DEFECTS OF THE DELAMINATION TYPE IN COMPOSITE STRUCTURES

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The high efficiency of laminated and laminofibrous composite structures is achieved by matching the fields of the strength characteristics with the external stress fields. By creating high strength in the direction of the laminations, we obtain low interlaminar shear and normal tensile strengths. In its turn, the loss of interlaminar monolithicity results in a fall in the strength and stiffness of the structure relative to the principle design loads. Laminated composite structures are especially sensitive to defects of the delamination type. These defects can arise both in the fabrication process (e.g., as a result of the action of interlaminar residual stresses [1]) and under the influence of service loads. In connection with the establishment of defect tolerances for acceptance and routine maintenance purposes it is necessary to investigate such questions as the maximum size of the safe (stable) defects, the rate of growth of defects under cyclic and long-term loading, and the loss of carrying capacity of structures with defects of the delamination type. A number of problems of the stability of bars and shells with such defects subjected to static compressive loads were examined in [2, 3]. A review of the work on the application of the methods of fracture mechanics to laminated composite structures can be found in [4, 5].

In fracture mechanics it is possible to distinguish between the global (energy) approach based on a consideration of the body—load system as a whole and various local approaches. The global approach has its origin in the classical work of Griffith. Though the premises may differ, the local approaches (Irwin criterion, various models of the end zone, etc.) have a common feature: the fracture conditions are expressed in terms of the parameters of the processes taking place directly at the crack front. The local criteria should also be taken to include all the semiempirical crack growth equations, since these relate the crack growth rate exclusively to the values of the parameters at the front. In the local approach we disregard the method of loading and how the loads behave during crack development. At the same time, experimenters know how strongly the results of mechanical tests depend on the method of loading. For example, a low-cycle fatigue crack develops differently depending on whether the method of loading is "hard" (given displacements) or "soft" (given loads). In the overwhelming majority of fracture mechanics problems studied so far this difference is of no significance, since the stiffness of the body as a whole is little affected by crack growth. However, this factor may be decisive in connection with thin exfoliations and, in general, with all problems where geometric nonlinearity must be taken into account.

In this article problems of the mechanics of bodies with defects of the delamination type are examined on the basis of the axiomatic approach [6]. This approach, which generalizes Griffith's energetic concept, includes both the case of multiparameter cracks under conditions of multiparameter loading and the description of the process of quasiequilibrium crack growth under cyclic and (or) long-term loads. Here the theory of quasiequilibrium growth is based on a joint consideration of two mechanisms — the development of the macroscopic crack and the accumulation of diffuse (microscopic) damage along the path of the growing crack.

1. Let us consider a body with cracks, the size, shape, and distribution of which within the body are given correct to the m parameters $l_1, \ldots, l_m$. We choose these parameters in such a way that the conditions of irreversibility have the form $dl_j \geq 0$ $(j = 1, \ldots, m)$. Let the body be subject to a monotonic loading process given correct to the μ parameters $s_1, \ldots, s_\mu$. These parameters may include the values of the external forces, the displacements, the temperature parameters, etc. For brevity, in what follows we will employ the vector notation $\mathbf{l} = \{l_1, \ldots, l_m\}^T$ and $\mathbf{s} = \{s_1, \ldots, s_\mu\}^T$. The system of cracks thus specified we will call m-parametric, and the loading μ-parametric. The loading process and the response of the body

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will be assumed to be relatively slow: at each moment of time at each point of the body the equilibrium conditions are fulfilled. The material of the body may be elastic or viscoelastic, elastoplastic, etc.

Starting from Griffith's concept, we will give axiomatic definitions of the notions of equilibrium and stability for bodies with cracks. Let I be a certain functional characterizing the state of the load—body system at a certain fixed moment of time. We assign to the parameters $l_1, \ldots, l_m$ arbitrary small increments $\delta l_1 \geq 0, \ldots, \delta l_m \geq 0$ at $|\delta 1| > 0$. Let us consider the arbitrary small increment δI of the functional I calculated for the following conditions: time and the given loads and external influences are not varied, there is no supply of heat, and at each point of the body, except perhaps for a small neighborhood of the crack fronts, the equilibrium and strain compatibility conditions are satisfied. We will call δI the Griffith variation of the functional I. Obviously, this variation is a linear form of the increments δl_j. At the same time, we introduce a second Griffith variation $\delta^2 I = \delta(\delta I)$, which is a quadratic differential form. These forms play a central role in the Griffith theory extended to the case of multiparameter cracks and multiparameter loading.

We will calculate the Griffith variation from the total energy of the body—external load system taken with the opposite sign:

$$\delta I = -\delta U + \delta A - \delta \Phi - 2 \sum_i \int_{S_i} \gamma |\delta \lambda \times d\sigma|. \tag{1.1}$$

On the right we have the Griffith variation δU of the strain energy of the body U. the elementary work δA done by the external forces, and the elementary dissipation $\delta \Phi$ in the material of the body. The last term on the right is equal to the elementary work expended on crack growth. Here, γ is the amount of energy going toward the formation of unit crack surface; $\delta\lambda$ is the crack size increment; $d\sigma$ is an arc element of its contour. We carry out the summation over the contours S_i of all the cracks in the body. In general, the quantity γ may be a variable even within the limits of a single crack. By a suitable choice of the sum $\delta U + \delta \Phi$ we can take into consideration materials with different mechanical properties.

We will introduce a classification of systems of cracks in a loaded body. We will call a system subequilibrium if for all $|\delta 1| > 0$ the inequality $\delta I > 0$ is satisfied, and equilibrium (according to Griffith) if for all $|\delta 1| > 0$ we have $\delta I = 0$. If among the variations there is at least one for which $\delta I > 0$, we will call the system of cracks nonequilibrium.

The Griffith equilibrium condition means that upon a small change in the sizes of the cracks the total increment in the work done by the external forces and the potential energy liberated is exactly compensated by the dissipation in the body and the work expended on crack growth. We introduce the following notation for the group of terms on the right side of relation (1.1):

$$\begin{gathered} -\delta U + \delta A - \delta \Phi \equiv \sum_{j=1}^{m} G_j(\mathbf{l}, \mathbf{s}) \delta l_j; \\ \delta A_\gamma \equiv 2 \sum_i \int_{S_i} \gamma |\delta\lambda \times d\sigma| \equiv \sum_{j=1}^{m} \Gamma_j(\mathbf{l}, \mathbf{s}) \delta l_j. \end{gathered} \tag{1.2}$$

The quantities G_j and Γ_j are the generalized forces corresponding to the generalized coordinates l_j. We will call G_j the generalized forces driving the cracks, and Γ_j the generalized forces of resistance. Relating the dissipation $\delta\Phi$ to the forces driving the crack is conventional, but convenient, since in dissipative materials it is often impossible to draw a line between δU and $\delta \Phi$. Moreover, $\delta\Phi$ takes into account the dissipation throughout the body whereas the work δA_γ corresponds to the processes localized in a small neighborhood of the crack fronts. In terms of the generalized forces, using (1.1) and (1.2), we find that for subequilibrium cracks all $G_j \leqslant \Gamma_j$, and for equilibrium cracks all $G_j = \Gamma_j$ $(j = 1, \ldots, m)$. If for at least one of the generalized coordinates $G_j > \Gamma_j$, then the system of cracks as a whole is regarded as nonequilibrium.

Clearly, subequilibrium cracks are stable, nonequilibrium cracks are unstable. Equilibrium cracks may be either stable or unstable depending on the sign of the second variation $\delta^2 I$. If $\delta^2 I < 0$, then the system of equilibrium cracks is stable; if $\delta^2 I > 0$, it is unstable. When $\delta^2 I = 0$, it is necessary to investigate the sign of the third variation and so on. In applications it is usual for some of the cracks to be subequilibrium and for some, or at least one, to be equilibrium. If a crack is given correct to two or more parameters, it may be subequilibrium with respect to some parameters and equilibrium with respect to the rest. Since the subequilibrium cracks are stable, in analyzing the stability of the system of cracks as a whole we should exclude from the variation those generalized coordinates for which $G_j < \Gamma_j$. Thus, the dimension of the vector $\mathbf{l}$ is reduced to the number of generalized coordinates for which $G_j = \Gamma_j$, and the remaining generalized coordinates play the part of parameters.

The approach outlined is, essentially, the original Griffith method extended to multiparameter problems and explained in terms of analytical mechanics. Significantly, this approach makes it possible to describe slow (quasiequilibrium) crack growth. In [6] the author formulated an hypothesis to the effect that, when the stress reaches a local extremum, a slowly growing crack satisfies the Griffith equilibrium condition with allowance for the fact that the resistance of the material to the propagation of the crack is reduced as a result of the previously accumulated damage. In special cases this approach leads to equations similar in structure to the known semiempirical equations of fatigue crack growth. However, it is also suitable in more general situations, in particular in connection with multiparameter loading processes and combinations of long-term quasistatic and cyclic loading processes.

2. Let us consider a quasistatic loading process slowly progressing in time t. We denote the set of characteristic crack dimensions at time t by $\mathbf{l}(t)$, and the set of loading parameters by $\mathbf{s}(t)$. Let the maximum loads and crack dimensions be such that at any moment of time all the crack dimensions are less than the equilibrium values calculated for the undamaged material. We introduce the basic assumption of the theory: the cracks grow in such a way that at the moments when the parameters $s_1, \ldots, s_\mu$ reach their most dangerous (usually maximum) values the condition $\delta I = 0$ is satisfied, with allowance for the damage accumulated along the path of the cracks throughout the previous loading process. As a result we obtain the system of equations

$$\underset{\tau=0}{\overset{\tau=t}{G_j}}[\mathbf{l}(\tau), \mathbf{s}(\tau)] = \underset{\tau=0}{\overset{\tau=t}{\Gamma_j}}[\mathbf{l}(\tau), \mathbf{s}(\tau)] \quad (j=1, \ldots, m). \tag{2.1}$$

Equations (2.1) relate the generalized forces driving the crack to the corresponding generalized resistances. The equations contain memory functionals that take into account the loading history on the interval [0, t]. For the purposes of direct numerical solution Eqs. (2.1) are subjected to discretization. As a result we arrive at the equations

$$G_j[\mathbf{l}(1), \mathbf{s}(1), \ldots, \mathbf{l}(n), \mathbf{s}(n)] = \Gamma_j[\mathbf{l}(1), \mathbf{s}(1), \ldots, \mathbf{l}(n), \mathbf{s}(n)] \\ (j=1, \ldots, m). \tag{2.2}$$

If the loading is cyclic, then the discrete argument n entering into Eqs. (2.2) represents the number of loading cycles or units. In this case the vector parameters $\mathbf{s}(n)$ are referred to the n-th cycle or unit and include the extreme values of the loads, their amplitudes, etc., together with the duration of the given cycle or unit measured in units of time t. Equations (2.2) can be solved for $\mathbf{l}(1)$, $\mathbf{l}(2)$, ... by a recurrence method, if the initial condition $\mathbf{l}(0) = \mathbf{l}_0$ is given. In fatigue crack growth calculations the numbers of cycles are usually very large. In these circumstances it is convenient to treat the number of cycles from the start of loading as a continuous argument, and the crack dimensions as continuous functions of that argument. If the loading vector $\mathbf{s}(n)$ is a slowly varying function of n, then instead of (2.2) we again obtain the system of functional equations (2.1).

The form of the functionals entering into Eqs. (2.1) depends on the memory properties of the material and on the choice of damage accumulation model. The memory properties of the material are introduced via the parameters of the undisturbed state in computing the variation (1.1) at the moment of time in question. The damage accumulated at the crack front can be taken into account by introducing the phenomenological damage criteria $\psi_1(t), \ldots, \psi_\nu(t)$. If the material does not possess memory properties, then instead of (2.1) we have

$$G_j[\mathbf{l}(t), \mathbf{s}(t), \boldsymbol{\psi}(t)] = \Gamma_j[\mathbf{l}(t), \mathbf{s}(t), \boldsymbol{\psi}(t)] \quad (j=1, \ldots, m), \tag{2.3}$$

where $\boldsymbol{\psi} = \{\psi_1, \ldots, \psi_\nu\}^T$ is the damage vector. In general, the damage is scattered throughout the body. However, we are primarily interested in the damage at the crack fronts. Let us arithmetize the location of the fronts at time τ by means of the vector $\boldsymbol{l}(\tau)$ (Fig. 1). We assume that the rate of damage accumulation at points at which the front arrives at future moments of time $t > \tau$ depends on the position of the fronts at time τ and also on the loading vector and the damage vector $\varphi(t, \tau)$ at that time. As a result we arrive at a system of differential equations

$$\frac{\partial \varphi_k(t, \tau)}{\partial \tau} = f_k[\mathbf{l}(t), \mathbf{l}(\tau), \mathbf{s}(\tau), \boldsymbol{\varphi}(t, \tau)] \quad (k=1, \ldots, \nu), \tag{2.4}$$

on the right sides of which stand the functions $f_k(\ldots)$ of the above-mentioned arguments. System of equations (2.3), (2.4) is closed by the relations

$$\varphi_k(t, t) = \psi_k(t) \quad (k=1, \ldots, \nu). \tag{2.5}$$

If for the given initial conditions $\mathbf{l}(0)=\mathbf{l}_0$; $\boldsymbol{\varphi}(0,0)=\boldsymbol{\psi}_0$ system (2.3)-(2.5) does not have real solutions for at least some of the components of the vector $\boldsymbol{l}$, then the crack parameters corresponding to these components are subcritical. This corresponds to the incubatory stage of damage accumulation at the crack fronts. The quasiequilibrium growth of these parameters begins when the damage at the front, whose accumulation is described by Eq. (2.4), reaches a certain level.

We will demonstrate how the system (2.3)-(2.5) can be approximately reduced to a system of ordinary differential equations in the components of the process $\boldsymbol{l}(t)$. Let $m = \mu$ and to each crack parameter l_j let there correspond the damage criterion ψ_j. Owing to the heavy stress concentration at the crack fronts accumulation is most intense in the neighborhood of the fronts. Accordingly, from (2.5) there follow the approximations

$$\psi_j(t) \approx \psi_j(0) + \int_{t-\Delta\tau_j}^{t} f_j[\mathbf{l}(t), \mathbf{l}(\tau), \mathbf{s}(\tau), \boldsymbol{\psi}(\tau)] d\tau \quad (j=1, \ldots, m). \tag{2.6}$$

Here, $\Delta\tau_j$ is the characteristic time of passage of the front through the intense damage accumulation zone. We will denote the characteristic dimension of this zone by ρ_j (see Fig. 1). Let the average velocities $dl_j/dt \neq 0$. Then $\Delta\tau_j \sim \rho_j (dl_j/dt)^{-1}$. This approximation includes averaging over a large number of zones of dimension $\sim\rho_j$ simultaneously located at the crack fronts [7]. Instead of (2.6) we obtain

$$\psi_j(t) \approx \psi_j(0) + \rho_j (dl_j/dt)^{-1} f_j[\mathbf{l}(t), \mathbf{l}(t), \mathbf{s}(t), \boldsymbol{\psi}(t)] \quad (j=1, \ldots, m). \tag{2.7}$$

Solving Eqs. (2.7) for $\psi_j(t)$ and substituting the results in (2.3), we arrive at the desired system of differential equations. We can achieve further simplifications by assuming that the damage accumulation rate $\psi_j(t)$ depends on the corresponding generalized forces $G_j[\mathbf{l}(t), \mathbf{S}(t)]$ and not on the processes $\boldsymbol{l}(t)$ and $\mathbf{s}(t)$ separately. This assumption, analogous to the self-similarity postulate in linear fracture mechanics, makes it possible to narrow considerably the class of functions in relations (2.7). In what follows we assume that

$$\psi_j(t) \approx \psi_j(0) + \rho_j (dl_j/dt)^{-1} F_j[G_j(t), \psi_j(t)] \quad (j=1, \ldots, m). \tag{2.8}$$

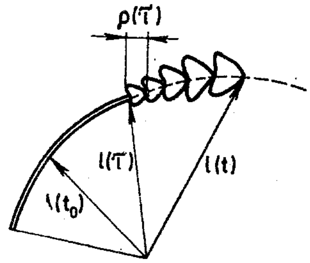

Fig. 1

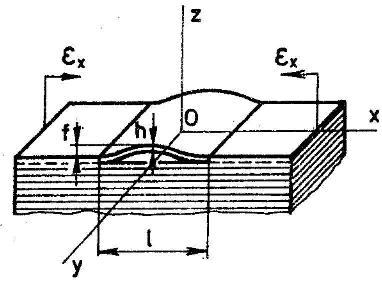

Fig. 2

3. The arrangement of the laminations and the structure of the composite predetermine the orientation and direction of development of the defect. If the delamination is in the interior of a compressed structural element and is sufficiently extensive, then the defect will have an important influence on the carrying capacity [2]. If the defect is close to the surface, then its development may have an adverse effect on the functional properties of the main element (airtightness, aerodynamic quality, etc.) and it may also become a source of further damage leading to complete failure. In what follows we will consider defects located close to the surface. We will refer to such defects as exfoliation.

Usually, the relations between the dimensions of the structural element and the size of the defect are such that it is permissible to consider the loading of the defect "hard," i.e., corresponding to given strains. The magnitude of these strains is determined by the loads applied to the main element and is practically independent of the exfoliation behavior. Particularly important is exfoliation that develops in compression. Such conditions may arise both in the manufacturing stage (compressive residual stresses) and during service (main element in compression or in bending with the exfoliation in the compression zone, nonuniform temperature fields of external origin). In composites with a layered structure, where the thermal expansion coefficients change sharply on transition from one layer to another, compressive strains may develop even if the temperature field is uniform.

The behavior of the exfoliation, especially when small, depends on the initial warping and the initial stresses in the middle surface. As a rule, these factors are already present in the fabrication stage. It is necessary to distinguish the unstressed exfoliation undeformed in the starting state, the deformed exfoliation without initial stresses, and the deformed exfoliation with initial stresses. In the latter case the initial stresses in the middle surface may vary widely depending on the fabrication history and, possibly, preloading. In the first approximation the initial stresses are close to the critical values determined from the corresponding elastic stability problem. Precisely these values are required to maintain the exfoliation in the slightly buckled state. During the growth of the exfoliation the stresses in its middle surface change. To determine them it is necessary at each stage to solve the nonlinear problem of the postcritical strains. In order to simplify the calculations we will assume that at each moment the stresses in the middle surface coincide with the critical values of the linear theory for an exfoliation of the given configuration.

Let us consider the conditions corresponding to the plane problem and assume that the composite is linear-elastic and macroscopically orthotropic with the principal directions of elasticity oriented along the coordinate axes. The element is subjected to "hard" loading with relative deformation $\varepsilon_x = -\varepsilon$ in the direction of the Ox axis (Fig. 2). We will treat the exfoliation as a thin elastic plate clamped at the ends $x = \pm l/2$. This corresponds to the "beam" approximation introduced into fracture mechanics as early as 1930 by I. V. Obreimov. For the deflection w(x) developed during loading we take the expressions

$$w(x) = f\cos^2(\pi x/l). \tag{3.1}$$

Here $f \sim h$, where the thickness of the exfoliation $h \ll l$. Outside the exfoliation $w(x) \equiv 0$. In this schematization the problem of finding the Griffith equilibrium forms of the exfoliation was examined in [8, 9]. Below, following the general theory, we will investigate the quasiequilibrium growth of the exfoliation.

In the initial state let the exfoliation be plane and unstressed, and during the buckling of the exfoliation and its quasiequilibrium growth let the deformation of the middle surface be equal to the critical value, i.e., $\varepsilon_x = -\varepsilon_*(l)$. Here

$$\varepsilon_*(l) = (\pi^2/3)\,(h/l)^2. \tag{3.2}$$

Equating the mutual displacements of the ends of the exfoliation, expressed in terms of its deformation, in the given approximation $u(l/2) - u(-l/2) = -\varepsilon l$, we find the relation between f, l, and ε:

$$f^2 = (4l^2/\pi^2)\,[\varepsilon - \varepsilon_*(l)]. \tag{3.3}$$

We will compute the potential energy of deformation of the exfoliation together with its prolongation over the entire length of the main element:

$$U = \text{const} - \frac{1}{2}\tilde{E}_x hl[\varepsilon^2 - \varepsilon_*^2(l)] + \frac{1}{2}\int_{-l/2}^{l/2} D_x w_{,xx}^2\,dx. \tag{3.4}$$

Here $\tilde{E}_x = E_x(1-\nu_{xy}\nu_{yx})^{-1}$ is the modulus of elasticity for plane strain; $D_x = \tilde{E}_x h^3/12$. The energy is referred to unit width of the exfoliation. The right side of Eq. (3.4) contains a constant equal to the potential energy of compression of the unexfoliated continuation of the surface lamination. With account for Eqs. (3.1)-(3.3), Eq. (3.4) gives $U = \text{const} - {}^1/_2\tilde{E}_x hl[\varepsilon - \varepsilon_*(l)]^2$. Since in the case of "hard" loading the external forces do no work, the generalized force $G(l, \varepsilon)$ driving the exfoliation is linked with the potential strain energy $U(l, \varepsilon)$ by the expression

$$G = -\partial U/\partial l. \tag{3.5}$$

Hence

$$G = \frac{1}{2}\tilde{E}_x h[\varepsilon^2 + 2\varepsilon\varepsilon_*(l) - 3\varepsilon_*^2(l)]. \tag{3.6}$$

Equation (3.6) is valid as long as $G(l, \varepsilon) \geq 0$. If $G(l, \varepsilon) < 0$, then it is necessary to set $G(l, \varepsilon) = 0$. This corresponds to the exfoliation "collapsing". In fact, the boundary values of the strain ε_f determined from the conditions $f = 0$ and $G(l, \varepsilon) = 0$ coincide and constitute $\varepsilon = \varepsilon_*(l)$, which has an obvious mechanical interpretation.

Let the work necessary to form a unit of new surface be constant and equal to γ_0. Then the generalized resistance $\Gamma_0 = 2\gamma_0$. We find the equilibrium dimensions of the exfoliation from the equation $G(l, \varepsilon) = \Gamma_0$. Substituting the expression (3.6) gives

$$\varepsilon^2 + 2\varepsilon\varepsilon_*(l) - 3\varepsilon_*^2(l) = \varepsilon_\infty^2, \tag{3.7}$$

where we have introduce the notation

$$\varepsilon_\infty = (4\gamma_0/\tilde{E}_x h)^{1/2}. \tag{3.8}$$

The value of the strain ε_∞ is fairly large. Typical values of the specific fracture energy for epoxy bonds [5] are $\gamma_0 \sim 10^2$ N/m for a modulus of elasticity of the composite in the direction of the laminations $\tilde{E}_x \sim 10^{11}$ N/m². Substituting these values in Eq. (3.8), taking h ~ 1 mm, gives $\varepsilon_\infty \sim 10^{-3}$.

When $(\sqrt{3}/2)\varepsilon_\infty < \varepsilon < \varepsilon_\infty$ Eq. (3.7) has two real roots l, and when $\varepsilon > \varepsilon_\infty$ one real root. This is illustrated by the right-hand curve in Fig. 3. The descending branch of the curve is unstable, the ascending branch stable. This is apparent, in particular, from the fact that to the right of this curve we have the inequality $G(l, \varepsilon) > \Gamma_0$, i.e., the corresponding exofoliation dimensions are nonequilibrium. The left-hand curve corresponds to the critical strain (3.2). Thus in this formulation quasiequilibrium growth of the exfoliations is possible only in the region left unshaded in Fig. 3.

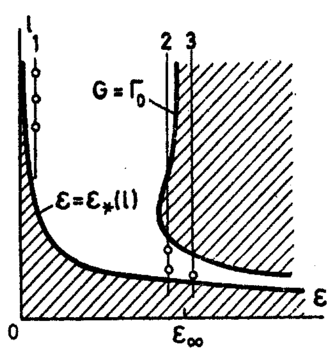

Fig. 3

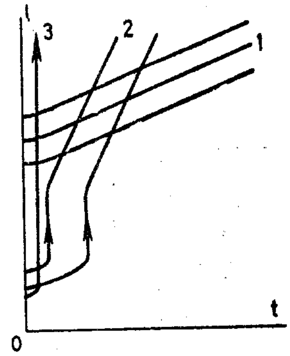

Fig. 4

In order to describe the growth of an exfoliation under long-term loads it is necessary to introduce additional assumptions concerning the effect of the damage criterion ψ on the generalized forces $G(l, \varepsilon)$ and Γ, and to postulate an equation of the type (2.4) describing the process of damage accumulation. Let $G(l, \varepsilon)$ not depend on ψ, whereas

$$\Gamma=\Gamma_0(1-\psi^{\alpha}), \tag{3.9}$$

where $\alpha > 0$. Damage accumulates only in the neighborhood of the lines $x = \pm l/2$, the derivative $d\psi(t)/dt$, equal to the value of $\partial\varphi(t, \tau)/\partial\tau$ at $\tau = t$, being a power function of the generalized force $G(l, \varepsilon)$:

$$\frac{d\psi}{dt}=\frac{1}{t_c}\left(\frac{G}{\Gamma_\psi}\right)^{m/2}. \tag{3.10}$$

Here t_c is the time constant; Γ_ψ is a certain characteristic of the resistance to damage accumulation, which we assume to be a material constant. The exponent m has been introduced into Eq. (3.10) and the analogous equations with the coefficient 1/2. This has been done to preserve for m a significance analogous to that of the exponents in the semiempirical fatigue crack growth equations (Paris—Erdogan, Forman, etc.). In fact, for sufficiently developed exfoliations the generalized force $G(l, \varepsilon)$ is approximately proportional to ε^2, so that the damage accumulation rate (3.10) is approximately proportional to ε^m.

In what follows, we assume that $\Gamma_\psi = \beta^2\Gamma_0$. In (2.8), taking into account (3.10), we must set $F=t_c^{-1}(G/\beta^2\Gamma_0)^{m/2}$. For initial damage $\psi(0) = 0$ we obtain $\psi(t)\approx(\rho/t_c)(dl/dt)^{-1}(G/\beta^2\Gamma_0)^{m/2}$, where ρ is the characteristic dimension of the damaged zone. Substituting this expression in (3.9), we construct the basic equation $G(l, \varepsilon) = \Gamma$, describing the quasiequilibrium growth of the exfoliation. Solving it for dl/dt, we obtain

$$\frac{dl}{dt}=\frac{\rho}{t_c\beta^m}\frac{g^{m/2}(l,\varepsilon)}{[1-g(l,\varepsilon)]^{1/\alpha}};\quad g(l,\varepsilon)=\frac{G(l,\varepsilon)}{\Gamma_0}. \tag{3.11}$$

Equation (3.11) is valid as long as $g(l, \varepsilon) < 1$, i.e., the size of the exfoliation is subequilibrium. If according to Eq. (3.6) $G(l, \varepsilon) < 0$, then it is necessary to put $g(l, \varepsilon) = 0$.

Examples of the integration of the equation for various initial conditions are presented in Fig. 4. For the initial conditions 1 (see Fig. 3) we obtain the family of curves 1 describing the quasiequilibrium growth of the exfoliations. Curves 2 in Fig. 4 correspond to initial conditions 2 in Fig. 3 for which an abrupt increase in the exfoliations to new subequilibrium values is possible. A method of calculating these values was indicated in [6]. Curve 3 corresponds to the initial condition for which after the attainment of a certain size quasiequilibrium growth becomes impossible. Dependences of the same type as curves 2 and 3 are observed in the region of $\varepsilon \sim \varepsilon_\infty$ and sufficiently small l_0/h. This is typical of exfoliations with a very low resistance to interlaminar fracture. Otherwise the initial values of l_0/h will fall in a region where the "beam" approximation is inappropriate.

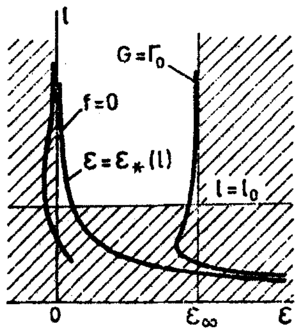

Fig. 5

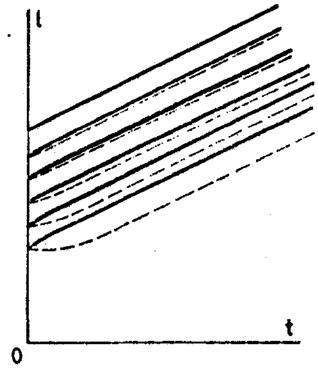

Fig. 6

4. Let the exfoliation have an initial elastic deflection:

$$w_0(x) = f_0 \cos^2(\pi x/l_0)$$

with a deformation of the middle surface $\varepsilon_x^o \approx -\varepsilon_*(l_0)$, where $\varepsilon_*(l)$ is the critical strain (3.2). Instead of (3.3) we obtain the relation

$$f^2 = \frac{4l^2}{\pi^2}[\varepsilon - \varepsilon_*(l)] + \left[\frac{l}{l_0}\left(f_0^2 + \frac{4}{3}h^2\right)\right]. \tag{4.1}$$

Relations (3.4) and (3.5) remain valid. The generalized force $G(l, \varepsilon)$ is given by

$$G = \frac{1}{2}E_x h\left\{\varepsilon^2 + 2\varepsilon\varepsilon_*(l) + 3\varepsilon_*^2(l)\left[\frac{l}{l_0}\left(\frac{f_0^2}{h^2} + \frac{4}{3}\right) - 1\right]\right\}. \tag{4.2}$$

For all ε and $l > l_0$ Eq. (4.3) gives $G(l, \varepsilon) > 0$. However, the expression loses its significance at $f < 0$. To this there corresponds a certain limiting strain of the main element. The magnitude of this strain ε_f may be found from (4.1) by setting $f = 0$. Using (3.2), we obtain

$$\varepsilon_f = -\varepsilon_*(l)\left[\frac{l}{l_0}\left(1 + \frac{3f_0^2}{4h^2}\right) - 1\right]. \tag{4.3}$$

If $f_0 \to 0$, $l = l_0$, then (4.3) gives $\varepsilon_f = 0$. In general, an exfoliation with an initial deflection "collapses" in the presence of tensile strains (see left-hand curve in Fig. 5).

When (3.2), (3.8), and (4.2) are taken into account, the equation $G(l, \varepsilon) = \Gamma_0$ takes the form

$$\varepsilon^2 + 2\varepsilon\varepsilon_*(l) + 3\varepsilon_*^2(l)\left[\frac{l}{l_0}\left(\frac{f_0^2}{h^2} + \frac{4}{3}\right) - 1\right] = \varepsilon_\infty^2. \tag{4.4}$$

The solution of Eq. (4.4) is represented schematically by the right-hand curve in Fig. 5. Since, by condition, $l \geq l_0$, only those segments of the curves for which this inequality is satisfied are meaningful. Therefore, despite the qualitative difference between the diagrams in Figs. 3 and 5, large discrepancies in the numerical results are not to be anticipated in the region left unshaded in Fig. 5.

This is confirmed by an analysis of the quasiequilibrium growth of the exfoliations. Retaining assumptions (3.9) and (3.10), we again arrive at Eq. (3.11) with the difference that in this case the generalized force is determined from (4.2). The graphs of the growth of the exfoliations are presented in Fig. 6. The dashed curves correspond to the case of a plane and initially unstressed exfoliation. With increase in l, the curves asymptotically approach parallel straight lines. This corresponds to the fact that in both cases we have the limits

$$\lim_{l/h\to\infty} g(l, \varepsilon) = \varepsilon^2/\varepsilon_\infty^2, \tag{4.5}$$

so that at sufficiently large l/h, or more precisely at $\varepsilon \gg \varepsilon_*(l)$, Eq. (3.11) takes the form

$$\frac{dl}{dt} = \frac{\rho}{t_c} \frac{(\varepsilon/\beta\varepsilon_\infty)^m}{[1-(\varepsilon/\varepsilon_\infty)^2]^{1/\alpha}}. \tag{4.6}$$

If ε = const, this gives a constant rate of growth of the exfoliation.

Similar results are obtained for cyclical loading. For the fatigue damage criterion, instead of (3.10) it is natural to take the equation

$$\frac{d\psi}{dn} = \left(\frac{\Delta G}{\beta_1^2\Gamma_0}\right)^{m_1/2} \tag{4.7}$$

in the function $\psi(n)$ of the time-smoothed number of cycles $n(t)$. Here ΔG is the amplitude of the generalized force $G(l, \varepsilon)$ within the cycle, which we assume to be a slowly varying function of n. The constants m_1 and β_1 are generally different from the constants m and β in Eq. (3.11). As a result, instead of (3.11) we obtain the fatigue delamination equation

$$\frac{dl}{dn} = \frac{\rho}{\beta_1^{m_1}} \frac{[g(l, \varepsilon_{max}) - g(l, \varepsilon_{min})]^{m_1/2}}{[1-g(l, \varepsilon_{max})]^{1/\alpha}}, \tag{4.8}$$

where ε_{max} and ε_{min} are the extreme values of the strains within the cycle. We take into account the existence of a fatigue damage accumulation threshold by taking instead of (4.7) the equation

$$\frac{d\psi}{dn} = \left(\frac{\Delta G - G_{th}}{\beta_1^2\Gamma_0}\right)^{m_1/2} \quad ; \quad \Delta G > G_{th}. \tag{4.9}$$

Here G_{th} is a constant equal to the threshold value of the amplitude ΔG. When $\Delta G < G_{th}$, we have $d\psi/dn = 0$.

If the processes of fatigue and quasi-static damage accumulation proceed in parallel, then the damage summation principle leads to the equation, generalizing Eqs. (3.11) and (4.8),

$$\frac{dl}{dt} = \frac{\rho}{[1-g(l, \varepsilon_{max})]^{1/\alpha}} \{g^{m/2}(l, \varepsilon_m)/(t_c\beta^m) + [g(l, \varepsilon_{max}) - g(l, \varepsilon_{min})]^{m_1/2}(dn/dt)/\beta_1^{m_1}\},$$

where $\varepsilon_m(t)$ are the mean long-term strains; dn/dt is the number of deformation cycles per unit time. For those cycles for which $\varepsilon_{min} < \varepsilon_f$, where ε_f is the limiting strain of the main element, at which the exfoliation "collapses," it is necessary to take $g(l, \varepsilon_{min}) = 0$ in Eqs. (4.8) and (4.9). We proceed similarly if $\varepsilon_m < \varepsilon_f$ and so on.

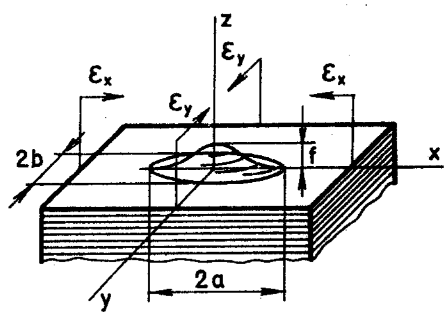

Fig. 7

5. In the earlier sections we have explained the general approach to the investigation of the conditions of stability of exfoliations in laminated composite structures and have also obtained the equations of quasiequilibrium exfoliation growth under cyclic and/or long-term loads. As an example we considered exfoliation under the conditions of the plane problem. The proposed approach is especially effective in the case of multiparameter exfoliation and multiparameter loading regimes. Let us consider an exfoliation which in plan has the shape of an ellipse with semiaxes a and b. The center of the ellipse coincides with the coordinate origin, and the principal axes are oriented along the coordinate axes Ox and Oy (Fig. 7). We will regard the exfoliation as an elastic orthotropic plate clamped along the edges and subjected to displacements corresponding to the strains of the main element ε_x and ε_y. The thickness of the exfoliation h is assumed to satisfy the condition $h \ll a$, b. If the main element is not loaded, then the deflection of the plate will be zero, and there will be no stresses in its middle surface. In the presence of deformations of the main element exceeding the critical values for loss of elastic stability, buckling will occur. We denote the deflection of the exfoliation at the coordinate origin by f. In general, the deflection f may be of the order of the thickness of the plate or even greater. However, we will assume that $f \ll a$, b.

It is necessary to solve the secondary problem of the buckling of exfoliations with fixed dimensions a and b. For the deflection function we will take the expression

$$w(x, y) = f\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2, \tag{5.1}$$

which satisfies the edge clamping conditions. The critical forces in the middle surface N_x and N_y are found from the condition $U_b = W$. Here U_b is the potential bending strain energy of the plate,

$$U_b = \frac{1}{2}\iint_\Omega (D_x w_{,xx}{}^2 + 2D_{xy} w_{,xx} w_{,yy} + D_y w_{,yy}{}^2 + 4D_t w_{,xy}{}^2)\, d\Omega; \tag{5.2}$$

W is a quadratic functional of the linear theory of elastic stability,

$$W = \frac{1}{2}\iint_\Omega (N_x w_{,x}{}^2 + N_y w_{,y}{}^2)\, d\Omega. \tag{5.3}$$

Here $w_{,x} \equiv \partial w/\partial x$ and so on; D_x and D_y are the cylindrical stiffnesses of the exfoliation; D_{xy} is the combined and D_t the torsional stiffness. The forces N_x and N_y are assumed to be positive in compression. The integration in (5.2) and (5.3) is carried out over the area of the exfoliation $\Omega = \pi a b$. Substitution of (5.1) in (5.2) and (5.3) after equating U_b and W gives an equation in the critical strains $\varepsilon_x^*(a, b)$ and $\varepsilon_y^*(a, b)$:

$$(\varepsilon^*_x + \nu_{xy}\varepsilon^*_y) E_x b^2 + (\varepsilon^*_y + \nu_{yx}\varepsilon^*_x) E_x a^2 = (h/ab)^2 H(a, b). \tag{5.4}$$

Here, for brevity, we have introduced the following notation:

$$H(a, b) = 12(1 - \nu_{xy}\nu_{yx}) h^{-3}\left[D_x b^4 + \frac{2}{3}(D_{xy} + 2D_t) a^2 b^2 + D_y a^4\right]. \tag{5.5}$$

The elastic constants in (5.4) and (5.5) are so selected that the symmetry condition has the form $\nu_{xy}/E_y = \nu_{yx}/E_x$.

Equation (5.4) contains two unknowns and in what follows will be supplemented by other equations. For the present we will evaluate this equation for the particular case of an isotropic plate circular in plan under conditions such that $N_x = N_y = N$. Setting $D_x = D_y = D$; $\nu_{xy} = \nu_{yx} = \nu$; $D_{xy} = \nu D$; $D_t = {}^1/_2(1 - \nu)D$, for the critical force we obtain the expression $N = 16D/a^2$. The exact solution is $14.68D/a^2$. For the purposes of this analysis an error of about 7% may be considered acceptable.

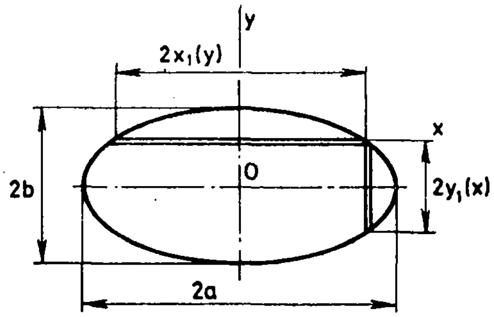

Fig. 8

In order to obtain a closed system of equations we will consider the shortening of the chords of the ellipse parallel to the coordinate axes during the deformation of the main element. The deformation of the middle surface of the exfoliation satisfies the approximation $u_{,x} + {}^1/_2 w_{,x}{}^2 \approx -\varepsilon^*{}_x$. Integrating this relation over the interval $[-x_1(y), x_1(y)]$, where $x = \pm x_1(y)$ is the equation of the boundary of the ellipse, and noting that $u[x_1(y), y] - u\cdot[-x_1(y), y] = -2x_1(y)\varepsilon_x$, we obtain (Fig. 8)

$$\frac{1}{2}\int_{-x_1(y)}^{x_1(y)} w_{,x}{}^2(x, y)\,dx = (\varepsilon_x - \varepsilon^*{}_x)2x_1(y).$$

Integrating again with respect to y over the interval $[-b, b]$ gives

$$\frac{1}{2}\iint_\Omega w_{,x}{}^2 d\Omega = (\varepsilon_x - \varepsilon^*{}_x)\Omega;\quad \frac{1}{2}\iint_\Omega w_{,y}{}^2 d\Omega = (\varepsilon_y - \varepsilon^*{}_y)\Omega \tag{5.6}$$

(the second equation was obtained from the first by index substitution). Substituting expression (5.1) in (5.6), we arrive at the relations

$$f^2 = 3a^2(\varepsilon_x - \varepsilon^*{}_x) = 3b^2(\varepsilon_y - \varepsilon^*{}_y). \tag{5.7}$$

Using (5.4) and (5.7), we compute the critical strains

$$\begin{aligned}\varepsilon^*{}_x &= [E_y(a^2+\nu_{yx}b^2)(\varepsilon_x a^2 - \varepsilon_y b^2) + (h/a)^2 H(a, b)]/H_1(a, b);\\ \varepsilon^*{}_y &= [E_x(b^2+\nu_{xy}a^2)(\varepsilon_y b^2 - \varepsilon_x a^2) + (h/b)^2 H(a, b)]/H_1(a, b).\end{aligned} \tag{5.8}$$

Here, together with (5.5), we have used the notation

$$H_1(a, b) = E_x b^4 + 2\nu_{xy}E_x a^2 b^2 + E_y a^4. \tag{5.9}$$

For the isotropic case we find that for $\nu = 1/3$ the functions $H(a, b)$ and $H_1(a, b)$ coincide.

Substitution of expressions (5.8) in one of relations (5.7) gives

$$f^2 = \frac{3a^2b^2}{H_1(a, b)}\left[E_x\varepsilon_x(b^2+\nu_{xy}a^2) + E_y\varepsilon_y(a^2+\nu_{yx}b^2) - \frac{h^2H(a, b)}{a^2b^2}\right]. \tag{5.10}$$

Taken together, relations (5.8) and (5.10) now make it possible to construct the equations determining the equilibrium and stable dimensions of the exfoliations.

6. Let the surface layers of the main element be subjected to compressive strains $\varepsilon_x > \varepsilon^*{}_x(a, b)$ and $\varepsilon_y > \varepsilon^*{}_y(a, b)$. We will compute the potential strain energy of an element with an exfoliation as a function of the dimensions of the exfoliation $U = U_b + U_c + U_0$. Here U_b is the bending energy of the exfoliation determined from (5.2) by substituting in it expressions (5.1) and (5.10); U_c is the compressive energy of the exfoliation; U_0 is the potential strain energy of the main element. Considering that within the exfoliation the strains

of the middle surface are assumed to be constant and equal to the critical strains $\varepsilon^*_x(a, b)$ and $\varepsilon^*_y(a, b)$, we find that, irrespective of the shape of the structure and the way it is supported and loaded, we have

$$U_c+U_0=\text{const}-\frac{\pi abh}{2(1-\nu_{xy}\nu_{yx})}\left[E_x(\varepsilon_x^2-\varepsilon^*{}_x^2)+2\nu_{xy}E_x(\varepsilon_x\varepsilon_y-\varepsilon^*{}_x\varepsilon^*{}_y)+E_y(\varepsilon_y^2-\varepsilon^*{}_y^2)\right]. \tag{6.1}$$

Here we have written down explicitly only that part of the energy which makes the main contribution to the generalized forces responsible for the growth of the exfoliation. These generalized forces we determine from the expressions

$$G_a=-\partial U/\partial a;\quad G_b=-\partial U/\partial b, \tag{6.2}$$

which follow from the general relations (1.2). Considering that the elementary work done by the resistance forces $dA_\gamma=2\gamma_0[\pi(a+da)(b+db)-\pi ab]$, we obtain

$$\Gamma_a=2\pi\gamma_0 b;\quad \Gamma_b=2\pi\gamma_0 a.$$

The equilibrium dimensions a and b are found from the equations

$$G_a(a, b)=2\pi\gamma_0 b;\quad G_b(a, b)=2\pi\gamma_0 a.$$

In the general case the further computations are very clumsy and do not permit general conclusions of a qualitative kind to be drawn. Accordingly, we will consider the particular case of an exfoliation circular in plan. In this case the load-carrying layers of the composite will be considered isotropic and the compression uniform. Thus, a = b; $E_x = E_y = E$; $\nu_{xy} = \nu_{yz} = \nu$; $\varepsilon_x = \varepsilon_y = \varepsilon$. We obtain a further simplification by setting $\nu = 1/3$, so that the right sides of (5.5) and (5.9) coincide. In this case $\varepsilon_*(a)=(h/a)^2$; $f^2=3(\varepsilon a^2-h^2)$; $H(a)=H_1(a)=8Ea^4/3$. Substituting these expressions in (5.2) and (6.1), we calculate the force driving the exfoliation:

$$G=-\partial U/\partial a=3\pi Eha[\varepsilon^2-\varepsilon_*^2(a)]. \tag{6.3}$$

Equation (6.3) is valid as long as $\varepsilon \geqslant \varepsilon_*(a)$. In order to find the corresponding resistance Γ_0, we find the work which must be done to increase the radius of the exfoliation a to the value a + da: $dA_\gamma = 2\gamma_0[\pi(a + da)^2 - \pi a^2]$.

$$\Gamma_0=4\pi\gamma_0 a. \tag{6.4}$$

The Griffith equilibrium dimensions of the exfoliation are determined from the condition $G(a, \varepsilon) = \Gamma_0$. Substitution of expressions (6.3) and (6.4) gives

$$a=h(\varepsilon^2-\varepsilon_\infty^2)^{-1/4}, \tag{6.5}$$

where as distinct from (3.8) we have introduced the notation

$$\varepsilon_\infty=(4\gamma_0/3Eh)^{1/2}.$$

If $\varepsilon < \varepsilon_\infty$, then there are no Griffith equilibrium exfoliations. At $\varepsilon > \varepsilon_\infty$ the dependence $a(\varepsilon)$ is a monotonically decreasing one. Thus, equilibrium dimensions (6.5) correspond to unstable exfoliations (Fig. 9). When $\varepsilon \leqslant \varepsilon_*(a)$, the exfoliation remains plane. Quasiequilibrium growth is possible only in the unshaded region of Fig. 9.

The equations describing the growth of the exfoliations are obtained from general considerations (2.3)-(2.8) by introducing special assumptions concerning the laws of damage accumulation and the effect of the damage on the properties of the composite. Thus, if we retain assumptions (3.9) and (3.10), we arrive at the system of differential equations

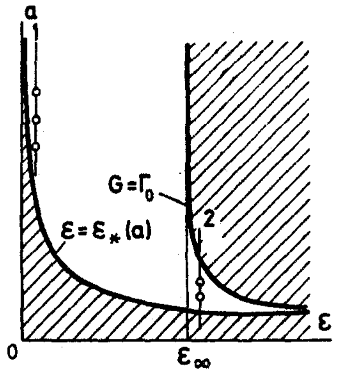

Fig. 9

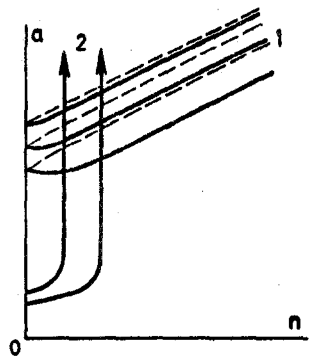

Fig. 10

$$\frac{da}{dt}=\frac{\rho_a}{t_c}\frac{(G_a/\beta_a^2\Gamma_a)^{m_a/2}}{(1-G_a/\Gamma_a)^{1/\alpha_a}};\quad \frac{db}{dt}=\frac{\rho_b}{t_c}\frac{(G_b/\beta_b^2\Gamma_b)^{m_b/2}}{(1-G_b/\Gamma_b)^{1/\alpha_b}}. \tag{6.6}$$

In Eqs. (6.6) we have taken into account the essential anisotropy of the elasticity and strength properties of the composite.

We will examine in more detail a circular exfoliation in a composite with isotropic load-carrying layers. We write down the growth equation of the exfoliation for cyclic loading (the duration of the cycle is identical with the time constant t_c):

$$\frac{da}{dn}=\frac{\rho[g(a,\varepsilon_{max})-g(a,\varepsilon_{min})]^{m/2}}{\beta^m[1-g(a,\varepsilon_{max})]^{1/\alpha}}. \tag{6.7}$$

In this case

$$g(a,\varepsilon)=\frac{G(a,\varepsilon)}{\Gamma_0}=\frac{1}{\varepsilon_\infty^2}[\varepsilon^2-\varepsilon_*^2(a)].$$

If $\varepsilon_{max} < \varepsilon_*(a)$, it is necessary to set $a(a, \varepsilon) = 0$. Certain results of integrating Eq. (6.7) are presented in Fig. 10. In this case the initial conditions for curves 1 were taken from the unshaded region in Fig. 9 at $\varepsilon < \varepsilon_\infty$. Curves 2 were obtained for initial conditions from the same region at $\varepsilon > \varepsilon_\infty$. These conditions correspond to small values of a/h, at which the approximation of thin plate theory is inapplicable. Accordingly, the corresponding numerical results are of the nature of a model.

7. So far we have assumed that in the initial state the exfoliation is plane, and that there are no initial stresses in it. In reality, the initial exfoliations of technological origin are buckled and the middle surface is under stress. We will show what changes must be introduced into the results obtained above in order to take into account the initial buckling. Let the initial buckling have the form

$$w_0(x,y)=f_0\left(1-\frac{x^2}{a_0^2}-\frac{y^2}{b_0^2}\right)^2, \tag{7.1}$$

where a_0 and b_0 are the dimensions of the initial exfoliation in plan; f_0 is the initial rise. We assume that $f_0 \ll a_0, b_0$, but $f_0 \sim h$. The initial stresses in the middle surface mainly depend on how the product was formed. We will assume that these stresses are equal to the critical values from the corresponding linear problem of elastic stability. In this case the initial strains $\varepsilon_x^\circ = \varepsilon_x^*(a_0, b_0)$ and $\varepsilon_y^\circ = \varepsilon_y^*(a_0, b_0)$ are related by expression (5.4). We will neglect the initial strains in the adjacent region of the main element.

In order to obtain the analog of Eq. (5.7), we write out the expressions for the components of the finite strain tensor and equate them to the difference between the critical strains in the state in question and the initial strains, referred to the dimensions of the exfoliation a and b:

$$u_{,x}+{}^1/_2 w_{,x}{}^2-{}^1/_2 w_{0,x}{}^2=-(\varepsilon^*{}_x-\varepsilon_x{}^0 a_0/a);$$
$$v_{,y}+{}^1/_2 w_{,y}{}^2-{}^1/_2 w_{0,y}{}^2=-(\varepsilon^*{}_y-\varepsilon_y{}^0 b_0/b). \qquad (7.2)$$

Here u(x, y) and v(x, y) are the displacements in the middle surface in the direction of axes O_x and O_y, respectively. The strains have been considered positive if they correspond to compression. On the other hand, the displacements at the ends of the chords of the ellipse y = const and x = const (see Fig. 8) are related by the expressions $u[x_1(y),y]-u[-x_1(y),y]=-2x_1(y)\varepsilon_x$; $v[x,y_1(x)]-v[x,-y_1(x)]=-2y_1(x)\varepsilon_y$. Here $x = \pm x_1(y)$ and $y = \pm y_1(x)$ are the equations of the boundary of the exfoliation; ε_x and ε_y are the strains of the main element. We integrate relations (7.2) over the lengths of the chords. As a result we arrive at equations of type (5.6). Repeat integration with respect to y gives

$$\frac{1}{2}\iint\limits_{\Omega} w_{,x}{}^2 d\Omega=\frac{1}{2}\iint\limits_{\Omega_0} w_{0,x}{}^2 d\Omega+\left(\varepsilon_x-\varepsilon^*{}_x+\varepsilon_y{}^0\frac{a_0}{a}\right)\Omega; \qquad (7.3)$$
$$\frac{1}{2}\iint\limits_{\Omega} w_{,y}{}^2 d\Omega=\frac{1}{2}\iint\limits_{\Omega_0} w_{0,y}{}^2 d\Omega+\left(\varepsilon_y-\varepsilon^*{}_y+\varepsilon_y{}^0\frac{b_0}{b}\right)\Omega.$$

The equations are consistent in the sense that for the initial state, when $\varepsilon_x=\varepsilon_y=0$; $a=a_0$; $b=b_0$; $\varepsilon_x{}^0=\varepsilon^*{}_x(a_0,b_0)$; $\varepsilon_y{}^0=\varepsilon^*{}_y(a_0,b_0)$, they are satisfied identically. In this connection, it is important that we took the initial forces in the exfoliation to be equal to the critical values of the linear problem. Substitution of (7.1) in Eqs. (7.3) gives

$$f^2=3a^2(\varepsilon_x-\varepsilon^*{}_x)+(ab_0/a_0b)f_0{}^2+3aa_0\varepsilon_x{}^0= \qquad (7.4)$$
$$=3b^2(\varepsilon_y-\varepsilon^*{}_y)+(a_0b/ab_0)f_0{}^2+3bb_0\varepsilon_y{}^0.$$

Jointly solving Eqs. (5.4) and (7.4), we express the strains $\varepsilon_x{}^*(a, b)$ and $\varepsilon^*{}_y(a, b)$ and the rise f in terms of the given parameters of the problem. It is then necessary to calculate the potential strain energy from (5.2) and (6.1), after which we determine the generalized forces $G_a(a, b)$ and $G_b(a, b)$ from Eqs. (6.2).

As in the previous case of a plane exfoliation, for the purposes of obtaining more transparent results we will examine in greater detail a circular isotropic exfoliation. We construct the expression for f. Using (7.4), we have $f^2=3(\varepsilon a^2-h^2)+f_0{}^2+3(a/a_0)h^2$, where we have borne in mind that $\varepsilon_0 = \varepsilon_*(a_0) = (h/a_0)^2$. Instead of (6.3) we obtain the expression

$$G=3\pi Eha\left[\varepsilon^2+\varepsilon_*{}^2(a)\left(\frac{a}{a_0}+\frac{2f_0{}^2}{3h^2}-1\right)\right] \qquad (7.5)$$

and instead of (6.5) the equilibrium dimension equation

$$\varepsilon^2+\left(\frac{h}{a}\right)^4\left(\frac{a}{a_0}+\frac{2f_0{}^2}{3h^2}-1\right)=\varepsilon_\infty{}^2. \qquad (7.6)$$

The growth equations have the same form as in the case of an initially plane exfoliation. Thus, for a circular exfoliation in a composite with isotropic load-carrying layers Eq. (6.7) is applicable. However, in this case, taking into account Eq. (7.5),

$$g(a,\varepsilon)=\frac{1}{\varepsilon_\infty{}^2}\left[\varepsilon^2+\varepsilon_*{}^2(a)\left(\frac{a}{a_0}+\frac{2f_0{}^2}{3h^2}-1\right)\right].$$

For all $a \geq a_0$ we have $g(a, \varepsilon) \geqslant 0$. If

$$\varepsilon<\varepsilon_f=-\varepsilon_*(a)\left(\frac{a}{a_0}+\frac{f_0{}^2}{3h^2}-1\right),$$

then it is necessary to set $g(a, \varepsilon) = 0$. The condition $\varepsilon = \varepsilon_f$ corresponds to the vanishing of the rise f. Thus, the region of initial values for a in the plane $\{\varepsilon, a\}$ is bounded by the curve of equilibrium exfoliations whose dimensions satisfy Eq. (7.6), by the curve $\varepsilon = \varepsilon_f(a)$ and by the straight line $a = a_0$. As a result, we arrive at a diagram very similar to the diagram for the plane problem (see Fig. 5).

At $\varepsilon >> \varepsilon_*(a)$ the difference in the results given by the two models considered ceases to be important. In this case $g(a, \varepsilon) \approx \varepsilon^2/\varepsilon_\infty^2$, which is analogous to the limit (4.5) for the plane problem. The $a(n)$ curves for exfoliations with an initial deflection are shown dashed in Fig. 10.

This simplification is also valid in the general case. Let the following condition be satisfied:

$$\varepsilon_x^2+2\nu_{xy}\varepsilon_x\varepsilon_y+(E_y/E_x)\varepsilon_y^2 \gg \varepsilon^{*2}_x+2\nu_{xy}\varepsilon^*_x\varepsilon^*_y+(E_y/E_x)\varepsilon^{*2}_y. \tag{7.7}$$

We will use the notation for the reduced strain $\varepsilon_r=[\varepsilon_x^2+2\nu_{xy}\varepsilon_x\varepsilon_y+(E_y/E_x)\varepsilon_y^2]^{1/2}$. Then, if (7.7) is satisfied, we obtain $G_a/\Gamma_a \approx G_b/\Gamma_b \approx E_x h\varepsilon_r^2/4\gamma = (\varepsilon_r/\varepsilon_\infty)^2$, where ε_∞ is determined from (3.8). Thus, Eqs. (6.6) take the form

$$\frac{da}{dt}=\frac{\rho_a}{t_c}\frac{(\varepsilon_r/\beta_a\varepsilon_\infty)^{m_a}}{[1-(\varepsilon_r/\varepsilon_\infty)^2]^{1/\alpha_a}};\quad \frac{db}{dt}=\frac{\rho_b}{t_c}\frac{(\varepsilon_r/\beta_b\varepsilon_\infty)^{m_b}}{[1-(\varepsilon_r/\varepsilon_\infty)^2]^{1/\alpha_b}}. \tag{7.8}$$

Although Eqs. (7.8) were obtained for a plane exfoliation in a plane structural element, under certain conditions they also apply to exfoliations on the surface of curved elements. These conditions take the form

$$\varepsilon_r \ll \varepsilon_*(a, b, R_1, R_2),\ \max\{a, b\} \ll \min\{R_1, R_2\},$$

where ε_r is the reduced strain (7.8); ε_* is the critical strain for the exfoliation; R_1 and R_2 are the principal radii of curvature of the exfoliation. In this case it is sufficient to project the exfoliation onto a tangent plane with a local rectangular Cartesian coordinate system, and in computing the generalized forces G_a and G_b to take into account only the release of energy of the moment-free state at the exfoliation front. From Eqs. (7.8) it follows that when $\rho_a=\rho_b$; $m_a=m_b$; $\alpha_a=\alpha_b$; $\beta_a=\beta_b$ the dimensions a and b will grow at the same rate, despite the anisotropy of the strain field: $a = a_0 + ct$; $b = b_0 + ct$. Thus, if these equalities are satisfied, as the exfoliation grows its shape will approach the circular.

8. The ultimate aim is to predict the growth of defects in composite structures under the action of operational and natural loads and to provide a basis for standards on permissible defects. By combining Eqs. (2.3) and (2.8) we arrive at a vector differential equation describing the growth of the defects in time:

$$\frac{d\mathbf{l}(t)}{dt}=\mathbf{f}[\mathbf{l}(t), \mathbf{s}(t)]. \tag{8.1}$$

Equations (3.11), (4.6), (6.6), etc. are particular cases of Eq. (8.1). Predicting the growth of defects reduces to integrating Eq. (8.1) for given initial conditions and a given loading process. In order to determine the permissible defects it is necessary to solve the inverse problem — to find the initial conditions for which in the course of a specified period of service the size of the defects will not exceed a predetermined limit.

Of particular interest is the solution of these problems for random loading. In many applications the loading process s(t) is quasistationary and quasiergodic. This means that any realization of the process s(t) during the period of service T can be divided into segments, each of which can be treated as a representative realization of a stationary ergodic random process. On transition from one segment to another the properties of the process change only slightly. Following [10], we introduce two arguments for describing the loading process $s(t, \tau)$ — "fast" time t and "slow" time τ. The vector l varies slowly with time and hence can be regarded as a certain function of "slow" time τ. Within each segment we will treat τ as a parameter. The probability density of the values of the process $s(t, \tau)$ will be denoted by $p(s; \tau)$. Averaging Eq. (8.1) gives an approximate expression for the mean rate of of growth of the defects

$$\frac{dl(\tau)}{d\tau} = \int f[l(\tau), s]\, p(s; \tau)\, ds.$$

The integration on the right is carried out over the entire region of values of the process $s(t, \tau)$.

As an example, consider Eqs. (7.8) with $\varepsilon_r \ll \varepsilon_\infty$. The equations are simplified as follows:

$$\frac{da}{dt} = \frac{\rho_a}{t_c}\left(\frac{\varepsilon_r}{\beta_a \varepsilon_\infty}\right)^{m_a}; \quad \frac{db}{dt} = \frac{\rho_b}{t_c}\left(\frac{\varepsilon_r}{\beta_b \varepsilon_\infty}\right)^{m_b}. \tag{8.2}$$

Let the reduced strain $\varepsilon_r(t, \tau)$ be a quasistationary and quasiergodic random process, whose values follow a Rayleigh distribution with slowly varying parameter $\varepsilon_c(\tau) > 0$. Then

$$p(\varepsilon_r, t) = \frac{\varepsilon_r}{\varepsilon_c^2(\tau)} \exp\left[-\frac{\varepsilon_r^2}{2\varepsilon_c^2(\tau)}\right]. \tag{8.3}$$

Averaging the right sides of Eqs. (8.2) with allowance for distribution (8.3), we find that

$$\frac{da(\tau)}{d\tau} = \frac{\rho_a}{t_c}\left[\frac{\varepsilon_c(\tau)}{\beta_a \varepsilon_\infty}\right]^{m_a} 2^{m_a/2}\, \Gamma\left(1 + \frac{m_a}{2}\right);$$
$$\frac{db(\tau)}{d\tau} = \frac{\rho_b}{t_c}\left[\frac{\varepsilon_c(\tau)}{\beta_b \varepsilon_\infty}\right]^{m_b} 2^{m_b/2}\, \Gamma\left(1 + \frac{m_b}{2}\right),$$

where $\Gamma(x)$ is the gamma function. If T is the specified life, and a_* and b_* the maximum permissible dimensions of the exfoliation, then the maximum permissible initial dimensions are

$$a_0 = a_* - \int_0^T \frac{da(\tau)}{d\tau} d\tau; \quad b_0 = b_* - \int_0^T \frac{db(\tau)}{d\tau} d\tau.$$

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