7. Yu. S. Urzhumtsev and R. D. Maksimov, Prediction of the Deformability of Polymer Materials [in Russian], Riga (1975).
8. A. F. Kregers, "Active and passive loading of nonlinearly viscoelastic material," in: Abstracts of Papers of the All-Union Symposium "Creep in Structures," Dnepropetrovsk (1982), pp. 170-171.
9. A. F. Kregers and U. K. Vilks, "Computerized approximation of families of curves of physically nonlinear creep of polymer materials," Mekh. Polim., No. 2, 220-229 (1976).
10. A. F. Kregers and M. R. Kilevits, "Complex investigation of high-density polyethylene under conditions of nonlinear creep and stress relaxation," Mekh. Kompozitn. Mater., No. 2, 195-201 (1985).
11. U. K. Vilks, "Grip for mounting tubular plastic specimens," Inventors' Certificate No. 289334, Otkrytiya. Izobret. Prom. Obraztsy. Tov. Zn., No. 1 (1971).
12. U. K. Vilks, "Device for measuring deformations," Inventors' Certificate No. 355486. Otkrytiya. Izobret. Prom. Obraztsy. Tov. Zn., No. 31 (1972).
13. A. F. Kregers and Yu. O. Yanson, "Variant of the phenomenological model of deformation of viscoelastic aging material," Mekh. Kompozitn. Mater., No. 3, 395-400 (1985).
14. A. K. Malmeister, V. P. Tamuzh, and G. A. Teters, Resistance of Polymer and Composite Materials [in Russian], 3rd ed., Riga (1980).
15. A. F. Kregers, "Variants of the theory of localization of deformations," Mekh. Polim., No. 2, 233-240 (1970).

# MECHANICS OF BUILT-UP VISCOELASTIC BODIES SUBJECTED TO AGING IN FINAL DEFORMATIONS

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We obtained the basic equations of the mechanics of built-up viscoelastic bodies, subjected to aging, in final deformations. Problems of axisymmetric build-up of a sphere of incompressible viscoelastic material are solved.

1. Statement of the Problem. We examine the deformation of a built-up viscoelastic body subjected to aging. Before application of the load the body $K_0$ occupies the domain $\Omega_0$ with piecewise-smooth boundary $\partial\Omega_0$. At the instant $t = 0$ surface and volume forces are applied to the body, and the process of build-up begins which continues up to the instant $t = T$. We denote by $D(t)$ the domain taken up by the body at the instant $t$, and by $\Gamma(t)$ the piecewise-smooth boundary of this domain. The sets $D(t)$ and $\Gamma(t)$ are determined in the actual state.

First we examine the process of discrete build-up of a body. We fix the instants $t_1, t_2, \ldots, t_N \in [0, T]$, $t_1 = 0$, $t_N = T$. In discrete build-up the elements $K_j$ are added at the instants $t_j$ to the surface of the built-up body.

We introduce the notion of the initial state of the built-up element. We denote by $\gamma_0$ the part of the boundary of the domain $\Omega_0$ on which the process of building up is determined. By the initial state of the element $K_1$ we mean the state in which it can be superposed on $\gamma_0$ without deformation as a rigid whole, with continuity being maintained along $\gamma_0$. We denote by $\Omega_1$ the domain occupied by the body $K_1$ which is in the initial state and is superposed on $\gamma_0$. We put $\Omega' = \Omega_0 \cup \Omega_1$. By $\gamma_1$ we denote the part of the boundary of the domain $\Omega'$ on which the process of building up is determined. By the initial state of the element $K_2$ we mean the state in which it can be superposed on $\gamma_1$ without deformation as a rigid whole, with continuity being maintained along $\gamma_1$. The domain occupied by the body $K_2$, which is in the initial state and is superposed on $\gamma_1$, is denoted by $\Omega_2$. Analogously we determine the initial state of an element $K_j$ for any $j > 2$. Thus, whereas the built-up elements $K_j$ are in the initial state, the built-up body K at the instant T is composed of bodies $K_j$ as rigid bodies, with continuity maintained along the surface of building up. We denote by $\Omega$ the domain occupied by body K in the initial state. Into $\Omega$ we introduce the system of coordinates $\xi_1\xi_2\xi_3$. We adopt the coordinates $\xi = (\xi_j)$ as Lagrange coordinates of points of the basic body and of the built-up elements.

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The introduction of the initial state enables us to show which points of the boundaries of the basic body and of the built-up elements are joined in build-up. The law according to which the correspondence of the points of the boundaries is established is formulated implicitly, namely: the set of points of the boundary to which the new element $K_j$ is joined is specified *a priori*, but the actual form of the surface $\gamma_{j-1}$ is found in solving the problem of determining the state of stress and strain in the built-up body. In the simplest cases, when there is central or axial symmetry, the form of the surface $\gamma_j$ can be determined beforehand.

We denote by $r(\xi)$ the radius vector of the point $\xi$ in the initial state. The base vectors of the system of Lagrange coordinates in the initial state are denoted by $g_i = r_{,i}$ (the subscript after the comma denotes the derivative with respect to the corresponding Lagrange coordinate). By $g^i$ we denote the vectors of the mutual base: $g^i \cdot g_j = \delta_j^{\ i}$ ($\delta_j^i$ are Kronecker deltas), and by $g_{ij}$, $g^{ij}$ the metric tensors of the principal base and of the mutual base in the initial state: $g_{ij} = g_i \cdot g_j$; $g^{ij} = g^i \cdot g^j$.

We introduce the notion of the natural (reference) state of the element $K_j$. If an element $K_j$ is in the natural state, then in the absence of an external load it is not being deformed and there are no stresses in it. The domain occupied by the body $K_j$ in the natural state is denoted $\Omega_j^{\ o}$. By $r^o(\xi)$ we denote the radius vector of the point $\xi$ in the natural state. We introduce the base vectors of the principal base and of the mutual base and the corresponding metric tensors in the natural form by the formulas $g_i^{\ o} = r_i^{\ o}$; $g^{oi} \cdot g_j^{\ o} = \delta_j^{\ i}$; $g_{ij}^{\ o} = g_i^{\ o} \cdot g_j^{\ o}$ $g^{oij} = g^{oi} \cdot g^{oj}$.

The process of continuous building up is regarded as the limit of the process of discrete building up on condition that $\Delta t_j = t_{j+1} - t_j \to 0$. Going over to the limit, we obtain the Lagrange coordinates of the points of the body, and also the corresponding metric tensors in the initial and the natural state in continiuous building up.

We denote by $\Omega(t)$ the domain which at the instant t the built-up body $K(t)$ occupies, by $\gamma(t)$ the boundary of this domain, and by $v(\xi)$ the rate of build-up along the normal to the surface $\gamma(t)$ at the point $\xi \in \gamma(t)$ in the initial state. In continuous build-up the element $dK(t)$ is added within time dt to the surface $\gamma(t)$; this element occupies in the initial state the domain $d\Omega(t)$, and it is a film with variable thickness $v(\xi)dt$ (the thickness of the film is measured in the initial state).

In various problems it is expedient to specify the rate of build-up along the normal to the boundary in the actual state $V(\xi)$, and not in the initial state. It is obvious that with the function $v(\xi)$ specified, $V(\xi)$ can be found after the problem of determining the displacements of the points of the built-up body has been solved, and conversely, with the function $V(\xi)$ specified, we can determine the magnitude of $v(\xi)$. The two approaches are therefore equivalent.

By $R(t, \xi)$ we denote the radius vector of the point $\xi$ in the actual state at the instant t. We introduce the base vectors of the principal and the mutual base and the corresponding metric tensors in the actual state: $G_i = R_{,i}$; $G^i \cdot G_j = \delta_j^{\ i}$; $G_{ij} = G_i \cdot G_j$; $G^{ij} = G^i \cdot G^j$.

Thus at the instant t the built-up body may be regarded as a three-dimensional manifold referred to Lagrange coordinates on which the three metric tensors $g_{ij}$, $g_{ij}^{\ o}$ and $G_{ij}$ are determined.

The construction of the theory of building up viscoelastic bodies is based on the introduction of three states of the elements of the body: the natural, the initial, and the actual states. We note that in the choice of the initial state there is a certain arbitrariness possible. If the technological process of building up a given object or equipment is specified, then the initial state expresses the true picture of this process (plan of build-up). If the technological process of build-up is not specified, then the initial state may be chosen by proceeding from the conditions of the production of the body or from design considerations or other considerations.

2. Principal Equations and Boundary Conditions. We denote by $\varepsilon_{ij}$ the Almansi strain components $\varepsilon_{ij} = (G_{ij} - g_{ij})/2$. These magnitudes are expressed through the displacement components $u = R - r$; $u = u^i G_i = u_i G^i$ by the formulas [1]

$$\varepsilon_{ij} = (\nabla_i u_j + \nabla_j u_i - \nabla_i u^k \nabla_j u_k) / 2. \tag{2.1}$$

Here, $\nabla_i$ is the operator of covariant differentiation with respect to the coordinate $\xi_i$ in the base of the actual state, and summing is carried out according to the repeated subscripts.

We denote by $\sigma^{ij}$ the Cauchy stress components measured per unit surface area of the deformed body, in the base of the actual state. If the processes of applying the external load and of build-up are sufficiently slow, then the forces of inertia may be neglected and the problem of deformation of the body may be examined in the quasistatic statement. Then the equations of equilibrium of an element of the body have the form

$$\nabla_i \sigma^{ij} + \Phi^j = 0. \tag{2.2}$$

Here $\Phi = \Phi^i G_i$ is the vector of mass force acting on unit volume.

In a viscoelastic aging body, the stress components $\sigma^{ij}(t, \xi)$ are correlated with the strain components $\varepsilon_{ij}{}^{\circ} = (G_{ij} - g_{ij}{}^{\circ})/2$ by the relations

$$\sigma^{ij}(t, \xi) = F^{ij}[t - \varkappa(\xi), \tau - \varkappa(\xi), \xi, \varepsilon_{ij}{}^{0}(\tau, \xi)]; \quad \tau \in [\varkappa(\xi), t]. \tag{2.3}$$

Here, $F^{ij}$ are some functionals based on the system of axioms of the mechanics of the continuum [1-3] whose form is determined by the elastic and rheological properties of the built-up viscoelastic body, and $\varkappa(\xi)$ is the instant of origin (production) of the element of the body with the coordinate $\xi$ [4].

The equation of state in the form (2.3) describes simultaneously the effect of two kinds of inhomogeneity of the viscoelastic body. The first kind is the age inhomogeneity which exists in aging material only. The second kind of inhomogeneity, when elements of the built-up body are made of different materials, exists also in nonaging bodies.

Putting $\varepsilon_{ij}{}^{\circ} = \varepsilon_{ij} + \varepsilon^*_{ij}$, where $\varepsilon^*_{ij} = (g_{ij} - g_{ij}{}^{\circ})/2$, we write the equation of state (2.3) in the form

$$\sigma^{ij}(t, \xi) = F^{ij}[t - \varkappa(\xi), \tau - \varkappa(\xi), \xi, \varepsilon_{ij}(\tau, \xi) + \varepsilon^*_{ij}(\xi)]. \tag{2.4}$$

For the sake of simplicity we assume that the surface load has been specified on the entire boundary of the domain D(t)

$$\sigma^{ij} n_i = f^j; \quad R \in \Gamma(t). \tag{2.5}$$

Here $f = f^j G_j$ is the vector of surface forces per unit surface area of the deformed body; $n = n_j G^j$ is the vector of the unique outer normal to the surface $\Gamma(t)$.

The system of equations (2.1)-(2.4) with the boundary conditions (2.5) determines the state of stress and strain in the built-up body.

In investigations of the deformation of a body with fixed boundary it is usually assumed that the initial state coincides with the natural state, i.e., $\varepsilon^*_{ij} = 0$. In problems of build-up this assumption is correct for the basic body but it need not apply to the elements that are being built up. In build-up with preload, the built-up element is first transferred from the natural state to the initial state, and it is maintained in this by an additional load which is relieved when this element grows together with the body. Here the state of stress and strain in the body depends on the tensor $\varepsilon^*_{ij}$ which has to be specified at each point of the built-up element.

In the general case the strain tensor $\varepsilon^*_{ij}$ is a function of the Lagrange coordinates of the points of the built-up elements $\xi$. It is technically expedient to specify the tensor $\varepsilon^*_{ij}$ either as a function of the radius vector of the point in the initial state $r(\xi)$ or as a function of the radius vector of the point in the actual state at the instant of growing together with the basic body $R[\varkappa(\xi), \xi]$. We note that these approaches lead to fundamentally different results. If the material of the body is elastic, then in the first case the state of stress and strain in the body does not depend on the history of build-up and loading, and it is determined solely by the shape of the body at the running instant. In the second case the state of stress and strain in the body depends substantially on the rate of build-up v and on the history of loading. Below both approaches are illustrated on actual examples.

To describe the process of build-up with preload we use the preload strain tensor $e^*_{ij}$. We note that this approach differs somewhat from the approach known from the literature [4]. In describing the process of build-up it is usual to specify the preload strain tensor [4]. However, for the specified regularity of the state of the material, both approaches are equivalent.

3. Build-Up of a Viscoelastic Pipe Subjected to Aging. We examine the deformation of a circular pipe made of incompressible viscoelastic material. Before deformation, the pipe is a cylinder $K_0$ whose inner radius is $a$, and outer radius is $r_1$. At the instant $t = 0$ external pressure $q_1$ and internal pressure $q_0$ are applied to the body, and the build-up process begins. It consists in the following: within time dt a cylindrical shell dK(t) is laid on the outer contour of the pipe; the thickness of the shell is proportional to dt, and it grows together with the body. The initial state of the built-up element need not coincide with the natural state. We assume that in the transition from the natural state to the initial state and from the initial state to the actual state plane axisymmetric deformation is effected.

The problem of continuous build-up is the limit problem of the following problem of discrete build-up. We have the basic body $K_0$ and a finite set of built-up elements $K_j$ ($j = 1, 2, \ldots, N$). In the initial state the body $K_j$ occupies the domain $\Omega_j$, a cylinder whose inner radius is equal to $r_j$, and the outer radius is equal to $r_{j+1}$. All the cylinders are coaxial. The process of build-up concludes when at the instant $t_j$ the cylinder $\Omega_j$ is "slipped over" $\bigcup_{i=0}^{j-1}\Omega_i$. Assume that the build-up process is concluded and there is no deformation of the cylinders. As a result of the build-up we obtain a body K which occupies the domain $\Omega$. In the domain $\Omega$ we introduce the system of coordinates $r\vartheta z$ whose z axis coincides with the cylinder axis. We take the coordinates $r, \vartheta, z$ as the Lagrange coordinates of the points of the bodies $K_j$. We denote by $\Omega_j^{\,0}$ the domain occupied by the body $K_j$ in the natural state. In symmetric deformation the domain $\Omega_j^{\,0}$ is also a cylinder. We denote by $r^0$, $\vartheta^0$, and $z^0$ the cylindrical coordinates of the points of the domain $\Omega_j^{\,0}$ whose $z^0$ axis has the direction of the cylinder axis.

Going over to the limit with $N \to \infty$, we obtain the coordinates of the points of the body $K_0$ and of the built-up elements $K_j$ in the initial and in the natural states during continuous build-up.

Let R, $\Theta$, Z be the cylindrical coordinates of the body K(t) in the actual state at the instant t. According to our assumption concerning the nature of the deformation $R = R(t, r)$; $r^0 = \psi(r)$; $\Theta = \vartheta^0 = \vartheta$; $Z = z^0 = z$.

The function R, determining the radial displacements in the transition from the initial state to the actual one, is continuously differentiable. The function $\psi$, characterizing the displacement in transition from the initial state to the natural state, may be discontinuous. In the problem of discrete build-up this function is continuously differentiable on the intervals $(r_j, r_{j+1})$, and it may have a discontinuity at the points $r_j$.

The nonzero components of the metric tensors in the initial, natural, and actual states have the form

$$\begin{gathered} g_{11}=1;\quad g_{22}=r^2;\quad g_{33}=1;\quad g^{11}=1;\quad g^{22}=r^{-2};\quad g^{33}=1;\quad g_{11}{}^0=(\psi')^2;\\ g_{22}{}^0=\psi^2;\quad g_{33}{}^0=1;\quad g^{011}=(\psi')^{-2};\quad g^{022}=\psi^{-2};\quad g^{033}=1;\quad G_{11}=(R')^2;\\ G_{22}=R^2;\quad G_{33}=1;\quad G^{11}=(R')^{-2};\quad G^{22}=R^{-2};\quad G^{33}=1. \end{gathered} \tag{3.1}$$

A prime indicates the derivative with respect to the argument r. At the points $r_j$ by derivative of the function $\psi$ we mean the corresponding one-sided derivative.

According to (3.1)

$$G=\det|G_{ij}|=(R'R)^2;\quad g^0=\det|g_{ij}{}^0|=(\psi'\psi)^2;\quad g=\det|g_{ij}|=r^2. \tag{3.2}$$

From (3.2) and the condition of incompressibility of the material $G = g^0 = g$ we obtain

$$R'R=r;\quad \psi'\psi=r. \tag{3.3}$$

When we integrate these equalities, we find

$$R^2(t,r)=r^2+C(t). \tag{3.4}$$

In discrete build-up

$$\psi^2(r)=r^2+X_N(r);\quad X_N(r)=C_j;\quad r\in(r_j,r_{j+1}). \tag{3.5}$$

Here, $C_j$ are the specified constants (in the general case different for each built-up element $K_j$); $C = C(t)$ is a function of time to be determined.

We denote by $X(r)$ the limit in the norm of the domain $L_2$ of the function $X_N(r)$ with $N \to \infty$. According to (3.5) in continuous build-up

$$\psi^2 = r^2 + X(r). \tag{3.6}$$

We denote by $r_0(t)$ the outer radius of the built-up body in the initial state, by $R_0(t)$ its outer radius in the actual state, and by $A(t)$ the inner radius of the built-up cylinder in the actual state at the instant $t$: $R_0(t) = R[t, r_0(t)]$; $A(t) = R(t, a)$. In continuous build-up

$$r_0(t) = r_1 + \int_0^t v(s)\,ds,$$

where $v(t) = dr_0/dt$ is the specified rate of build-up along the normal to the boundary in the initial state.

In continuous build-up the built-up element $dK(t)$ is transferred instantaneously from the natural state to the initial state and grows together with the basic body and changes into the actual state. We denote by $\varkappa(r)$ the instant of growing together of the element $dK(t)$ with the basic body. This value is determined from the equality $r_0[\varkappa(r)] = r$.

We take it that the material of the body satisfies the equation of state

$$\sigma^{ij} = pG^{ij} + \mu(I - L)(g^{0ij} - J_1 G^{ij}/3). \tag{3.7}$$

Here, $\mu$ is a constant determining the properties of the material; $p$ is the pressure; $J_1 = g^{0ij}G_{ij}$ is the first invariant of the measure of Cauchy deformation; $I$ is the unit operator; $L$ is the relaxation operator [4],

$$Ih = h(t, r); \quad Lh = \int_{\varkappa(r)}^t l[t - \varkappa(r), \tau - \varkappa(r)]\,h(\tau, r)\,d\tau,$$

where $l(t, \tau)$ is the relaxation kernel. Relation (3.7) was adopted on the basis of the results of investigations and experiments carried out with polymer and rubberlike materials [5].

In accordance with (3.1), (3.3), we obtain

$$J_1 = 1 + (R/\psi)^2 + (\psi/R)^2. \tag{3.8}$$

From relations (3.7), taking (3.1), (3.3) into account, we find

$$\begin{gathered} \sigma^{11} = p(R')^{-2} + \mu(I - L)[(\psi/r)^2 - J_1(R/r)^2/3]; \\ \sigma^{22} = pR^{-2} + \mu(I - L)(\psi^{-2} - J_1 R^{-2}/3); \\ \sigma^{33} = p + \mu(I - L)(1 - J_1/3). \end{gathered} \tag{3.9}$$

The remaining stress components are equal to zero.

We denote by $\sigma_r$, $\sigma_\vartheta$, $\sigma_z$ the physical stress components $\sigma_r = \sigma^{11}(R')^2$; $\sigma_\vartheta = \sigma^{22}R^2$; $\sigma_z = \sigma^{33}$. From relations (3.9) with the aid of (3.3) we obtain

$$\begin{gathered} \sigma_r = p + \mu R^{-2}(I - L)(\psi^2 - J_1 R^2/3); \quad \sigma_\vartheta = p + \mu R^2(I - L)(\psi^{-2} - J_1 R^{-2}/3); \\ \sigma_z = p + \mu(I - L)(1 - J_1/3). \end{gathered}$$

Eliminating from these relations the unknown pressure $p$, we find

$$\sigma_\vartheta = \sigma_r + \mu f(t, r); \tag{3.10}$$

$$f = R^2(I - L)(\psi^{-2} - J_1 R^{-2}/3) - R^{-2}(I - L)(\psi^2 - J_1 R^2/3). \tag{3.11}$$

The equation of equilibrium of an element of the body has the form [1]

$$\partial\sigma_r/\partial R + (\sigma_r - \sigma_\vartheta)/R = 0. \tag{3.12}$$

We substitute expression (3.10) into relation (3.12)

$$\partial\sigma_r/\partial R = \mu f(t, r)/R.$$

Integrating this equality with respect to R from A(t) to $R_0(t)$ and taking into account that $\sigma_r$ is continuous and that $\sigma_r(t, a) = -q_0$; $\sigma_r(t, r_0(t)) = -q_1$, we obtain

$$\int_{A(t)}^{R_0(t)} \frac{f(t,r)dR}{R} = \frac{q_0 - q_1}{\mu}.$$

We now go over to the integration with respect to the variable r, using equality (3.4):

$$\int_a^{r_0(t)} \frac{rf(t,r)dr}{r^2+C(t)} = \frac{q_0-q_1}{\mu}. \tag{3.13}$$

From (3.4), (3.6), (3.8), and (3.11), we find

$$f = \frac{1}{3}\left\{ \frac{r^2+C}{r^2+X}(I-L)\left[2 - \frac{r^2+X}{r^2+C} - \left(\frac{r^2+X}{r^2+C}\right)^2\right] - \frac{r^2+X}{r^2+C}(I-L)\left[2-\frac{r^2+C}{r^2+X} - \left(\frac{r^2+C}{r^2+X}\right)^2\right]\right\}$$

If we substitute this expression into (3.13), we obtain the integral equation for determining the unknown function C(t)

$$\begin{gathered}\int_{a^2}^{r_0^2(t)} \left\{(I-L)\left[2 - \frac{x+X_1}{x+C} - \left(\frac{x+X_1}{x+C}\right)^2\right] - \left(\frac{x+X_1}{x+C}\right)^2 (I-L)\times\right. \\ \left.\times\left[2 - \frac{x+C}{x+X_1} - \left(\frac{x+C}{x+X_1}\right)^2\right]\right\}\frac{dx}{x+X_1} = 6\mu^{-1}(q_0-q_1); \\ X_1(x) = X(x^{1/2}).\end{gathered} \tag{3.14}$$

Having determined the function C(t), we find the displacements of the points of the pipe in build-up from relation (3.4). If the material of the pipe is elastic (L = 0), then Eq. (3.14) becomes simplified:

$$\int_{a^2}^{r_0^2(t)} \left(\frac{x+C}{x+X_1} - \frac{x+X_1}{x+C}\right)\frac{dx}{x+C} = 2\mu^{-1}(q_0-q_1). \tag{3.15}$$

In build-up without preload, when the initial state of the built-up elements coincides with their natural state (X = 0), we obtain from (3.15) an algebraic equation for determining the function $\alpha(t) = C(t)r_0^{-2}(t)$:

$$\ln\left[1 + \frac{(1-\beta)\alpha}{1+\alpha\beta}\right] + \frac{(1-\beta)\alpha}{(1+\alpha)(1+\alpha\beta)} = \frac{2(q_0-q_1)}{\mu}; \quad \beta = (a/r_0)^2.$$

In build-up of a cylinder of viscoelastic material without preload, when the initial state of the built-up elements coincides with their natural state, we find from relation (3.14):

$$\int_{a^2}^{r_0^2(t)} \left\{(I-L)\left[2 - \frac{x}{x+C} - \left(\frac{x}{x+C}\right)^2\right] - \left(\frac{x}{x+C}\right)^2 (I-L)\left[2 - \frac{x+C}{x} - \left(\frac{x+C}{x}\right)^2\right]\right\}\frac{dx}{x} = 6\mu^{-1}(q_0-q_1).$$

With small deformations, when $Ca^{-2} \ll 1$, it follows from this equality that

$$\int_a^{r_0(t)} (I-L)Cr^{-3}dr = \frac{q_0-q_1}{2\mu}. \tag{3.16}$$

If we substitute the expressions for the operators I and L into (3.16), we obtain an integral equation for determining the function C(t)

$$\int_a^{r_0(t)} \left\{ C(t) - \int_{\varkappa(r)}^{t} l[t-\varkappa(r), \tau-\varkappa(r)] C(\tau) d\tau \right\} r^{-3} dr = \frac{q_0 - q_1}{2\mu}.$$

If we change the sequence of integration, we obtain a Volterra equation of the second kind

$$C(t) - \int_0^t l_1(t,\tau) C(\tau) d\tau = (q_0 - q_1) \{\mu [a^{-2} - r_0^{-2}(t)]\}^{-1};$$

$$l_1(t,\tau) = [a^{-2} - r_0^{-2}(t)]^{-1} \left\{ l(t,\tau)(a^{-2} - r_1^{-2}) + 2\int_{r_1}^{r_0(\tau)} l[t-\varkappa(r), \tau-\varkappa(r)] r^{-3} dr \right\}.$$

We denote by $l_2(t, \tau)$ the relaxation kernel in the form

$$l_2(t,\tau) = [a^{-2} - r_0^{-2}(\tau)]^{-1} [a^{-2} - r_0^{-2}(t)] l_1(t,\tau) =$$

$$= [a^{-2} - r_0^{-2}(\tau)]^{-1} \left[ l(t,\tau)(a^{-2} - r_1^{-2}) + 2\int_0^\tau l(t-s, \tau-s) v(s) r_0^{-3}(s) ds \right],$$

and by $k_2(t, \tau)$ the creep kernel corresponding to it. The function C(t) is determined by the formula

$$C(t) = \mu^{-1}(q_0 - q_1)[a^{-2} - r_0^{-2}(t)]^{-1} \left[ 1 + \int_0^t k_2(t,\tau) d\tau \right]. \tag{3.17}$$

If the relaxation kernel of the material is a difference kernel $l(t, \tau) = l(t - \tau)$, then $l_2(t, \tau) = l(t - \tau)$, and the kernel $k_2(t, \tau)$ coincides with the creep kernel $k(t - \tau)$ of the material of the cylinder. In this case we obtain in accordance with (3.17)

$$C(t) = \mu^{-1}(q_0 - q_1)[a^{-2} - r_0^{-2}(t)]^{-1} \left[ 1 + \int_0^t k(\tau) d\tau \right].$$

According to (3.4), radial displacement of points of the pipe with small deformations is determined by the formula

$$u_r = R - r = C(t) / (2r). \tag{3.18}$$

Relations (3.17), (3.18) are analogous to the expressions for the displacements of points of a built-up cylinder with small deformations obtained in [4].

If we determine build-up without preload as build-up in which the natural state of the built-up element coincides with its actual state at the instant of growing together with the basic body, we obtain from relations (3.4), (3.6)

$$X(r) = C[\varkappa(r)]; \quad r \geqslant r_1. \tag{3.19}$$

For the body $K_0$ the initial state coincides with the natural state:

$$X(r) = 0; \quad a \leqslant r \leqslant r_1. \tag{3.20}$$

In this case the equation for determining the function C(t) assumes the form

$$\int_{a^2}^{r_1^2} \left\{ (I-L)\left[ 2 - \frac{x}{x+C} - \left(\frac{x}{x+C}\right)^2 \right] - \left(\frac{x}{x+C}\right)^2 (I-L) \times \right.$$

$$\left. \times \left[ 2 - \frac{x+C}{x} - \left(\frac{x+C}{x}\right)^2 \right] \right\} \frac{dx}{x} + \int_{r_1^2}^{r_0^2(t)} \left\{ (I-L)\left[ 2 - \frac{x+C^0}{x+C} - \right.\right.$$

$$-\left(\frac{x+C^0}{x+C}\right)^2\Big]-\left(\frac{x+C^0}{x+C}\right)^2(I-L)\left[2-\frac{x+C}{x+C^0}-\left(\frac{x+C}{x+C^0}\right)^2\right]\Big\}\frac{dx}{x+C^0}=6\mu^{-1}(q_0-q_1);\quad C^0=C[\varkappa(x^{1/2})]. \tag{3.21}$$

If the material of the cylinder is elastic ($L = 0$), then we find from (3.21)

$$\int_a^{r_1}\left[\frac{r^2+C(t)}{r^2}-\frac{r^2}{r^2+C(t)}\right]\frac{rdr}{r^2+C(t)}+\int_{r_1}^{r_0(t)}\left[\frac{r^2+C(t)}{r^2+C(\varkappa(r))}-\frac{r^2+C[\varkappa(r)]}{r^2+C(t)}\right]\frac{rdr}{r^2+C(t)}=\frac{q_0-q_1}{\mu}.$$

In the case of small deformations, when $Ca^{-2} \ll 1$, this equation becomes simplified:

$$\int_a^{r_1}\frac{C(t)dr}{r^3}+\int_{r_1}^{r_0(t)}\frac{C(t)-C[\varkappa(r)]}{r^3}dr=\frac{q_0-q_1}{2\mu}.$$

We transform this relation:

$$C(t)[a^{-2}-r_0^{-2}(t)]-2\int_{r_1}^{r_0(t)}\frac{C[\varkappa(r)]}{r^3}dr=\frac{q_0-q_1}{\mu},$$

and hence we find for $q_0$ = const, $q_1$ = const

$$C(t)=(q_0-q_1)[\mu(a^{-2}-r_1^{-2})]^{-1}. \tag{3.22}$$

It follows from (3.18), (3.22) that with a selected type of build-up, the radial displacements of the built-up layers do not depend on time, and they are inversely proportional to the distance from the cylinder axis.

Assume that the material of the body is viscoelastic. Then the equation for determining the function $C(t)$ with small deformations assumes the form

$$\int_a^{r_1}(I-L)\frac{Cdr}{r^3}+\int_{r_1}^{r_0(t)}(I-L)\frac{(C-C^0)dr}{r^3}=\frac{q_0-q_1}{2\mu}.\quad C^0=C[\varkappa(r)].$$

When we substitute the expressions for the operators I and L into this relation, we find

$$C(t)\int_a^{r_0(t)}\frac{dr}{r^3}-\int_a^{r_1}\frac{dr}{r^3}\int_0^t l(t,\tau)C(\tau)d\tau-\int_{r_1}^{r_0(t)}\frac{dr}{r^3}\int_{\varkappa(r)}^t l[t-\varkappa(r),\tau-\varkappa(r)]C(\tau)d\tau-$$
$$-\int_{r_1}^{r_0(t)}C[\varkappa(r)]\frac{dr}{r^3}+\int_{r_1}^{r_0(t)}\frac{C[\varkappa(r)]dr}{r^3}\int_{\varkappa(r)}^t l[t-\varkappa(r),\tau-\varkappa(r)]d\tau=\frac{q_0-q_1}{2\mu}.$$

Replacing the variable $r = r_0(s)$, $s = \varkappa(r)$ in the integrand, we obtain

$$C(t)\int_a^{r_0(t)}\frac{dr}{r^3}-\int_0^t l(t,\tau)C(\tau)d\tau\int_a^{r_1}\frac{dr}{r^3}-\int_0^t\frac{v(s)ds}{r_0^3(s)}\int_s^t l(t-s,\tau-s)C(\tau)d\tau-\int_0^t C(s)\frac{v(s)ds}{r_0^3(s)}+$$
$$+\int_0^t C(s)\frac{v(s)ds}{r_0^3(s)}\int_s^t l(t-s,\tau-s)d\tau=\frac{q_0-q_1}{2\mu}. \tag{3.23}$$

Introducing the notation

$$H_1(t)=\int_t^{r_0(t)}\frac{dr}{r^3};\quad H_2(t,\tau)=l(t,\tau)\int_a^{r_1}\frac{dr}{r^3}+\int_0^{\tau}l(t-s,\tau-s)\frac{v(s)ds}{r_0^3(s)}+\frac{v(\tau)}{r_0^3(\tau)}\left[1-\int_{\tau}^t l(t-\tau,s-\tau)ds\right],$$

we rewrite Eq. (3.23) in the form

$$H_1(t)C(t)-\int_0^t H_2(t,\tau)C(\tau)d\tau=\frac{q_0-q_1}{2\mu}. \tag{3.24}$$

With $q_1 = 0$, Eqs. (3.18), (3.24) coincide with an accuracy within the notation with the analogous equations obtained for the problem of build-up of a hollow cylinder with small deformations in [4, p. 119].

4. Build-Up of a Hollow Sphere of Aging Viscoelastic Material. We examine the deformation of a hollow sphere of incompressible viscoelastic material. Before deformation, the body $K_0$ is a hollow sphere whose inner radius is $a$, and the outer radius is $r_1$. At the instant $t = 0$ the external pressure $q_1$ and the internal pressure $q_0$ are applied to the body, and the process of build-up begins; it ends when within time dt the film $dK(t)$, whose thickness is proportional to dt, is superposed on the outer contour of the sphere. After being superposed, the film grows instantaneously together with the sphere. The initial state of the built-up element need not coincide with its natural state. We assume that upon transition from the natural state to the initial state and from the initial to the actual state, centrally symmetric deformation is effected.

The problem of continuous build-up is the limit problem for the following problem of discrete build-up. We have the basic body $K_0$ and a finite set of built-up elements $K_j$ ($j = 1, 2, \ldots, N$). In the initial state the body $K_j$ occupies the domain $\Omega_j$, a hollow sphere whose inner radius is $r_j$ and outer radius is $r_{j+1}$. The process of build-up consists in the following: at the instant $t_j$ the domain $\Omega_j$ is joined to the domain $\bigcup_{i=0}^{j-1}\Omega_i$. We assume that the process of build-up is completed and that there is no deformation of the bodies $K_j$. As a result of the build-up we obtain the body K which occupies the domain $\Omega$. We introduce into $\Omega$ the spherical coordinates $r, \vartheta, \lambda$; the origin of coordinates coincides with the center of the sphere. We regard the coordinates $r$, $\vartheta$, and $\lambda$ as the coordinates of the points of the bodies $K_j$ in the initial state.

We denote by $\Omega_j{}^0$ the domain occupied by the body $K_j$ in the natural state. In accordance with our assumptions concerning the nature of the deformation, the domain $\Omega_j{}^0$ is a hollow sphere. We denote by $r^0, \vartheta^0, \lambda^0$ the spherical coordinates of the points of the domain $\Omega_j{}^0$. Going over to the limit $N \to \infty$, we obtain the coordinates of the points of the initial body and of the built-up elements in the initial and in the natural state in continuous build-up.

Let R, $\Theta$, and $\Lambda$ be the spherical coordinates of a point in the actual state at the instant t. From the condition of central symmetry of the deformation these equalities follow: $R = R(t, r)$; $r^0 = \psi(r)$; $\Theta = \vartheta^0 = \vartheta$; $\Lambda = \lambda^0 = \lambda$.

The nonzero components of the metric tensors in the initial, natural, and actual states have the form

$$\begin{gathered} g_{11}=1;\quad g_{22}=r^2;\quad g_{33}=r^2\sin^2\vartheta;\quad g^{11}=1;\quad g^{22}=r^{-2};\quad g^{33}=r^{-2}\sin^{-2}\vartheta;\\ g_{11}{}^0=(\psi')^2;\quad g_{22}{}^0=\psi^2;\quad g_{33}{}^0=\psi^2\sin^2\vartheta;\quad g^{011}=(\psi')^{-2};\quad g^{022}=\psi^{-2};\\ g^{033}=\psi^{-2}\sin^{-2}\vartheta;\\ G_{11}=(R')^2;\quad G_{22}=R^2;\quad G_{33}=R^2\sin^2\vartheta;\quad G^{11}=(R')^{-2};\quad G^{22}=R^{-2};\\ G^{33}=R^{-2}\sin^{-2}\vartheta. \end{gathered} \tag{4.1}$$

In accordance with (3.2), (4.1), we find

$$G=(R'R^2\sin\vartheta)^2;\quad g^0=(\psi'\psi^2\sin\vartheta)^2;\quad g=(r^2\sin\vartheta)^2 . \tag{4.2}$$

From relations (4.2) and the condition of incompressibility of the material we obtain

$$R'R^2=r^2;\quad \psi'\psi^2=r^2. \tag{4.3}$$

When we integrate these equalities, we find in analogy to (3.4)-(3.6)

$$R^3(t,r)=r^3+C(t), \tag{4.4}$$

with discrete build-up

$$\psi^3(r)=r^3+X_N(r); \quad X_N(r)=C_j; \quad r\in(r_j,r_{j+1});$$

with continuous build-up

$$\psi^3(r)=r^3+X(r). \tag{4.5}$$

Here, $C_j$ are the specified constants; $X(r) = \lim_{N\to\infty} X_N(r)$; $C = C(t)$ is a function of time to be determined.

We denote by $r_0(t)$ the outer radius of the built-up sphere in the initial state, by $R_0(t)$ its outer radius in the actual state, and by $A(t)$ the inner radius of the sphere in the actual state at the instant t:

$$R_0=R[t,r_0(t)]; \quad A=R(t,a); \quad r_0(t)=r_1+\int_0^t v(s)ds,$$

where $v(t) = dr_0/dt$ is the specified rate of build-up of the sphere along the normal to the boundary in the initial state.

We assume that the material of the sphere satisfies the equation of state (3.7). In accordance with (4.1), (4.3), we find

$$J_1=(\psi/R)^4+2(\psi/R)^2. \tag{4.6}$$

From relations (3.7), (4.1), (4.3), we obtain

$$\sigma^{11}=p(R')^{-2}+\mu r^{-4}(I-L)(\psi^4-J_1R^4/3);$$
$$\sigma^{22}=pR^{-2}+\mu(I-L)(\psi^{-2}-J_1R^{-2}/3);$$
$$\sigma^{33}=[pR^{-2}+\mu(I-L)(\psi^{-2}-J_1R^{-2}/3)]\sin^{-2}\vartheta.$$

The remaining stress components are equal to zero. We denote by $\sigma_r = \sigma^{11}(R')^2$; $\sigma_\vartheta = \sigma^{22}R^2$; $\sigma_\lambda = \sigma^{33}(R\sin\vartheta)^2$ the physical stress components

$$\sigma_r=p+\mu R^{-4}(I-L)(\psi^4-J_1R^4/3); \quad \sigma_\vartheta=\sigma_\lambda=p+\mu R^2(I-L)(\psi^{-2}-J_1R^{-2}/3).$$

When we eliminate the unknown pressure p from these equalities, we find

$$\sigma_\vartheta=\sigma_r+\mu f(t,r);$$
$$f(t,r)=R^2(I-L)(\psi^{-2}-J_1R^{-2}/3)-R^{-4}(I-L)(\psi^4-J_1R^4/3). \tag{4.7}$$

The equation of equilibrium of an element of the body has the form [1]

$$\partial\sigma_r/\partial R+2(\sigma_r-\sigma_\vartheta)/R=0.$$

Integrating this equality with respect to R from $A(t)$ to $R_0(t)$ and taking into account the continuity of $\sigma_r$ and the boundary conditions $\sigma_r(t, a) = -q_0$; $\sigma_r[t, r_0(t)] = -q_1$, we obtain

$$\int_{A(t)}^{R_0(t)} \frac{f(t,r)dR}{R}=\frac{q_0-q_1}{2\mu}.$$

In this relation we go over to integrating with respect to the variable r with the aid of (4.3):

$$\int_a^{r_0(t)} \frac{r^2 f(t,r)dr}{r^3+C(t)}=\frac{q_0-q_1}{2\mu}. \tag{4.8}$$

We transform expression (4.8) using relations (4.4)-(4.7):

$$\int_{a^3}^{r_0^3(t)} \left(\frac{x+C}{x+X_1}\right)^{2/3}\left\{(I-L)\left[1-\left(\frac{x+X_1}{x+C}\right)^2\right]-2\left(\frac{x+X_1}{x+C}\right)^2(I-L)\times\right.$$
$$\left.\times\left[1-\left(\frac{x+C}{x+X_1}\right)^2\right]\right\}\frac{dx}{x+C}=\frac{9(q_0-q_1)}{2\mu}; \quad X_1(x)=X(x^{1/3}). \tag{4.9}$$

If the material of the body is elastic (L = 0), Eq. (4.9) becomes simplified:

$$\int\limits_{a^3}^{r_0^3(t)} \left(\frac{x+C}{x+X_1}\right)^{2/3} \left[3-2\left(\frac{x+C}{x+X_1}\right)^2-\left(\frac{x+X_1}{x+C}\right)^2\right]\frac{dx}{x+C}=\frac{9(q_0-q_1)}{2\mu}.$$

With small deformations ($Ca^{-3} \ll 1$) we find from Eq. (4.9) that

$$\int\limits_a^{r_0(t)} (I-L)(C-X)\frac{dr}{r^4}=\frac{q_0-q_1}{4\mu}. \quad (4.10)$$

In accordance with (4.4) the radial displacement of the points of the sphere is then determined by the formula

$$u_r=R-r=C(t)/(3r^2). \quad (4.11)$$

Assume that the build-up proceeds in such a way that the natural state of the built-up element coincides with its actual state at the instant of growing together with the basic body. Then the function X(r) has the form of (3.19), (3.20), From (4.10), we have

$$\int\limits_a^{r_0(t)} (I-L)Cr^{-4}dr-\int\limits_{r_1}^{r_0(t)} (I-L)C^0r^{-4}dr=\frac{q_0-q_1}{4\mu}; \quad C^0=C[\varkappa(r)].$$

Into this equality we substitute the expressions for the operators I and L:

$$\begin{gathered} C(t)\int\limits_a^{r_0(t)}\frac{dr}{r^4}-\int\limits_a^{r_1}\frac{dr}{r^4}\int\limits_0^t l(t,\tau)C(\tau)d\tau-\int\limits_{r_1}^{r_0(t)}\frac{dr}{r^4}\int\limits_{\varkappa(r)}^t l[t-\varkappa(r),\tau-\varkappa(r)]C(\tau)d\tau- \\ -\int\limits_{r_1}^{r_0(t)}\frac{C[\varkappa(r)]dr}{r^4}+\int\limits_{r_1}^{r_0(t)}\frac{C[\varkappa(r)]dr}{r^4}\int\limits_{\varkappa(r)}^t l[t-\varkappa(r),\tau-\varkappa(r)]d\tau=\frac{q_0-q_1}{4\mu}. \end{gathered} \quad (4.12)$$

We substitute the variables of integration $r = r_0(s)$, $s = \varkappa(r)$. Then we obtain from (4.12):

$$\begin{gathered} C(t)\int\limits_a^{r_0(t)}\frac{dr}{r^4}-\int\limits_a^{r_1}\frac{dr}{r^4}\int\limits_0^t l(t,\tau)C(\tau)d\tau-\int\limits_0^t\frac{v(s)ds}{r_0^4(s)}\int\limits_s^t l(t-s,\tau-s)C(\tau)d\tau- \\ -\int\limits_0^t\frac{C(s)v(s)ds}{r_0^4(s)}+\int\limits_0^t\frac{C(s)v(s)ds}{r_0^4(s)}\int\limits_s^t l(t-s,\tau-s)d\tau=\frac{q_0-q_1}{4\mu}. \end{gathered} \quad (4.13)$$

We introduce the notation

$$H_1(t)=\int\limits_a^{r_0(t)}\frac{dr}{r^4}; \quad H_2(t,\tau)=l(t,\tau)\int\limits_a^{r_1}\frac{dr}{r^4}+\int\limits_0^\tau l(t-s,\tau-s)\frac{v(s)ds}{r_0^4(s)}+\frac{v(\tau)}{r_0^4(\tau)}\left[1-\int\limits_\tau^t l(t-\tau,s-\tau)ds\right]. \quad (4.14)$$

In accordance with (4.14), Eq. (4.13) assumes the form

$$H_1(t)C(t)-\int\limits_0^t H_2(t,\tau)C(\tau)d\tau=\frac{q_0-q_1}{4\mu}. \quad (4.15)$$

For $q_1 \equiv 0$, Eqs. (4.11), (4.15) coincide accurately within the notation with the analogous expressions obtained in [4, p. 111] for the problem of build-up of a hollow sphere with small deformations.

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## LITERATURE CITED

1. A. I. Lur'e, Theory of Elasticity [in Russian], Moscow (1970).
2. K. Truesdell, Beginner's Course of Rational Mechanics of the Continuum [Russian translation], Moscow (1975).
3. B. D. Coleman and W. Noll, "Foundations of linear viscoelasticity," Rev. Mod. Phys., 33, No. 2, 239-249 (1961).
4. N. Kh. Arutyunyan and V. B. Kolmanovskii, Theory of Creep of Inhomogeneous Bodies [in Russian], Moscow (1983).
5. A. A. Adamov, "Identification of the model of inherited elasticity in finite deformations," in: Structural Mechanics of Inhomogeneous Media [in Russian], Sverdlovsk (1982), pp. 8-11.

# EFFECT OF FILLERS AND RADIATION CROSS-LINKING ON STRESS RELAXATION IN POLYETHYLENE

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The processes of stress relaxation in polyethylene (PE) are due to rearrangement of the structure of different degrees of organization, and the value of the activation energy characterizes the degree of ordering of the structure of the polymer [1]. The addition of different additives to PE and the effect of ionizing radiation result in a change in the crystal structure and the appearance of new intermolecular chemical bonds in the polymer. The structural transformations should affect the mechanism of tensile deformation of samples of modified PE and the character of the rearrangement of the structure during stress relaxation in the polymer. The method of stress relaxation with initial deformations of $\varepsilon_0$ = 10% in the region where intense formation of submicron fissures is characteristic was used to evaluate the degree of modification of the properties of PE.

Brand 15303-003 high-pressure PE was used for the study. The rutile form of titanium dioxide, $TiO_2$, with a specific surface of $(5-7)\cdot 10^3$ $m^2/kg$ and particle size of less than $10^{-3}$ mm, and brand NSO-6 chopped glass fibers (GF) with an unfinished surface $(8-10)\cdot 10^{-3}$ mm in diameter and 5-6 mm long were used as the fillers. The choice of fillers was due to the fact that $TiO_2$ can be an efficient cross-linking agent in PE. Addition of $TiO_2$ in the volume fraction of approximately 0.1% primarily affects the formation of the small spherulite structure, and addition of approximately 2.0% also causes an increase in the degree of crystallinity and an increase in the cross sections of the crystallites [2]. Addition of GF improves the mechanical [3] and thermophysical [4] properties of the polymer, and samples of such polymers are not destroyed when $\varepsilon_0$ = 10%.

Filled PE composites were prepared on a Brabender plastograph at 443°K. The samples for the studies were prepared by hot molding; the working size of the samples was 1 × 8 × 50 mm. The samples were irradiated in air with an ÉlT-1.5 electron accelerator with a dose rate of ~10 kGr/sec. For elimination of postradiation effects in the PE, heat treatment at 403°K for 300 sec was immediately conducted after irradiation. Stress relaxation in the samples of PE was studied on a setup which ensured fixed deformation.

The experimental data were processed by the method of relaxation spectrometry, which permits determining the discrete relaxation time spectrum and distinguishing the most probable elementary processes [1, 5]. Five elementary processes of stress relaxation were observed for the samples of high-pressure PE during the observation. In this case, the relaxing stress $\sigma(t)$ is described by the expression [6]

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