(\frac{KC_0^2}{8} - 2\right) r_0^3 - \frac{KC_0 r_0^4}{6} + \frac{Kr_0^5}{4} = \left(1 - \frac{2M}{r_0} + K^2 r_0^2\right), \tag{4.12}$$

$$2a_1(a_0 + a_1 r_0) = \frac{2M}{r_0^2} + 4K^2 r_0^3 \,. \tag{4.13}$$

From Equations (4.11) and (4.13), we have

$$\frac{a_0}{a_1 r_0^2} + \frac{1}{r_0} = \frac{1 + K^2}{M} - \frac{2}{r_0}\,, \tag{4.14}$$

which shows that

$$\frac{a_0}{a_1 r_0^2} + \frac{3}{r_0} > 0 \tag{4.15}$$

or

$$a_0 > -3a_1 r_0 \,. \tag{4.16}$$

Case II

Subcase (*i*) $n = 1$In this case at the boundary $r = r_0$, we have

$$\exp\left[\sin^{-1}\left\{\frac{-C_1 + 2r_0^2}{\sqrt{C_1^2 + 4w_0^2(1 - K^2)}}\right\}\right] = \left(1 - \frac{2M}{r_0} + K^2\right), \tag{4.17}$$

$$w_0^2[C_1 r_0^2 - r_0^4 + w_0^2(1 - K^2)]^{-1} = \left(1 - \frac{2M}{r_0} + K^2\right)^{-1}, \tag{4.18}$$

$$\exp\left[\sin^{-1}\left\{\frac{-C_1 + 2r^2}{\sqrt{C_1^2 + 4W_0^2(1 - K^2)}}\right\}\right] \frac{r_0^3}{\sqrt{C_1 r_0^2 - r_0^4 + W_0^2(1 - K^2)}} = M \,. \tag{4.19}$$

With the aid of Equations (4.17), (4.18), and (4.19) we can easily obtain

$$M = \frac{r_0^3}{W_0^2}\,\sqrt{C_1 r_0^2 - r_0^4 + W_0^2(1 - K^2)}\,. \tag{4.20}$$

Since M, K, and W_0 are positive quantities,

$$r_0^2 > C_1 . \tag{4.21}$$

The value of r_0 is obtained by putting $p = 0$ in Equation (3.13).

Subcase (ii) $n = 3$

$$\exp\left[\frac{1}{\sqrt{1 - K^2 W_0^2}} \sin^{-1}\left\{\frac{2r_0^2(1 - K^2 W_0^2) - C_1}{C_1^2 - 4W_0^2(1 - K^2 W_0^2)}\right\}\right] = \\ = \left(1 - \frac{2M}{r_0} + K^2 r_0^4\right), \tag{4.22}$$

$$W_0^2[W_0^2 + C_1 r_0^2 - (1 - W_0^2 K^2) r_0^4]^{-1} = \left(1 - \frac{2M}{r_0} - K^2 r_0^4\right)^{-1}, \tag{4.23}$$

$$\exp\left[\frac{1}{\sqrt{1 - K^2 W_0^2}} \sin^{-1}\left\{\frac{2r_0^2(1 - K^2 W_0^2) - C_1}{C_1^2 - 4W_0^2(1 - K^2 W_0^2)}\right\}\right] \times \\ \times \frac{r_0^3}{\sqrt{C_1 r^2 + W_0^2 - r_0^4(1 - K^2 W_0^2)}} = M + 2K^2 r_0^5 . \tag{4.24}$$

By use of Equations (4.22) and (4.23) in (4.24) we have

$$\frac{r_0^3}{W_0^2} \sqrt{W_0^2 + C_1 r_0^2 - (1 - W_0^2 K^2) r_0^4} = M + 2K^2 r_0^5 . \tag{4.25}$$

Again for positive M, K, and W_0 one has

$$C_1 > (1 - W_0^2 K^2) r_0^2 . \tag{4.26}$$

Furthermore, putting the value of $p = 0$ in Equation (3.17) the value of r_0 is obtained.

Case III $n = 1$

The boundary conditions are

$$C_4^2\left(C_3 - \frac{2}{r_0}\right) = \left(1 - \frac{2M}{r_0} + K^2\right), \tag{4.27}$$

$$[(K^2 - 1)(C_3 r - 2)^2]^{-1} = \left(1 - \frac{2M}{r_0} + K^2\right)^{-1}, \tag{4.28}$$

$$C_4^2 = M . \tag{4.29}$$

From these equations we obtain

$$C_3 = \frac{1 + K^2}{M} . \tag{4.30}$$

Case IV

Subcase (*i*) $n = 1$

$$C_6^2(3C_5 + r_0^3)^2 = \left(1 - \frac{2M}{r_0} + K^2\right), \tag{4.31}$$

$$(3C_5 + 4r_0^3)\,[3C_5 + (C_7 - 2r_0)r_0^2 - 3C_5K^2 + 2K^2r_0^3]^{-1} = \left(1 - \frac{2M}{r_0} + K^2\right)^{-1}, \tag{4.32}$$

$$3C_6^2(3C_5 + r^3)r_0^4 = M\,. \tag{4.33}$$

From Equations (4.31)–(4.33) we can obtain

$$\frac{C_5}{r_0^4} + \frac{7}{3r_0} = \frac{1 + K^2}{M}\,. \tag{4.34}$$

Hence,

$$\frac{C_5}{r_0^4} + \frac{7}{3r_0} > 0 \tag{4.35}$$

or

$$C_5 > -\frac{7}{3r_0^3}\,. \tag{4.36}$$

Similarly for subcase (ii) one can obtain

$$C_5 > -\frac{7}{3r_0^3}\,. \tag{4.37}$$

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